Introduction
Saturn’s rings represents one of the Solar System’s great mysteries. The origin of these rings, as well as their past and present evolution, are all poorly understood. The youthful appearance of these rings is particularly puzzling; the purity of these water-ice rings due to the lack of contamination by dark interplanetary dust suggest an “exposure” age of order 100 million years (Doyle et al. 1989). Similarly, the small satellites orbiting just beyond the main A ring seem even younger, ~10 million years, due to their interactions with the rings (Poulet & Sicardy 2001). The main challenge then is to understand why these rings appear to be so much younger than the rest of the Solar System. To address this, the following describes the development of a model that will ultimately track the dynamical evolution of Saturn’s coupled ring-satellite system. The goal of this effort will be to use this model to determine the past and future histories of this ring system, and to infer its origin.

The ring’s radial evolution
Saturn’s rings evolve radially due to the viscous torque that is the result of the ring particles’ frequent collisions; and also due to the torques that are exerted by Saturn’s satellites. The ring is treated here as a thin fluid disk whose evolution is described by a mass continuity equation (left Eqn. 1), and Euler’s equation for the fluid velocity \( \mathbf{v} \); these equations can then be combined to yield another conservation equation for the ring’s angular momentum (right Eqn. 1, see also Pringle 1981):

\[
\dot{J}_r = \int (\mathbf{v} \times \mathbf{r}) \cdot \mathbf{n} \, \mathrm{d}A
\]

where \( \mathbf{r} \) is the ring’s mass surface density and \( \mathbf{n} \) is the ring normal vector. The ring’s angular momentum density \( \dot{J}_r \) is the source term on the right side of (1) is the torque density \( \tau = \partial \mathbf{v} / \partial t \). For a nearly keplerian system, the ring’s radial velocity is proportional to the torque, \( \tau = \mu v^2 \) with \( \mu \) the satellite’s mass, and \( v = ( \mathbf{v} \cdot \mathbf{r} ) / \mathbf{r} \). Inserting this into Eqs. (1) then yields a single diffusion equation for the disk’s angular momentum surface density (Pringle & Lin 1986):

\[
\frac{\partial J}{\partial t} = \frac{\partial}{\partial r} \left[ \frac{1}{\rho} \frac{\partial (\rho r^2 \dot{J}_r)}{\partial r} \right]
\]

The ring’s evolution is driven by the torque surface density \( \dot{J}_r = \tau = t \), which has two parts: the viscous torque \( \tau_v \), and the satellite torque \( \tau_s \) that is due to resonant interactions with more massive satellites, like the Mimas 2:1 resonance. Most of the time, the satellite’s orbit stays fixed, while the computation time \( t \) = 1/10 of years.

Satellite migration
The total torque that the satellite exerts upon the disk is

\[
\tau = \int_{-\infty}^{\infty} t \, \mathrm{d}r
\]

so \( \tau_s \) is the torque that the disk exerts upon the satellite of mass \( m_s \), which causes its orbit to evolve at the rate \( \dot{a}_s = \frac{32}{5} \frac{\mu m_s}{m_r} a_s^2 \tau_s \). This same calculation is also employed in studies of type II planet migration (Ward 1997).

The evolution equations
The torques that a small satellite (such as Pan) exerts in a planetary ring system to change is very narrow gap in the ring whose half-width is \( a_e = a_j \Delta \), this is our problem’s natural unit of length, when this quantity is the time scale \( T_a = \sqrt{a_j} \tau_s \). This process can be studied using the dynamical equations (2-7), which we make linear by assuming that the viscosity \( \nu \) is constant. We also ignore changes in slowly varying quantities like \( \mathbf{v} \), etc., and make the equations dimensionless by replacing the small distance \( a \) with \( a_0 = \Delta a/j \), and time \( t_0 = t \Delta \tau_s/2 \).

This calculates serves as a test of the code, since it shows that the code does indeed settle into the expected equilibrium configuration, which occurs when the viscous and resonant torques are just right of Prometheus is the F ring, which in reality would be confined by Pandora, but is absent from this 1:2 satellite simulation. (The next generation model work handle multiple satellites.) In this simulation, the A ring spreads radially outwards, which in turn drives Prometheus and the F ring further outwards. Note also that Pandora’s absence from this simulation means that the simulated F ring remains unconfined, so it too spreads radially outwards.

The numerical solution
The disk evolution is obtained from Eqs. (8), which is solved numerically using a Cranck-Nicholson finite difference scheme (Richtmyer & Morton 1967). A Runge Kutta integrator (Press et al. 1994) is used to solve Eqs. (9) and track the satellite’s position \( (s, s) \) as it evolves over time. The boundary conditions are such that the torque \( \tau = 0 \) at the disk’s inner and outer edges, which keeps disk material from flowing out of the computational domain.

Test case: Pan opens the Encke gap
The above model can be tested by using it to examine how a small satellite embodied in a planetary ring can widen a gap about its orbit, which we apply to Pan and the Encke gap. Pan orbits Saturn at \( a_p = 20 \) km, and has mass \( m_p = 8 \times 10^2 \) Saturn masses, and resides in the A ring having a surface density of \( \rho = 50 \) gm/cm^2. If we assume that Pan formed via coagulation, then it would have accreted all material in its feeding zone whose half-width is \( a_p = 3 \times 10^5 \) Pan’s Hill radius \( h_p \), which is about 40% of the Encke gap’s half-width. Figure 1 illustrates how a suddenly-formed Pan, which would initially inhabit a very narrow gap due to its own accretion, tends to widen that gap further due to its resonant torque \( \tau_s \). Note also that the satellite’s orbit stays fixed, \( s = s_0 \), due to the disk’s radial symmetry (see Eqn. 9).

This calculates serves as a test of the code, since it shows that the disk indeed does settle into the expected equilibrium configuration, which occurs when the viscous and resonant torques are just right of Prometheus is the F ring, which in reality would be confined by Pandora, but is absent from this 2:1 satellite simulation. (The next generation model work handle multiple satellites.) In this simulation, the A ring spreads radially outwards, which in turn drives Prometheus and the F ring further outwards. Note also that Pandora’s absence from this simulation means that the simulated F ring remains unconfined, so it too spreads radially outwards.

Action Plan
- improve the numerical method, particularly so that angular momentum is well preserved as satellites migrate
- generalize the code so that it can handle 2+ satellites
- adapt the code so that can handle discrete resonances from more massive satellites, like the Minas 2:1 which maintains the inner edge of the Cassini Division
- use the code to see if small satellites (necessarily orbiting deep within Saturn’s Roche limit...) might account for gaps and plateaus seen in the rings (see Fig. 3)
- use the model to attempt to infer from the ring’s early origin from observations of its current configuration

Figure 2: Snapshots of a system composed of the outer A ring with Prometheus just beyond, and a narrow F ring nestled between. This simulation is a computa- tional experiment, a comet-like density \( \rho \) = 10^12 km^-3, while the computation time \( t \) = 1/10 of years, or 5 million years.

Figure 3: Cassini press release image of the outer A ring and a Cassini Division gap (top Images) and gap + planetoc triad in the F ring (lower Image) from the Celebration website.