Lecture Notes for PHY 405
Classical Mechanics

From Thorton & Marion's *Classical Mechanics*

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Chapter 7: Lagrangian & Hamiltonian Dynamics

Problem Set #4
due Tuesday November 1
at start of class

text problems 7–7, 7–10, 7–11, 7–12, 7–20. Please *derive* all solutions—don't simply show that the text's solutions satisfy your EOM.

Newton's Law $\mathbf{F} = m\dot{\mathbf{p}}$ can be problematic at times.

For instance, the resulting EOM can at times be messy in spherical, cylindrical, or other coordinate systems:

$$\begin{aligned}\mathbf{F}/m &= \ddot{x}\hat{\mathbf{x}} + \ddot{y}\hat{\mathbf{y}} + \ddot{z}\hat{\mathbf{z}} \quad \text{in Cartesian coordinates} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\theta} + \ddot{z}\hat{\mathbf{z}} \quad \text{in cylindrical coord's}\end{aligned}$$

Newton's law requires knowing *all* the forces acting on a particle. In particular, constraints are in additional forces to be accounted for in $F = ma$.

Some forces of constraint are easy
ex.: a particle on a flat plane has $\mathbf{F}_{constraint} = +mg\hat{\mathbf{z}}$

However other problems may have constraining forces that are too complicated or difficult to formulate.
ex.: motion on a curved surface, motion of a bead along a curved wire, etc.

Lagrange equations of motion

An alternate approach is to use *Lagrangian dynamics*,
which is a reformulation of Newtonian dynamics
that can (sometimes) yield simpler EOM.

Another advantage of Lagrangian dynamics is that it can easily account for the forces of constraint.

Begin by noting that the solution to many physics problems can be solved by first invoking a minimization principle.

Ex.: in 1657 Fermat postulated that light rays always travel along the path that requires the least amount of time. From the principle of least time, one can derive the law of reflection (eg, the angle of reflection at a mirror = angle of incidence) and Snell's law of refraction.

We will derive Langrangian mechanics by invoking Hamilton's Principle (1834), which asserts that a dynamical system follows the path that minimizes the time integral of the Lagrangian $L = T - U$,
where T and U are the system's kinetic and potential energies.

Chapter 6 tells us that this integral, sometimes called the *action*, is

$$J = \int_{t_1}^{t_2} L(x_i, \dot{x}_i; t)dt$$

where the $x_i(t)$ with $i = 1, 2, \ldots, N$ are the system's trajectories in this N–dimensional problem, and $\dot{x}_i(t)$ are the velocities.

For example, $x_i(t)$ could represent the $x(t), y(t), z(t)$ of a single particle.

Hamilton's Principle implies that the action J has a minimum along the system's trajectory $x_i(t)$.

Consequently, each of the trajectories $x_i(t)$ obey the Euler–Lagrange eqn's:

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0$$

These equations are usually called the Lagrange eqn's.

Note that Newton's Law can be recovered from the Lagrange eqn's:
Consider the 1D motion of a particle moving in the potential $U = U(x)$:

$$\begin{aligned} L(x, \dot{x}) &= T - U = \frac{1}{2}m\dot{x}^2 - U(x) \\ \text{so} \quad \frac{\partial L}{\partial x} &= -\frac{\partial U}{\partial x} = F \\ \text{thus} \quad F &= \frac{d}{dt}m\dot{x} = m\ddot{x} \quad \text{as expected.} \end{aligned}$$

Note that the Lagrange EOM are a *reformulation* of Newtonian mechanics. They do not introduce any new physics. Rather, they merely provide an alternate approach to solving physical problems.

Note also that Lagrangian dynamics does not deal with forces,
which are *vector* quantities;
rather, it deals with energies, which are scalars
(which can also be simpler to formulate).

Generalized coordinates

Suppose your 3D system has N particles,
and there are m equations of constraint. Then this problem has

$$s = 3N - m \quad \text{degrees of freedom}$$

which means that the problem can be described by
$s = 3N - m$ *generalized coordinates* q_i where $i = 1, 2, \ldots, s$.

The set of $\{q_i\}$ is the smallest possible set of coordinates that can completely specify the state of the system.

Note that the q_i need not have units of length—they might instead be some combination of lengths, energies, angles, dimensionless coordinates, etc.

Note also that the $\dot{q}_i$, which are known as the *generalized velocities*, may or may not have units of length or angle per time.

