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The Viscous Evolution of Saturn's Ring–Satellite System

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Introduction

Saturn's rings represents one of the Solar System's great mysteries. The origin of these rings, as well as their past and present evolution, are all poorly understood. The youthful appearance of these rings is particularly puzzling; the purity of these water–ice rings due to the lack of contamination by dark interplanetary dust suggest an 'exposure' age of order 100 million years (Doyle *et al.* 1989). Similarly, the small satellites orbiting just beyond the main A ring seem even younger, ∼10 million years, due to their interactions with the rings (Poulet & Sicardy 2001). The main challenge then is to understand why these rings appear to be so much younger than the rest of the Solar System. To address this, the following describes the development of a model that will ultimately track the dynamical evolution of Saturn's coupled ring-satellite system. The goal of this effort will be to use this model to determine the past and future histories of this ring system, and to infer its origin.

The ring's radial evolution

Saturn's rings evolve radially due to the viscous torque that is the result of the ring particles' frequent collisions, and also due to the torques that are exerted by Saturn's satellites. The ring is treated here as a thin fluid disk whose evolution is described by a mass continuity equation (left Eqn. 1), and Euler's equation for the fluid velocity $\mathbf{v}$; those equations can then be combined to yield another conservation equation for the ring's angular momentum (right Eqn. 1, see also Pringle 1981):

$$\frac{\partial\sigma}{\partial t}+\nabla\cdot(\sigma\mathbf{v})=0 \quad \text{and} \quad \frac{\partial\ell}{\partial t}+\nabla\cdot(\ell\mathbf{v})=\tau \qquad (1)$$

where $\sigma(r,t)$ is the ring's mass surface density and $\vec{\ell}(r,t)=\sigma\mathbf{r}\times\mathbf{v}$ is the ring's angular momentum surface density. The source term on the right side of (1) is the torque surface density $\vec{\tau}=\sigma\mathbf{r}\times d\mathbf{v}/dt$.

For a nearly keplerian system, the ring's radial velocity is proportional to the torque, $v_r\simeq\tau/\sigma\partial(rv_\theta/\partial r)$. Inserting this into Eqns. (1) then yields a single diffusion equation for the disk's angular momentum surface density (Pringle 1981, Ruden & Lin 1986):

$$\frac{\partial\ell}{\partial t}\simeq-\frac{2}{\sqrt{r}}\frac{\partial}{\partial r}(r^{3/2}\tau). \qquad (2)$$

The rings' evolution is driven by the torque surface density $\tau=\tau_\nu+\tau_s$ which has two parts: the viscous torque τ_ν, and the satellite torque τ_s that is due to resonant perturbations of the ring.

The viscous torque density

The viscous torque surface density is (from Lynden–Bell & Pringle 1974)

$$\tau_\nu=-\frac{3}{2r}\frac{\partial}{\partial r}(\ell\nu) \qquad (3)$$

where ν is the ring's viscosity. A ring of colliding particles has a viscosity $\nu\simeq\tau_d v_d^2/2\Omega=K(v_d/r\Omega)^2\ell/2$, where v_d is the ring particles dispersion velocity, $\Omega(r)$ is the ring's angular velocity, and $\tau_d=K\sigma$ is the ring optical depth where $K=$ ring opacity, and $\ell=\sigma r^2\Omega$ (Goldreich and Tremaine 1982). Note that this viscosity law has $\tau_\nu\propto\ell(\partial\ell/\partial r)$, which makes the diffusion Eqn. (2) nonlinear.

The resonant torque density

A satellite's resonant torque density τ_s is due to the wakes and/or spiral density waves it excites at its various Lindblad resonances (LRs, aka mean–motion resonances) that lie in the ring. A satellite of semimajor axis a_s has an m^{th} LR at radius $r_m=(1\pm1/m)^{2/3}a_s$. The satellite will also be gravitationally attracted to the density disturbance it excites in the ring, and consequently exerts a torque (Goldreich & Tremaine 1978)

$$T_m\simeq\pm8m^2\mu_s^2\sigma r^4\Omega^2=\pm8m^2\mu_s^2\ell r^2\Omega \qquad (4)$$

there, where μ_s is the satellite's mass in Saturn units, all quantities are evaluated that the m^{th} resonance, and the sign is chosen so that the torque is positive/negative for ring material orbiting exterior/interior to the satellite. Assuming this torque is distributed uniformly between adjacent resonances separated by a distance $\Delta r_m\simeq2a_s/3m^2=3\Delta^2/2a_s$, then the satellite's resonant torque density is

$$\tau_s=\frac{T_m}{2\pi r\Delta r_m}\simeq\pm0.4\mu_s^2\left(\frac{r}{\Delta}\right)^4\ell\Omega \quad \text{(Goldreich \& Tremaine 1980)} \qquad (5)$$

where $\Delta=$satellite–resonance distance.

