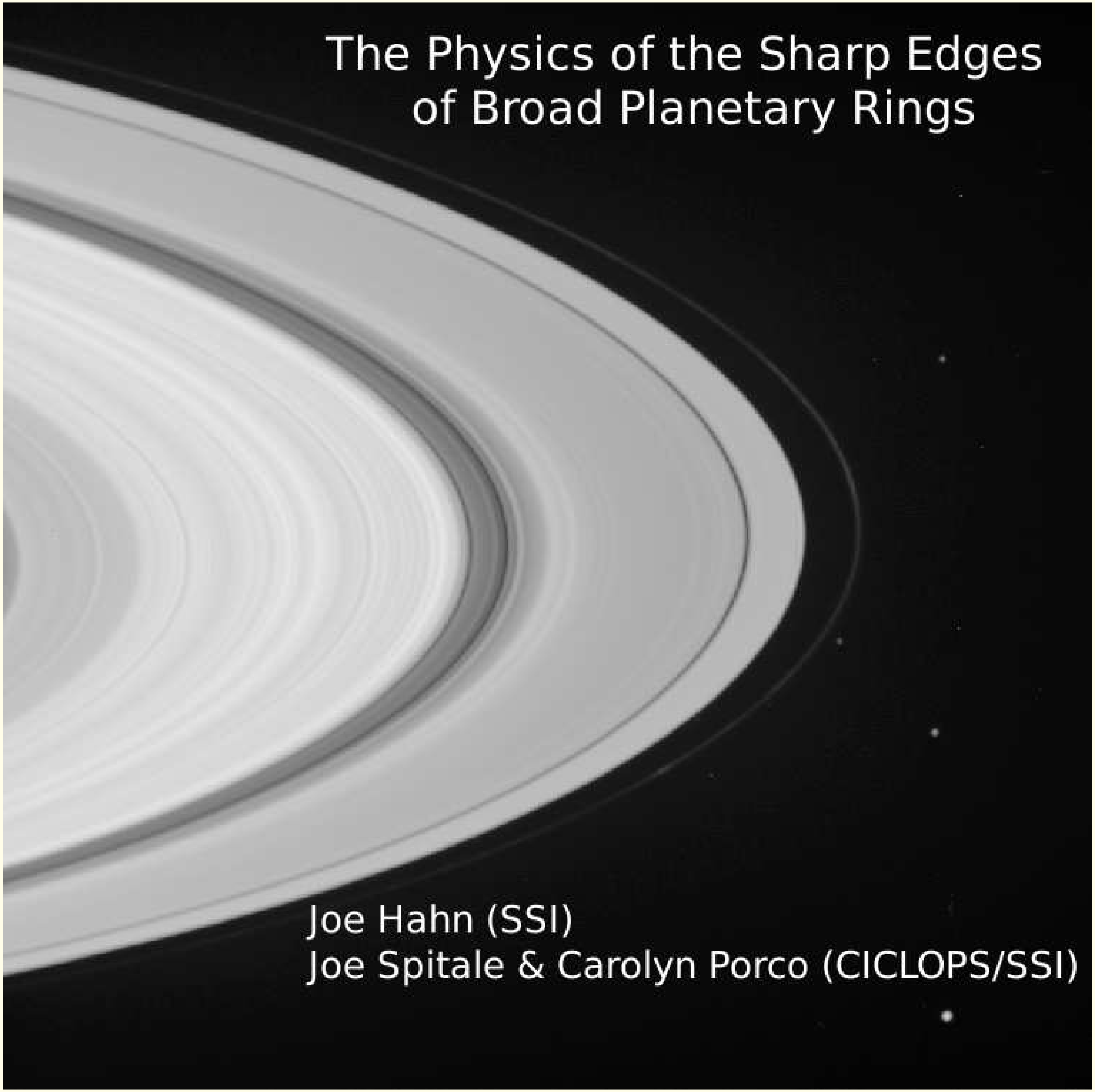


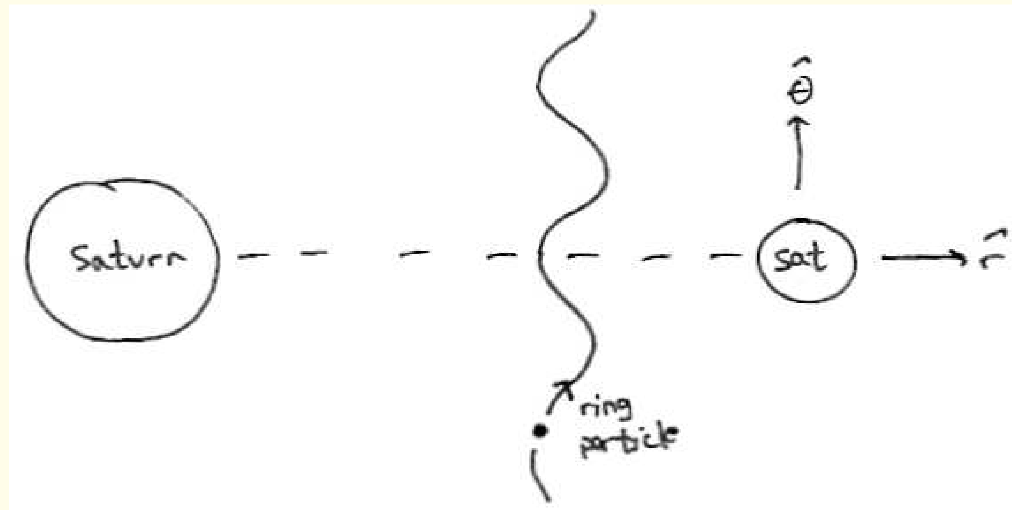
The Physics of the Sharp Edges of Broad Planetary Rings

Joe Hahn (SSI)
Joe Spitale & Carolyn Porco (CICLOPS/SSI)

Forced Epicyclic Motion

An orbiting particle's response to resonant perturbations is **epicyclic**, with its path tracing m radial excursion about its mean orbit (GT82, BT87):

$$\left|\frac{\Delta r}{r}\right| = e\cos[m(\theta - \theta_{sat}) - \tilde{\omega}]$$



GT82 show that

$$e \simeq \frac{M_{sat}/M_{planet}}{|x|}$$

where $x =$ fractional distance from Lindblad resonance (LR),