The Lagrange equations for the generalized coordinates are

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \quad \text{i=1,2,...,s}$$

Example 7.5

A pendulum is attached to a massless rim of radius a that rotates at a constant angular velocity ω. Obtain the Lagrange equation for mass m.

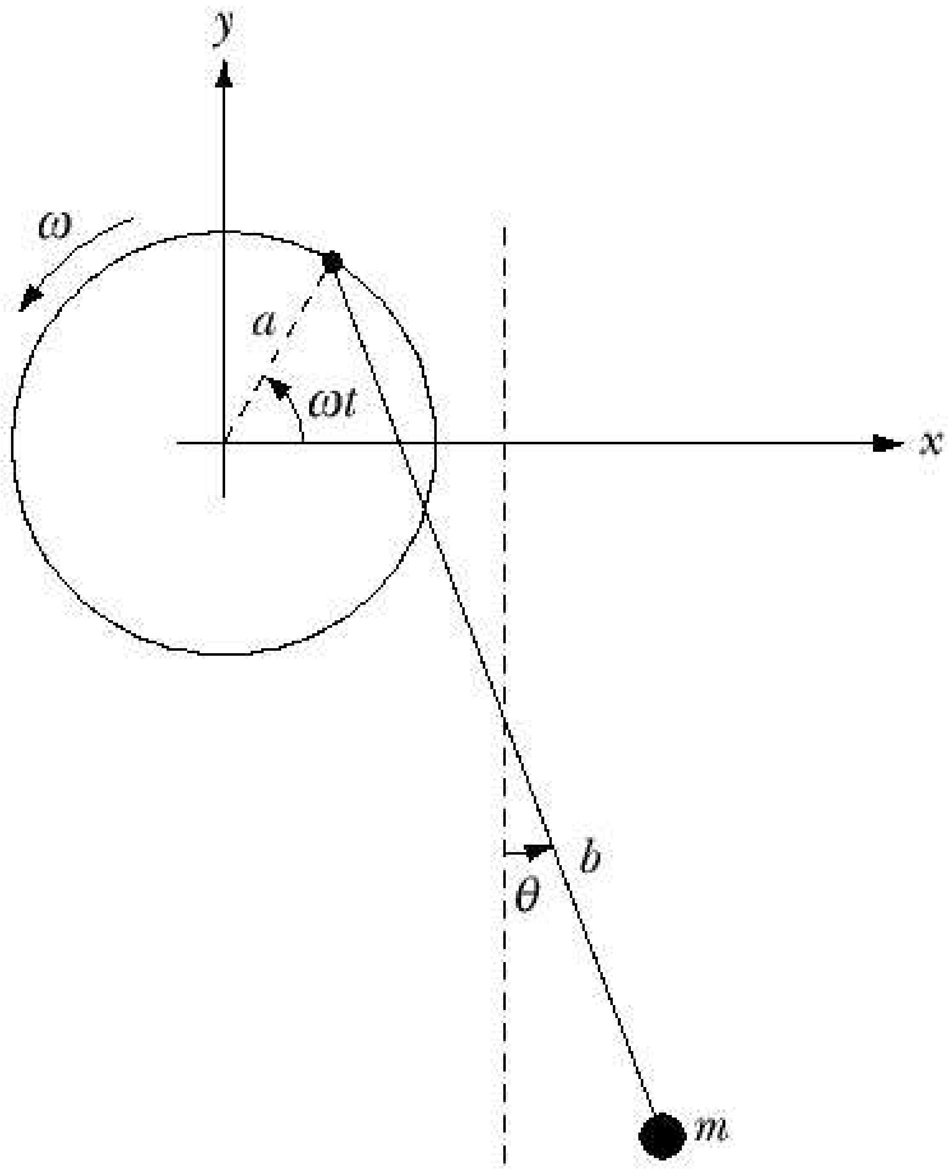

Fig. 7–3.

Begin by writing the Lagrangian $L = T - U$.

What is this system's potential energy U?

We will need m's Cartesian coordinates:

$$\begin{aligned} x &= a\cos\omega t + b\sin\theta \\ y &= a\sin\omega t - b\cos\theta \end{aligned}$$

What is T?