Satellite migration

The total torque that the satellite exerts upon the disk is

$$T=\int_{disk}2\pi\tau_s r dr, \qquad (6)$$

so $-T$ is the torque that the disk exerts upon the satellite of mass M_s, which causes its orbit to evolve at the rate

$$\frac{da_s}{dt}=-\frac{2T}{M_s a_s\Omega_s}. \qquad (7)$$

This same calculation is also employed in studies of type II planet migration (Ward 1997).

The evolution equations

The torque that a small satellite (such as Pan) exerts in a planetary ring tends to carve open a very narrow gap in the ring whose half–width is $r_{gap}\ll r$; this is our problem's natural unit of length, while the natural unit of time is the viscous timescale $T_\nu=r_{gap}^2/9\nu$. This process can be studied using the dynamical equations (2–7), which we make linear by assuming that the viscosity ν is a constant. We also ignore changes in slowly varying quantities like $r,\Omega(r)$, etc., and make the equations dimensionless by replacing the small distance Δ with $x=\Delta/r_{gap}$ and time $t\rightarrow t/T_\nu$:

$$\frac{\partial\ell}{\partial t}=-\frac{\partial\tau}{\partial x} \quad \text{where} \quad \tau=-\frac{1}{3}\frac{\partial\ell}{\partial x}+\frac{s\ell}{(x-x_s)^4} \qquad (8)$$

is the dimensionless viscous + resonant torque densities, and $s=\text{sgn}(x-x_s)$ where x_s is the satellite's dimensionless radial coordinate which evolves at the rate

$$\frac{dx_s}{dt}=-2\mu_{ds}x_{gap}\int_{disk}\frac{s\ell dx}{(x-x_s)^4} \qquad (9)$$

where $\mu_{ds}=\pi\sigma r^2/M_s$ is the dimensionless disk–satellite mass ratio, and $x_{gap}=r_{gap}/r=(\mu_s^2 r^2\Omega/9\nu)^{1/3}$ is the fractional gap half–width.

The numerical solution

The disk evolution is obtained from Eqn. (8), which is solved numerically using a Crank–Nicholson finite difference scheme (Richtmyer & Morton 1967). A Runga Kutta integrator (Press *et al* 1994) is used to solve Eqn. (9) and track the satellite's position $x_s(t)$ as it evolves over time. The boundary conditions are such that the torque $\tau=0$ at the disk's inner and outer edges, which keeps disk material from flowing out of the computation domain.