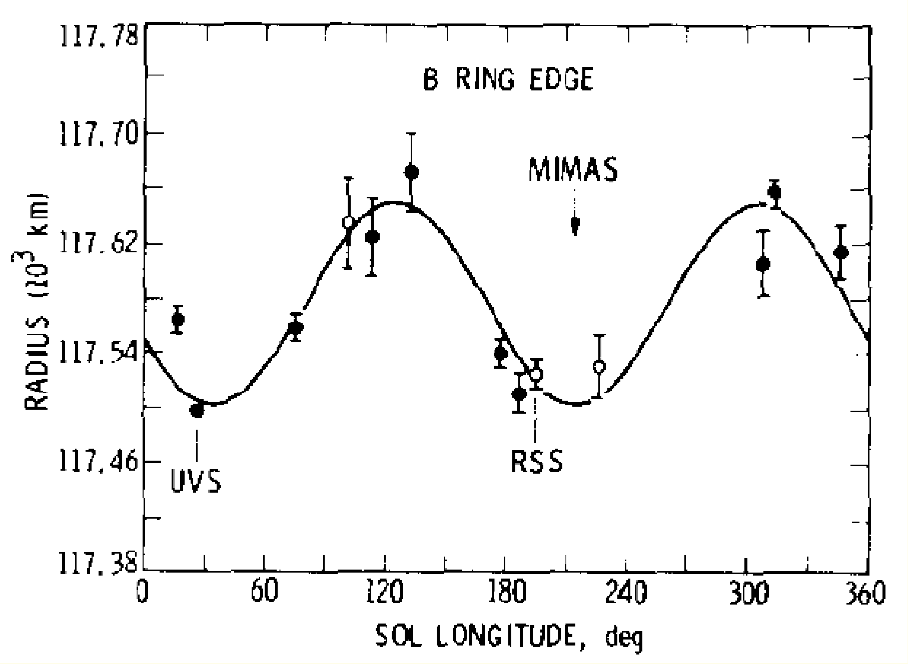
and longitude of periapse $\tilde{\omega}$, measured from the satellite's longitude θ_{sat}, is

$$\tilde{\omega} = \begin{cases} 0^\circ & \text{when particle orbits interior to LR} \Leftarrow \text{orbit is } \textit{peri-aligned} \text{ with sat'} \\ 180^\circ & \text{when particle orbits exterior to LR} \Leftarrow \text{orbit is } \textit{apo-aligned} \text{ with sat'} \end{cases}$$

This applies wherever the planetary ring's internal forces (gravity, pressure, etc) are negligible, which is *not* the case at the edge of a perturbed ring...