Also need m's velocities:

$$\begin{aligned} \dot{x} &= -a\omega\sin\omega t + b\cos\theta\dot{\theta} \\ \dot{y} &= a\omega\cos\omega t + b\sin\theta\dot{\theta} \\ \text{so}\quad v^2 &= \dot{x}^2 + \dot{y}^2 = (a\omega)^2 + (b\dot{\theta})^2 + 2ab\omega\dot{\theta}(-\cos\theta\sin\omega t + \sin\theta\cos\omega t) \\ \text{but right ()} &= \sin(\theta - \omega t) \\ \text{so}\quad T &= \frac{1}{2}mv^2 = \frac{1}{2}m[(a\omega)^2 + (b\dot{\theta})^2 + 2ab\omega\dot{\theta}\sin(\theta - \omega t)] \\ \text{and}\quad L &= \frac{1}{2}m[(a\omega)^2 + (b\dot{\theta})^2 + 2ab\omega\dot{\theta}\sin(\theta - \omega t)] - mg(a\sin\omega t - b\cos\theta) \end{aligned}$$

What are the generalized coordinates for this system?

The generalized velocities?

We could have written L in terms of x, y and $\dot{x}, \dot{y}$.
Are the generalized coordinates?

What is the Lagrange equation for this system?

$$
\begin{aligned}
\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= 0 \\
\text{where} \quad \frac{\partial L}{\partial \theta} &= mab\omega\dot{\theta}\cos(\theta - \omega t) - mgb\sin\theta \\
\text{and} \quad \frac{\partial L}{\partial \dot{\theta}} &= mb^2\dot{\theta} + mab\omega\sin(\theta - \omega t) \\
\text{so} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= mb^2\ddot{\theta} + mab\omega(\dot{\theta} - \omega)\cos(\theta - \omega t) \\
\text{thus} \quad ab\omega\dot{\theta}\cos(\theta - \omega t) &- gb\sin\theta - b^2\ddot{\theta} - ab\omega(\dot{\theta} - \omega)\cos(\theta - \omega t) = 0 \\
\text{so} \quad \ddot{\theta} + \frac{g}{b}\sin\theta &= \frac{a}{b}\omega^2\cos(\theta - \omega t)
\end{aligned}
$$

is the EOM.

This is the EOM for a pendulum that is driven
by an external torque (eg, the term on the right).

ie, the simple pendulum is recovered when $\omega = 0$.
How would you solve the EOM?

Always keep in mind the distinction in the meaning of a *partial* derivative:

$$\frac{\partial L}{\partial \dot{\theta}} \tag{1}$$

and a *total* derivative:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) \tag{2}$$

If you confuse the two, your EOM will be wrong.

Example 7.7—constrained motion

A bead of mass m slides along a parabolic wire where $z = cr^2$.
The wire rotates with angular velocity ω about the vertical axis.
Obtain the system's Lagrange eqn's.

Also, how fast should the wire rotate in order to suspend the bead at an equilibrium at height $z > 0$.

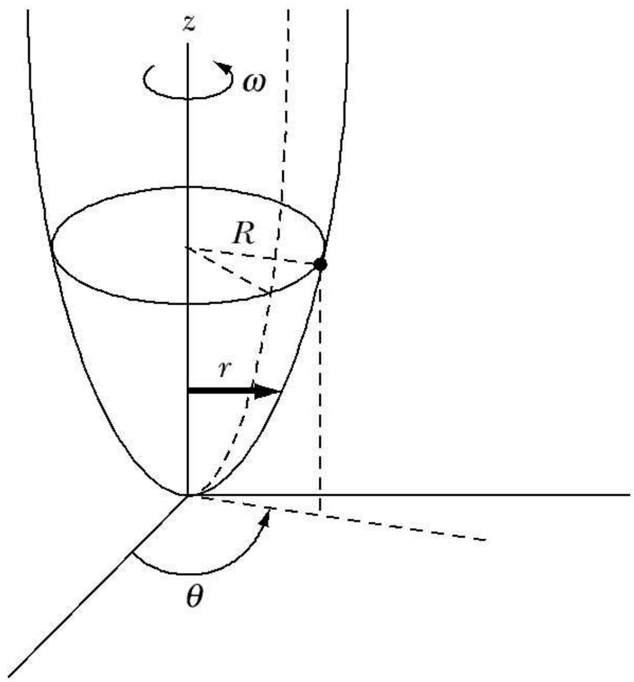

Fig. 7–5

The Lagrangian is

$$\begin{aligned} L &= T - U \\ \text{where} \quad U &= mgz \\ T &= \frac{1}{2}mv^2 \\ \mathbf{v} &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{\mathbf{z}} = \text{ bead's velocity in cylindrical coord's} \\ \text{so} \quad L &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz \end{aligned}$$

Is L written in terms of the system's generalized coordinates?