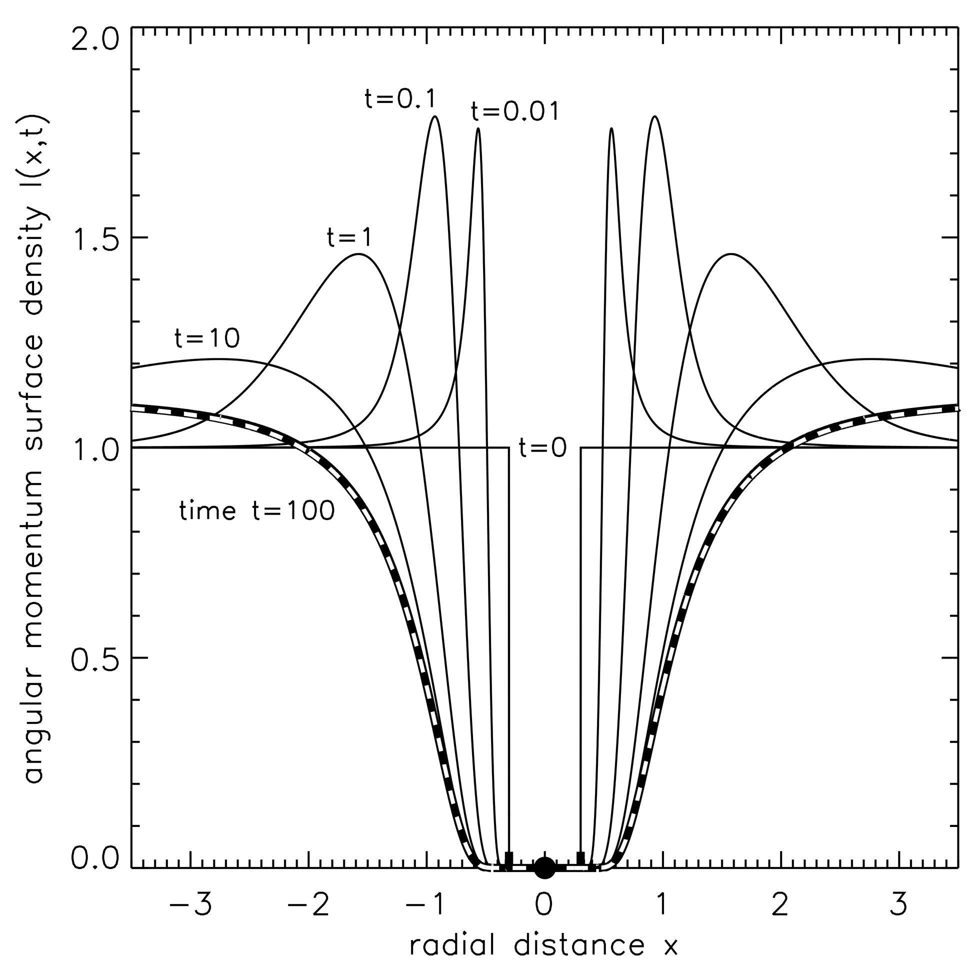

Figure 1: A small satellite initially inhabits a narrow gap at time $t=0$. The disk's angular momentum surface density ℓ (which is proportional to the mass surface density σ) is shown for later times t, which eventually settles into the equilibrium configuration, Eqn. (10), which is the dashed curve. The computational length unit $\Delta x=1$ corresponds to a physical distance of $\Delta r=150$ km (the half–width of the Encke gap), while one unit of time $\Delta t=1$ corresponds to 4×10^6 orbits, or about 5000 years.

Test case: Pan opens the Encke gap

The above model can be tested by using it to examine how a small satellite embedded in a planetary ring can widen a gap about its orbit, which we apply to Pan and the Encke gap. Pan orbits Saturn at $r=1.34\times10^5$ km, and has mass $\mu_s=8.7\times10^{-12}$ Saturn masses, and resides in the A ring having a surface density of $\sigma\sim50$ gm/cm^2. If we assume that Pan formed via runaway growth, then it would have accreted all material in its feeding zone whose half–width is $\Delta r_{acc}=2\sqrt{3}\times$ Pan's Hill radius $\simeq60$ km, which is about 40% of the Enck gap's half–width. Figure 1 illustrates how a suddenly–formed Pan, which would initially inhabit a very narrow gap due to its own accretion, tends to widen that gap further due to its resonant torque τ_s, Eqn. (5). Note also that the satellite's orbit stays fixed, *ie* $\dot{x}_s=0$, due to the disk's radial symmetry (see Eqn. 9).

This calculates serves as a test of the code, since it shows that the disk does indeed settle into the expected equilibrium configuration, which occurs when the viscous and resonant torques balance, *ie*, $\tau=0$ everywhere (see Eqn. 8), which has solution

$$\ell(x)=e^{-1/|x-x_s|^3}, \qquad (10)$$

which is the dashed curve in Fig. 1. Another test is angular momentum conservation, which is preserved here to $\Delta L/L\sim3\times10^{-5}$.

Application: Prometheus & the A ring

Figure 2 shows a prototype simulation of Prometheus, which orbits just outside the A ring. The matter orbiting just right of Prometheus is the F ring, which in reality would be confined by Pandora, but is absent from this 1–satellite simulation. (The next generation model will handle multiple satellites.) In this simulation, the A ring spreads radially outwards, which in turn drives Prometheus and the F ring further outwards. Note also that Pandora's absence from this simulation means that the simulated F ring remains unconfined, so it too spreads radially outwards.