B Ring Peculiarities

When Porco et al (1984) examined Voyager observations of the outer B ring, they found:



Saturn is far downwards, while Mimas is way up.

From Porco et al (1984).

- the ring's outer edge lies 24km *beyond* resonance

- yet the ring was peri-aligned with Mimas, NOT apo-aligned as might be expected...

Porco et al (1984) suggested that this unusual configuration might be due to the ring's internal forces (gravity, pressure, viscosity).

...this curious arrangement also piqued my interest, too...

My task:

- derive the equations of motion (EOM) for ring particles orbiting near a LR, accounting for the sat's perturbations (s), and rings' internal forces ($g, p,$ and ν).

- solve those EOM, which then provides a model that can predict the ring's epicyclic amplitude R and surface density $\sigma(r, \theta)$.

Once done, J. Spitale's task will be to:

- fit the model to Cassini observations of these rings

Since R and $\sigma(r, \theta)$ will be functions of $\sigma_0 =$ ring's undisturbed surface density,
$c =$ ring particles' dispersion velocity,
$\nu =$ ring's viscosity,

we expect to be able to infer the ring's physical properties from Cassini observations.

Note that the A ring provides a nice check on our work since σ_0, c, ν are known there.

But the B ring is particularly interesting, since its σ_0, c, ν are quite unknown.

Proceed by solving the EOM using the strategy given in GT82.

Newton's 2[nd] Law provides the acceleration on a single ring particle:

$$\ddot{\mathbf{r}} = -\nabla(\Phi_{\text{planet}} + \Phi_{\text{sat}}) + \mathbf{f}_{\text{ring}}$$

where $\mathbf{f}_{\text{ring}} = a_r\hat{\mathbf{r}} + a_\theta\hat{\theta}$ is the acceleration due to the ring's internal forces g, p, ν.

Then Fourier expand the perturbations:

$$\begin{aligned}
\Phi_{\text{sat}} &= \phi_s^m(r)e^{im(\theta-\theta_s)} \\
a_r &= A_r^0 + A_r^m e^{im(\theta-\theta_s)} \\
a_\theta &= A_\theta^0 + ...
\end{aligned}$$

and note that the particle motion traces a "streamline" in the ring

$$r(t) \rightarrow r(a,\theta) = a - R(a)\cos[m(\theta-\theta_s) - \tilde{\omega}(a)]$$

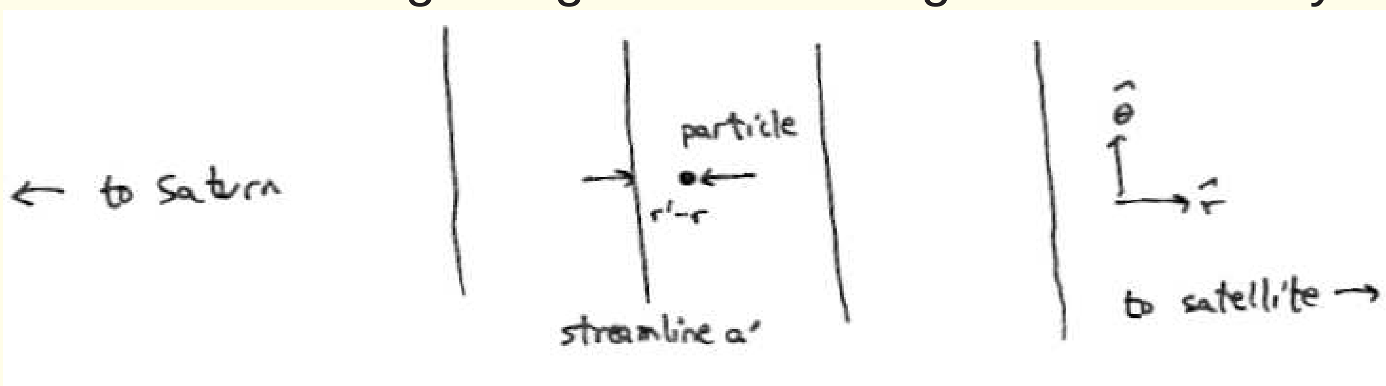
that appears stationary in the reference frame that corotates with the satellite.