How do I simplify this further using the constraint imposed by the wire?

First note that $\dot{\theta} = \omega$,
and that $z = cr^2$, so that $\dot{z} = 2cr\dot{r}$ and

$$L \;=\; \frac{1}{2}m(\dot{r}^2 + r^2\omega^2 + 4c^2r^2\dot{r}^2) - mgcr^2$$

What are this system's generalized coordinates?

The Lagrange eqn' is

$$\begin{aligned}
\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) &= 0 \\
\text{where} \quad \frac{\partial L}{\partial r} &= mr\omega^2 + 4mc^2r\dot{r}^2 - 2mgcr \\
\text{and} \quad \frac{\partial L}{\partial \dot{r}} &= m\dot{r} + 4mc^2r^2\dot{r} \\
\text{so} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) &= m\ddot{r} + 8mc^2r\dot{r}^2 + 4mc^2r^2\ddot{r} \\
\text{so} \quad (1 + 4c^2r^2)\ddot{r} \;+\; 4c^2r\dot{r}^2 - r\omega^2 + 2gcr &= 0
\end{aligned}$$

is the L' EOM.

What is the condition for 'floating' the bead
at some equilibrium height $z = cr^2 > 0$?
ie, how fast must the wire rotate for centrifugal force to balance gravity?

Since $\dot{r} = 0$ and $\ddot{r} = 0$,

$$w^2 \;=\; 2gc$$

is the angular at which the wire must spin in order to float the bead.

Example 7.9

A disk of mass M is constrained to roll down an inclined plane without slipping. Solve the Lagrange equations for motion.

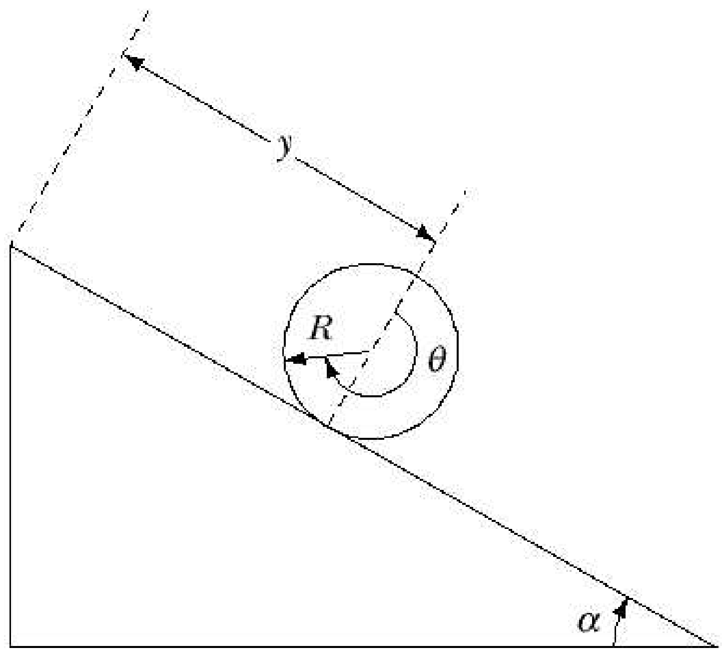

Fig. 6–7

First get the kinetic energy.

Recall from PHY305 that $T = T_{\text{center of mass}} + T_{rot} = \frac{1}{2}M\dot{y}^2 + T_{rot}$,
where $T_{rot} = \frac{1}{2}I\dot{\theta}^2$ is the KE due to the disk's rotation,
$I = \frac{1}{2}MR^2 =$ disk's moment of inertia:

$$T \;=\; \frac{1}{2}M\dot{y}^2 + \frac{1}{4}MR^2\dot{\theta}^2$$

What is U?

The Lagrangian is then

$$L \;=\; T - U = \frac{1}{2}M\dot{y}^2 + \frac{1}{4}MR^2\dot{\theta}^2 + Mgy\sin\alpha$$

What does the no–slip constraint tell us about the coordinates y and θ? What about the velocities?