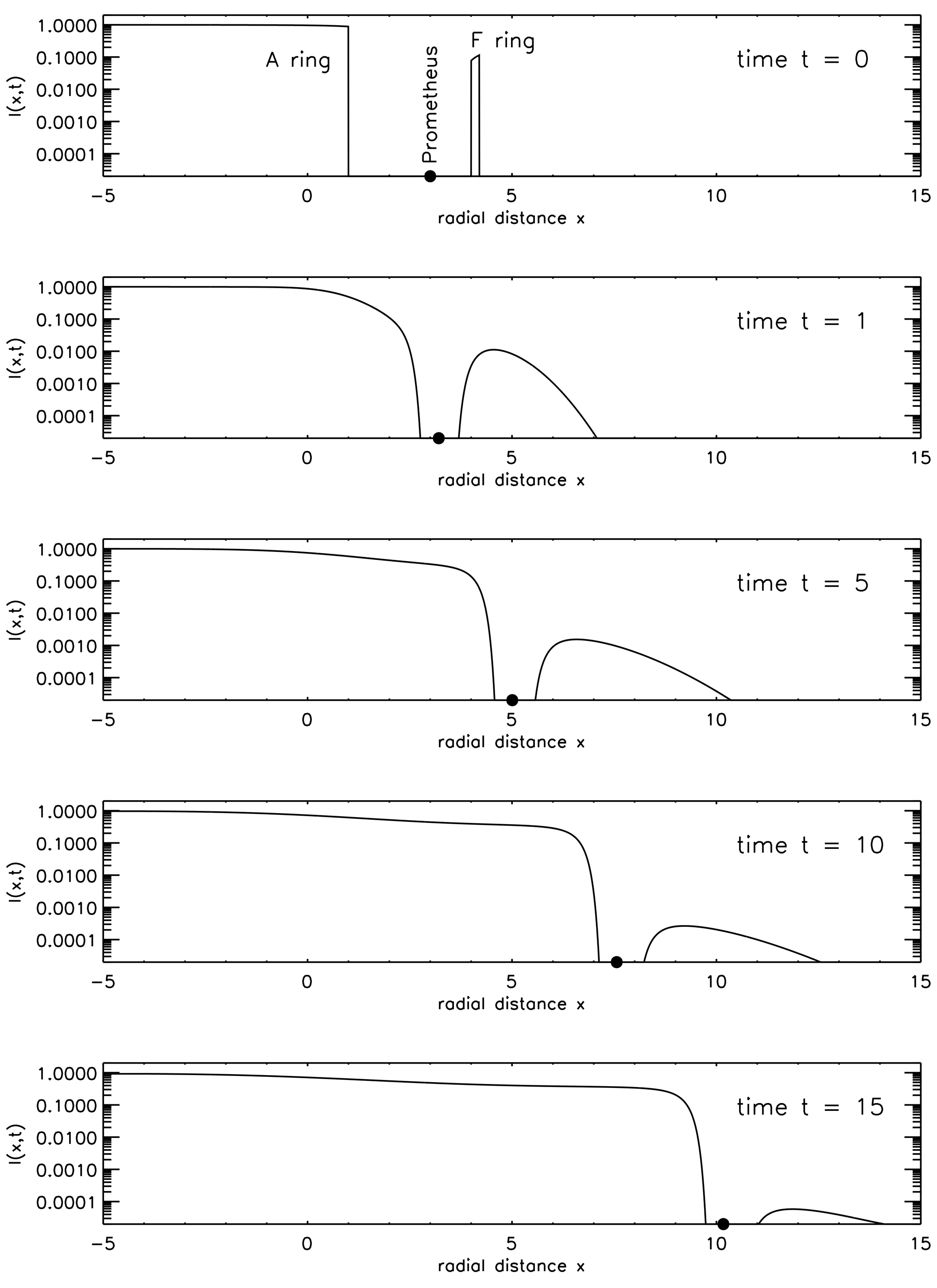

Figure 2: Snapshots of a system comprised of the outer A ring with Prometheus just beyond that, and a narrow F ring orbiting beyond that. In this simulation, a computational distance of $\Delta x=0.6$ corresponds to Prometheus' present distance from the A ring, $\Delta r=2600$ km, while the computation time $\Delta t=1$ corresponds to 3×10^9 orbits, or 5 million years.

Sadly, the current generation of this code is still problematic—it does not conserve the system's angular momentum well when the satellite migrates. In the above run, $\Delta L/L$ was only ~0.1. Note, however, that L is well preserved when the satellite's orbit is static—see Fig. 1.