Streamlines

A perturbed planetary ring can be regarded as being composed of numerous, nested *streamlines* (BGT85). This approach is useful since it simplifies the derivations of the accelerations that are due to the ring's internal forces:

radial acceleration	a_r	$=$	$a_{\text{gravity}} + a_{\text{pressure}}$
tangential acceleration	a_θ	$=$	$a_{\text{viscosity}}$

The particle is perturbed by nearby streamlines,
which are treated here as long straight wires having a linear density λ:



For gravity, $$\delta a_{\text{grav}} = \frac{2G\lambda}{r'-r} = \frac{2G\sigma_0(a')\delta a'}{r'-r} = \text{ gravity due to streamline } a',$$

so total gravity $$a_{\text{grav}} = \int_{\text{ring}} \delta a_{\text{grav}} = \text{ integral over } \sigma_0(a) \text{ and } e(a).$$

Pressure & Viscosity

To get the acceleration due to pressure,
treat the ring as a compressible, barotropic fluid:

$$a_{\text{pressure}} = -c^2 \left(\frac{1}{\sigma} \frac{\partial \sigma}{\partial r} \right)$$

where c is the particle's dispersion velocity.

The acceleration due to viscosity ν is obtained from the viscous couple g
of Lynden-Bell & Pringle's (1974):

$$a_{\text{viscosity}} = -\frac{\partial g/\partial r}{2\pi\sigma r^2} \simeq -\frac{3}{2}\nu\Omega \left(\frac{1}{\sigma} \frac{\partial \sigma}{\partial r} \right)$$

which conveniently has the same form as pressure.

Also have to Fourier-decompose those accelerations, which is quite a chore
due to σ's nonlinear dependence on the ring's eccentricity gradient...

Nonlinear Surface Density Variations

A streamline in the ring has the form

$$r(a,\theta) = a - \Re e[\mathcal{R}e^{im(\theta-\theta_s)}] \qquad \text{where} \qquad \mathcal{R}(a) = R(a)e^{-i\tilde{\omega}}$$

where the complex quantity $\mathcal{R}$ is convenient,
since it communicates the streamline's epicyclic amplitude R and orientation $\tilde{\omega}$.

BGT85 show that a perturbed ring's surface density varies as

$$\sigma(a,\theta) = \frac{\sigma_0(a)}{1 - \Re e[\mathcal{R}' e^{im(\theta-\theta_s)}]} \qquad \text{(due to mass conservation)}$$

$$\text{where} \qquad \mathcal{R}' = \frac{d\mathcal{R}}{da} = \left(R' - i\frac{\partial\tilde{\omega}}{\partial a}R\right)e^{-i\tilde{\omega}}$$

Noting that $|\mathcal{R}'| =$ BGT's nonlinearity parameter q.

When $|\mathcal{R}'| \ll 1$, the ring is said to be linear, since the ring's
fractional surface density variations $\Delta\sigma/\sigma \propto |\mathcal{R}'|$ are sinusoidal and small.

However, the perturbed edges of the A and B rings are nonlinear,
since $|\mathcal{R}'|$ is not small there...

Relationship to NL Density Waves

Set wavenumber $$k = -\frac{\partial\tilde{\omega}}{\partial a}$$

so the NL parameter is $$\mathcal{R}' = (R' + ikR)\, e^{-i\tilde{\omega}}$$

When $|kR| \gg |R'| \Rightarrow$ you have a tightly-wrapped spiral density wave.
This is known as the tight-winding approximation.

In a gravitating ring, these waves want to propagate radially outwards from ILR, but can't here—there is no ring material there, only a gap (e.g., Cassini Division just beyond B ring, or A ring's outer edge).