Tip: put a dot on the disk, and use it to relate y $\leftrightarrow$arclength.

$$y \;=\; R\theta \quad \text{and} \quad \dot{y} = R\dot{\theta}$$

This allows us to write L in terms of a single generalized coordinate:

$$L \;=\; \frac{3}{4}MR^2\dot{\theta}^2 + MgR\theta\sin\alpha$$

The Lagrange equation for this system is

$$\begin{aligned}
\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= 0 \\
\text{so} \quad MgR\sin\alpha &= \frac{3}{2}MR^2\ddot{\theta} \\
\text{ie} \quad \ddot{\theta} &= \frac{2g}{3R}\sin\alpha \\
\text{so} \quad \dot{\theta}(t) &= \frac{2gt}{3R}\sin\alpha \quad \text{assuming disk starts at rest} \\
\text{and} \quad \theta(t) &= \frac{gt^2}{3R}\sin\alpha
\end{aligned}$$

is the solution for the disk's motion.

Problem Set #5
due Thursday November 10
at start of class

text problems 7–17, 7–27, 7–28, 7–33.

Exam #2
on Chapter 7 & Problem Sets 4 & 5
Thursday Nov. 17

The Hamiltonian H

Now lets derive another set of equations of motion from the Hamiltonian H. This is usually obtained from the system's Lagrangian:

$$\text{begin by defining the generalized momentum} \quad p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$$

$$\text{the Lagrange Eqn. is then} \quad \frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \dot{p}_j$$

Example: 1D motion of a single particle:

$$L = \frac{1}{2}m\dot{q}^2 - U(q)$$

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$$

Note that $p =$ the customary mass $\times$ velocity only when q is a length. For other systems, p might instead be an angular momentum, or something else.

Now construct the Hamiltonian H via the following equation:

$$H(p_i, q_i, t) = \sum_j p_j \dot{q}_j - L(q_i, \dot{q}_i, t)$$

where the sum extends over all of the p_i & q_i.

a simple example:

$$
\begin{aligned}
\text{suppose} \quad L &= L(x, y, \dot{x}, \dot{y}) \\
\text{then} \quad p_x &= \frac{\partial L}{\partial \dot{x}} \quad p_y = \frac{\partial L}{\partial \dot{y}} \\
\text{and} \quad H(p_x, p_y, x, y) &= p_x \dot{x} + p_y \dot{y} - L(x, y, \dot{x}, \dot{y})
\end{aligned}
$$

But note that H is *defined* to be a function of the p's and q's, while L is ordinarily a function of q's and $\dot{q}$'s!

How do we exchange the q's and $\dot{q}$'s for p's and q's?

To write H as a function of the p's and q's, use $p_j = \partial L / \partial \dot{q}_j$ to obtain an equation for $\dot{q}_j$ in terms of the p's and q's, ie, $\dot{q}_j = \dot{q}_j(q_i, p_i, t)$.

Then replace each $\dot{q}_j$ appearing in L with the equivalent expression $\dot{q}_j(q_i, p_i, t)$ that depends on the p's and q's
$\Rightarrow$this yields the Hamiltonian $H(p_i, q_i, t)$ in its desired form.

Another set of EOM—Hamilton's equations

To obtain the H' EOM, start by calculating the total derivative of H:

$$\begin{aligned}
H(p_i, q_i, t) &= \sum_j p_j \dot{q}_j - L(q_i, \dot{q}_i, t) \\
\text{so by Chain Rule,} \quad dH &= \sum_j \left(\frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j \right) + \frac{\partial H}{\partial t} dt \\
\text{while derivative of RHS} \quad &= \sum_j \left(\dot{q}_j dp_j + p_j d\dot{q}_j - \frac{\partial L}{\partial q_j} dq_j - \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \right) - \frac{\partial L}{\partial t} dt \\
\text{Note that} \quad \frac{\partial L}{\partial \dot{q}_j} &= p_j \quad \text{and} \quad \frac{\partial L}{\partial q_j} = \dot{p}_j, \\
\text{so RHS} \quad &= \sum_j \left(\dot{q}_j dp_j - \frac{\partial L}{\partial q_j} dq_j \right) - \frac{\partial L}{\partial t} dt
\end{aligned}$$

Think of dH as the total change in H that results when you alter the p_j, q_j, and t by small, *arbitrary* displacements dp_j, dq_j, dt.

Next bring RHS→LHS:

$$\sum_j \left[\left(\frac{\partial H}{\partial p_j} - \dot{q}_j \right) dp_j + \left(\frac{\partial H}{\partial q_j} + \dot{p}_j \right) dq_j \right] + \left(\frac{\partial H}{\partial t} + \frac{\partial L}{\partial t} \right) dt = 0$$

Since the displacements dp_j, dq_j, dt are arbitrary,
what does that tell us about their coefficients?

