COORBITAL BENDING WAVES AND INCLINATION DECAY. Wm. R. Ward, JPL, Pasadena, CA 91109 and Joseph M. Hahn, Univ. of Notre Dame, Notre Dame, IN 46556.

A secondary orbiting in a self-gravitating disc on an inclined orbit will launch bending waves from vertical resonance sites where the Doppler shifted forcing frequency, $m|\Omega-\Omega_{ps}|$, of an m^{th} order Fourier component of the disturbing potential matches the natural frequency, μ, for vertical oscillations of the disc [1,2]. The pattern speed of an ℓ,m component is $\Omega_{ps} = \Omega_s + (\ell/m - 1)\mu_s$, where Ω_s and μ_s are the mean motion and vertical oscillation frequency of the secondary. The potential amplitudes, $\phi_{\ell,m}$, are of order $|\ell-m|$ in inclination, I; so that the $\ell = m \pm 1$ terms are first order in I and have pattern speeds slightly faster and slower than the perturber's mean motion, i.e., $\Omega_{ps} - \Omega_s = \pm\mu_s/m$. As a result, the inner (outer) resonance of the faster (slower) component falls inside (outside) the perturber's orbit (i.e., at $r_v \approx a(1 -(+)4/3m)$, designated *external*), while the outer (inner) resonance falls at the perturber's semi-major axis ($r_v \approx a$, designated *coorbiting*).

The response of the disc to an m^{th} order perturbation is to launch a spiral bending wave. The attraction of the perturber for this configuration gives rise to a reaction torque on the secondary of magnitude $T_m = \mathrm{sgn}(\Omega-\Omega_{ps})(\pi\sigma\mathcal{H})^2 mGr$, where σ is the surface density of the disc, Ω is the orbital frequency of the resonance site, and $\mathcal{H}$ is the maximum vertical displacement of disc material after the wave has achieved full amplitude [3]. The equations of motion for the perturber are $\dot{L}_\perp = T_m$, $\dot{E} = \Omega_{ps}T_m$, from which one finds the rate of change in the inclination to be [4]

$$(1) \qquad \frac{dI}{dt} = \left[\Omega_s - \Omega_{ps} + 2\Omega_{ps}\sin^2 I/2\right] \frac{T_m}{M(a\Omega_s)^2}$$

The amplitude of an m^{th} order bending wave in a self-gravitating disc is given by Shu et al [1,2]. Substituting their results into eqn (1) we find

$$(2) \qquad \sin I \frac{dI}{dt} = \pm[\cos I \pm 2m\sin^2 I/2]\left(\frac{\pi\sigma r^2}{M}\right) \frac{\pi\langle\partial\phi/\partial z\rangle^2}{a^2\Omega_s|\mathcal{D}|}$$

where $|\mathcal{D}| \equiv |rd/dr\{\mu^2-m^2(\Omega-\Omega_{ps})^2\}| \sim 3m\Omega^2$, $\langle\partial\phi/\partial z\rangle \equiv |\pi^{-1/2}\int_{-\infty}^{\infty}\partial\phi/\partial z \exp i\xi^2 d\xi|$, $\xi \equiv (r/r_v-1)\surd(r|\mathcal{D}|/4\pi G\sigma)$, and the subscripts m$\pm$1, m on ϕ are to be understood. Coorbital torques (lead negative sign) are to be evaluated in the vicinity of the perturber ($r \sim a$) and damp the inclination; while external torques (lead positive sign) are to be evaluated in the vicinity of $r-a \sim \pm 4a/3m$ and excite I. [The second $\pm$ sign refers to slow vs. fast pattern speeds.]

Using the direct portion of the disturbing potential, $\Phi = -(GM/a)\{1+\gamma^2-2\gamma\cos\Delta\theta+(z'-z_s')^2\}^{-1/2}$, the needed forcing amplitudes are

$$(3) \qquad \frac{\partial\phi}{\partial z} = \frac{m}{\pi}\frac{GM}{a^2}\left[I_1(\lambda_-)K_0(\lambda_+) - I_0(\lambda_-)K_1(\lambda_+)\right]\frac{\sin I}{\sqrt{x^2+\sin^2 I}}$$

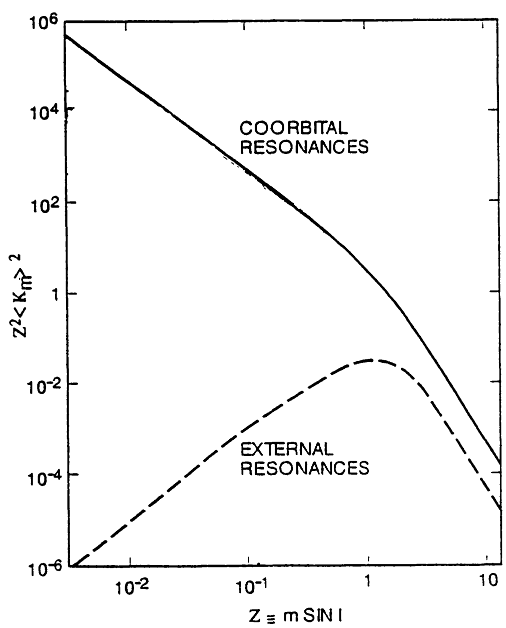
where M is the secondary's mass, $\gamma \equiv r/a$, $x \equiv \gamma-1$, and $\{I_n, K_n\}$ denote modified Bessel functions of the first and second kind with arguments $\lambda_\pm \equiv (m/2)[\surd(x^2+\sin^2 I) \pm |x|]$.

Substituting eqn (3) into (2) yields the desired expression for the rate of change of the inclination due to m^{th} order vertical resonances

$$(4) \qquad \frac{2}{\sin 2I}\frac{dI}{dt} = \pm\frac{2}{3}\frac{m^3}{\tau}\frac{M}{M_p}\left|\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathcal{K}_m \exp -i\xi^2 \, d\xi\right|^2$$

where M_p is the mass of the primary, $\tau \equiv (M_p/\sigma r^2)\Omega_s^{-1}$, and $\mathcal{K}_m \equiv [I_0K_1 - I_1K_0]m^{-1}(x^2+\sin^2 I)^{-1/2}$. In obtaining (4), it is assumed that the disc extends far enough interior and exterior to the secondary to contain resonances from both fast and slow pattern speeds for each m, which are then combined while ignoring any "slow" changes in the disc, i.e., for which d/dr ~ $O(1/r)$. If the spatial variation of the forcing potential is slow compared to the oscillation of the exponential, the quantity $\mathcal{K}_m$ can be pulled out of the integral with the remaining part reducing to unity. For $\sin I > \sqrt{(4\pi\sigma r^2/3M_p)}$, this procedure is valid for all orders.

In this case, eqn (4) can be rearranged to read $d/dt\{\ln(\cos I)\} = \mp (2/3)(m/\tau)(M/M_p)\{Z\mathcal{K}_m(Z)\}^2$, where the last bracketed quantity on the R.H.S. is a function of $Z \equiv m\sin I$ only. Figure 1 displays the behavior of $\{Z\mathcal{K}_m(Z)\}^2$ as a function of Z for both coorbital and external resonances. We expect these results should apply whenever $a\sin I$ exceeds both the thickness of the disc and the Hill's radius of the secondary. At small I, the inclination excitation rate found by Borderies et al. [4] for external resonances is recovered. However, figure 1 indicates that the damping strength of coorbital resonances exceeds the excitation strength of external resonances for all Z. Hence, if the secondary is embedded in a smooth disc without a local gap in the surface density, bending wave interaction should damp the orbital inclination [5,6].



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