This system is also quite interesting: Poulet & Sicardy (2001) report that the ring torques will drive Prometheus outwards until it crashes into Pandora in a few ~10's of million years. However their estimate is inferred from a static disk—one that does not evolve due to viscosity. But if the above results are to be believed, then Fig. 2 suggests that viscosity will allow the spreading A ring to 'chase' Prometheus. It is then possible that the ring's viscous timescale actually determines when Prometheus and Pandora collide. Alternately, this chase might instead spawn another generation of small satellites as the A ring spreads further beyond Saturn's Roche limit, which might facilitate additional satellite formation. These and other scenarios will be considered using an improved version of this model.

Action Plan

- improve the numerical method, particularly so that angular momentum is well preserved as satellites migrate
- generalize the code so that it can handle 2+ satellites
- adapt the code so that can also handle discrete resonances from more massive satellites, like the Mimas 2:1 which maintains the inner edge of the Cassini Division
- use the code to see if small satellites (necessarily orbiting deep within Saturn's Roche limit...) might account for gaps and plateaus seen in the rings (see Fig. 3)
- use the model to attempt to infer from the ring's early origin from observations of its current configuration

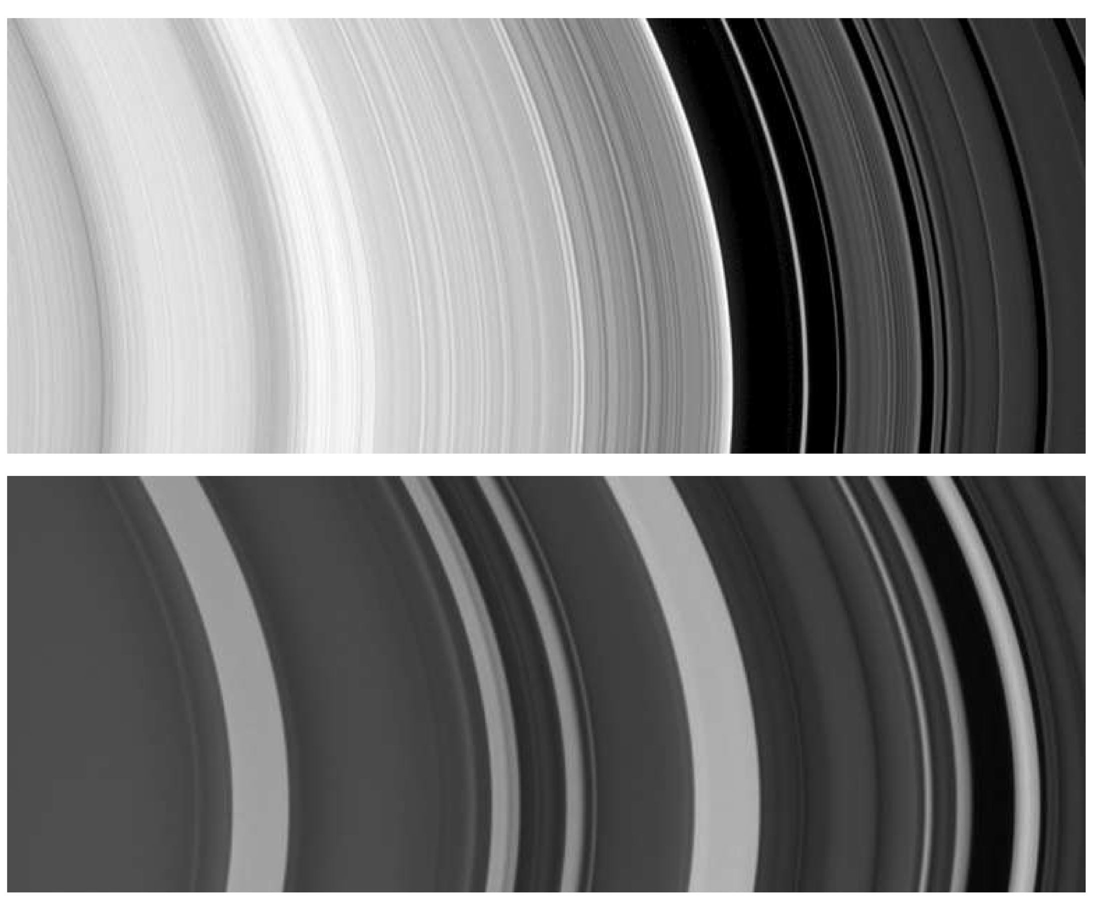

Figure 3: Cassini press release images of the outer B ring & Cassini Division (upper image), and gaps & plateaus in the C ring (lower image). Images from the CICLOPS website.