Thus we get Hamilton's equations:

$$\begin{aligned}
\dot{q}_j &= \frac{\partial H}{\partial p_j} \\
\dot{p}_j &= -\frac{\partial H}{\partial q_j} \\
\frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t}
\end{aligned}$$

⇒The system's H tells you how its p's and q'q evolve over time.

Suppose our system has s degrees of freedom,
ie, L is a function of s generalized coordinates.

The L' EOM would thus yield s second–order DE's,
while Hamilton's Eqn's would yield $2s$ first–order differential eqn's.

Hamilton's eqn's provide yet another distinct set of EOM that are equivalent to the Lagrange EOM and Newton's Laws of motion.

Hamilton's equations are especially useful in studies of nonlinear & chaotic systems.

They are also quite handy when you want to draw a system's phase diagram, plots of the p_i plotted versus the q_i,
and are simply curves of constant $H(p_i, q_i, t)$.

You will also need to know how to construct H in quantum mechanics, since H appears in the Schrödinger eqn'.

The 7 Steps of H

Using Hamilton's Eqn's requires 7 steps:

1. Write $L(q_i, \dot{q}_i, t)$.
2. get the generalizes momenta

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

3. Use the above to solve for $\dot{q}_j = \dot{q}_j(q_i, p_i, t)$.
4. Insert the above into L to express it as $L(q_i, p_i, t)$.
5. Construct the Hamiltonian

$$H(p_i, q_i, t) \;=\; \sum_j p_j \dot{q}_j - L(q_i, p_i, t)$$

6. Get Hamilton's Eqn's:

$$\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} \\ \dot{p}_j &= -\frac{\partial H}{\partial q_j} \end{aligned}$$

7. solve, if possible

Example 7.11

Use Hamilton's eqn's to solve for the motion of a particle of mass m that is subject to a spring–force $\mathbf{F} = -k\mathbf{r}$
while constrained to move on a cylinder of radius R.

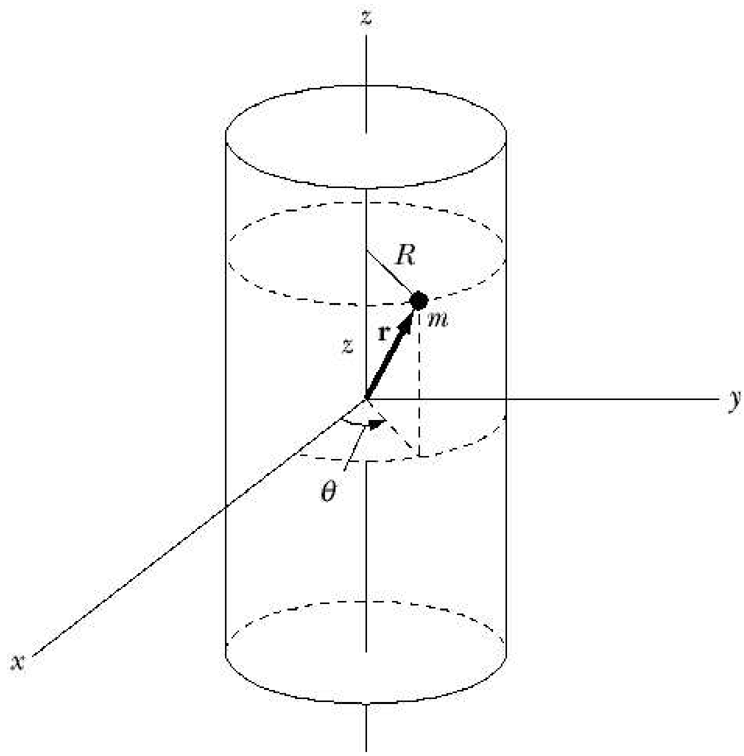

Fig. 7–9.

1. obtain $L = T - U$:

$$U = \frac{1}{2}kr^2 = \frac{1}{2}k(x^2 + y^2 + z^2)$$

How is U affected by the constraint?

$\Rightarrow x^2 + y^2 = R^2$ so $U = \frac{1}{2}k(R^2 + z^2)$.

The particle's KE in cylindrical coordinates is

$$T \;=\; \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$$

How does the constraint alter T?
$\Rightarrow r = R$ and $\dot{r} = 0$ so $T = \frac{1}{2}m(R^2\dot{\theta}^2 + \dot{z}^2)$.

$$\text{and} \quad L = T - U = \frac{1}{2}m(R^2\dot{\theta}^2 + \dot{z}^2) - \frac{1}{2}k(R^2 + z^2)$$

2. get the generalized momenta:

$$\begin{aligned} p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} = \text{ angular momentum} \\ p_z &= \frac{\partial L}{\partial \dot{z}} = m\dot{z} = \text{ vertical momentum} \end{aligned}$$

3. solve for the generalized velocities $\dot{q}_i$:

$$\begin{aligned} \dot{\theta} &= \frac{p_\theta}{mR^2} \\ \text{and} \quad \dot{z} &= \frac{p_z}{m} \end{aligned}$$

4. Write $L = L(q_i, p_i, t)$:

$$L = \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} - \frac{1}{2}k(R^2 + z^2)$$

5. Construct H:

$$\begin{aligned} H(p_i, q_i, t) &= \sum_j p_j\dot{q}_j - L(q_i, p_i, t) = p_\theta\dot{\theta} + p_z\dot{z} - L(q_i, p_i, t) \\ &= \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2}k(R^2 + z^2) \end{aligned}$$

6. Get Hamilton's Eqn's:

$$\begin{aligned}
\dot{q}_j &= \frac{\partial H}{\partial p_j} \quad \text{and} \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \\
\text{so} \quad \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \quad \text{and} \quad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \\
\text{while} \quad \dot{p_\theta} &= -\frac{\partial H}{\partial \theta} = 0 \Rightarrow \text{angular momentum is conserved} \\
\text{and} \quad \dot{p_z} &= -\frac{\partial H}{\partial z} = -kz
\end{aligned}$$

7. Solve for the motion.

Since $p_\theta =$ constant, the particle revolves
with a constant angular velocity $\dot{\theta} = p_\theta/mR^2 \Rightarrow \theta(t) = \dot{\theta}t$.

The equations for the vertical motions is that of a SHO:

$$\begin{aligned}
\ddot{z} &= \frac{\dot{p}_z}{m} = -\frac{k}{m}z = -\omega_0^2 z \\
\text{so} \quad z(t) &= A\cos\omega_0 t
\end{aligned}$$

H conservation

Recall that $L = L(q_i, \dot{q}_i, t)$. Thus by the Chain Rule,

$$\begin{aligned}
\frac{dL}{dt} &= \sum_j \left(\frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \right) + \frac{\partial L}{\partial t} \\
\text{but the Lagrange eqn' is} \quad \frac{\partial L}{\partial q_j} &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \\
\text{so} \quad \frac{dL}{dt} &= \sum_j \left(\dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \right) + \frac{\partial L}{\partial t} \\
&= \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) + \frac{\partial L}{\partial t} \\
\text{but} \quad \frac{\partial L}{\partial \dot{q}_j} &= p_j \\
\text{so} \quad \sum_j \frac{d}{dt}(\dot{q}_j p_j - L) &= \frac{d}{dt}\sum_j (\dot{q}_j p_j - L) = \frac{dH}{dt} = -\frac{\partial L}{\partial t}
\end{aligned}$$

thus if L is independent of $t \quad \Rightarrow \quad H$ is conserved

when does $H = E$?

Next, show that $H = E$ under *certain circumstances*:

$$\begin{aligned}
\text{now suppose} \quad \frac{\partial L}{\partial t} &= 0 \\
\text{and} \quad U &= U(q_i) \quad \text{ie, } U \text{ is independent of } t \text{ and } \dot{q}_i
\end{aligned}$$

what kind of system is this?

Also assume that T is a *quadratic* function of the $\dot{q}$'s:

$$T(\dot{q}_i) = \sum_{j=1}^{N}\sum_{k=1}^{N} a_{j,k}\dot{q}_j\dot{q}_k \quad \text{where } a_{j,k} = a_{k,j}$$

Examples include:

$$\begin{aligned}
T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{y}^2) \quad \text{in Cartesian coord's} \\
\text{or} \quad T &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) \quad \text{in spherical coord's}
\end{aligned}$$

Is $T = \frac{1}{2}m(\dot{x}^2 + \dot{x}\dot{y} + v\dot{z})$ quadratic in the generalized velocities?

Now show that $H = E$ in this case:

$$\text{since} \quad H = \sum_{i=1}^{N} p_i \dot{q}_i - L$$
$$= \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - T + U$$
$$= \sum_{i=1}^{N} \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i - T + U$$

Now calculate

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{N}\sum_{k=1}^{N} a_{j,k}\frac{\partial}{\partial \dot{q}_i}\dot{q}_j\dot{q}_k$$
$$= \sum_{j=1}^{N}\sum_{k=1}^{N} a_{j,k}(\delta_{i,j}\dot{q}_k + \dot{q}_j\delta_{k,i})$$
$$= \sum_{k=1}^{N} a_{i,k}\dot{q}_k + \sum_{j=1}^{N} a_{j,i}\dot{q}_j = 2\sum_{k=1}^{N} a_{i,k}\dot{q}_k \quad \text{since } a_{j,i} = a_{i,j}$$

$$\text{consequently,} \qquad \sum_{i=1}^{N} \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i = 2\sum_{i=1}^{N}\sum_{k=1}^{N} a_{i,k}\dot{q}_i\dot{q}_k = 2T$$

$$\text{and thus} \quad H = 2T - T + U = E$$

To summarize:
When (*i.*) $\partial L/\partial t = 0 \Rightarrow H = \text{constant}$

But when (*i.*) $\partial L/\partial t = 0$
and (*ii.*) $U = U(q_i)$
and (*iii.*) $T = \sum_{j,k} a_{j,k}\dot{q}_j\dot{q}_k \Rightarrow H = E = \text{constant}$

When conditions *i.*, *ii.*, and *iii.* hold, you can readily obtain the system's Hamiltonian H by simply writing down the system energy E.

Just make sure that E is written as a function of the p's and q's rather than the q's and $\dot{q}$'s—you still have to eliminate the $\dot{q}$'s in favor of the p's.

Nonetheless, using $H = E$ to construct the Hamiltonian can be a bit easier than using the formal definition of $H = \sum_i p_i\dot{q}_i - L$.

Cyclic coordinates

The pair (q_k, p_k) are called *canonical conjugates*

and the transformation from
$L(q_i, \dot{q}_i, t) \rightarrow H(q_i, p_i, t)$ is a *canonical transformation.*

A coordinate q_k that does not appear in L or H is said to be *cyclic.*

Cyclic coordinates are especially handy in Hamilton's eqn's since

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} = 0$$

$\Rightarrow$ the momenta of cyclic coordinates are constants of the motion.

Problem 7–28

A particle of mass m moves in the central force–field $F(r) = -k/r^2$. What are the Hamilton EOM?

What coordinate system should I use?

The PE for this system is $U(r) = -k/r$, which recovers $F = -\partial U/\partial r$.

What is the KE in this coordinate system?

The Lagrangian is

$$L \;=\; T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}$$

So the system's momenta are

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \text{so} \qquad \dot{r} = \frac{p_r}{m} \\ \text{and} \quad p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \text{so} \qquad \dot{\theta} = \frac{p_\theta}{mr^2} \end{aligned}$$

Can we simply use $H = E = T + U$ to construct the system's Hamiltonian?

What 3 conditions must be met?

(i) is $\partial L/\partial t = 0$?

(ii) is the system conservative, ie, is $U = U(q_i)$?

(iii) is T quadratic in the $\dot{q}_i$'s?

consequently, we can use $\quad H = T + U = \dfrac{p_r^2}{2m} + \dfrac{p_\theta^2}{2mr^2} - \dfrac{k}{r}$

Hamilton's EOM are

$$\begin{aligned}
\dot{q}_j &= \frac{\partial H}{\partial p_j} \\
\text{so} \quad \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\
\text{and} \quad \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \quad \text{which we already knew...} \\
\text{and} \quad \dot{p}_j &= -\frac{\partial H}{\partial q_j} \\
\text{so} \quad \dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2} \\
\text{and} \quad \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = 0
\end{aligned}$$

Which coordinate is cyclic? What quantity is then conserved?

Note that these results are consistent with our earlier results for the 2–body problem, upon replacing $m \rightarrow \mu$ and $k \rightarrow G(m_1 + m_2)$:

$$\begin{aligned}
p_\theta &= mr^2\dot{\theta} = \quad \text{angular momentum } \ell \text{ in Chapter 8} \\
\text{also note that} \quad \ddot{r} &= \frac{\dot{p}_r}{m} = \frac{p_\theta^2}{m^2 r^3} - \frac{k/m}{r^2}
\end{aligned}$$

which is the eqn' for m's radial acceleration,
which we derived earlier on page 26 of Chapter 8.