Rather, I'm interested in ring material orbiting on the non-wave side of the ILR.

Since $\tilde{\omega}(a)$ will vary slowly with a there, $|kR| \ll |R'|$ and $\mathcal{R}' \simeq R'e^{-i\tilde{\omega}} = (de/dx)e^{-i\tilde{\omega}}$

and $$\sigma(a,\theta) \simeq \frac{\sigma_0(a)}{1 - \frac{de}{dx}\cos[m(\theta - \theta_s) - \tilde{\omega}]}$$

$\Rightarrow$ the ring's surface density, as well as its internal forces, are all controlled by the eccentricity gradient that the satellite excites in the ring.

The Streamlines' Equations of Motion (EOM)

Newton's 2$^{\text{nd}}$ Law is a 2D vector equation, so it provides two EOM.

Those EOM include perturbations from satellite s, and ring g, p, and ν.

Those EOM are NL since $\sigma = \sigma_0/[1 - e' \cos m(\theta - \theta_s)]$.

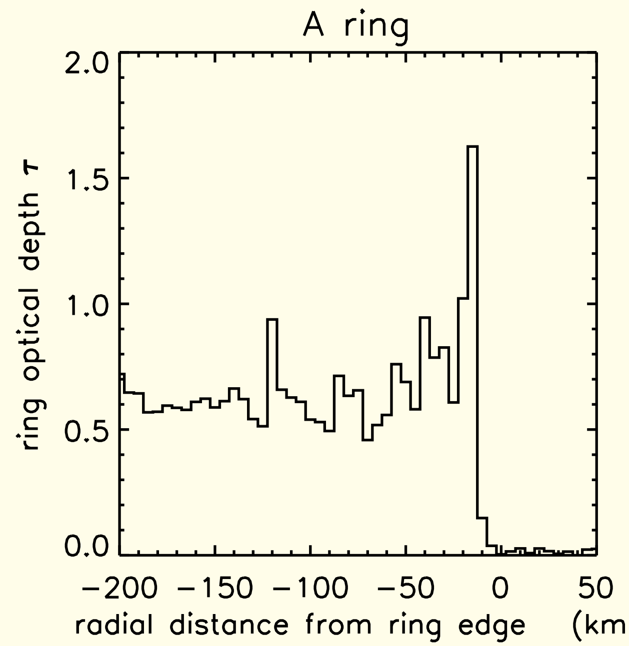
They are also integro-differential equations, since
a_{grav} requires integrating across the ring, while a_{press} and a_{visc} are $\propto \partial\sigma/\partial r$.

To solve these EOM:

- treat the ring as if it were composed of N discrete streamlines whose orbits are described by $e_j = e(a_j)$ and $\tilde{\omega}_j = \tilde{\omega}(a_j)$.
- then convert a_{grav} into a sum over all a_j,
- and use finite-difference methods to handle the derivatives.

$\Rightarrow$ this provides a system of $2N$ coupled NL eqn's for the $2N$ unknowns e_j and $\tilde{\omega}_j$.
Solving this system of coupled equations numerically is straightforward.

The A and B Ring's Sharp Edges



Outer A ring optical depth, from PDS rings node.

Another EOM is derived from the specific torque T that the s and ν exert on a particle/streamline:

$$\begin{aligned} T &= (\mathbf{r}\times\ddot{\mathbf{r}})\cdot\hat{\mathbf{z}} \\ &= T_\nu + T_s \simeq T_\nu \end{aligned}$$

since the satellite's torque $|T_s| \ll |T_\nu|$
is small compared to the viscous torque T_ν.

Note that static equilibrium requires $T = 0$ (or particles/streamlines drift radially),

so $$T \simeq T_\nu = rA_\nu^0 \propto \frac{a}{\sigma_0}\frac{\partial\sigma_0}{\partial a} + \frac{2e'e''}{1-e'^2} = 0$$

where $\sigma_0(a)$ is the unperturbed ring's intrinsic surface density.

$\Rightarrow$this equations says that the ring's edge is the site where $e'e''$ gets large, which in turn causes $\partial\sigma_0/\partial a$ to get very negative, causing the sharp edge.

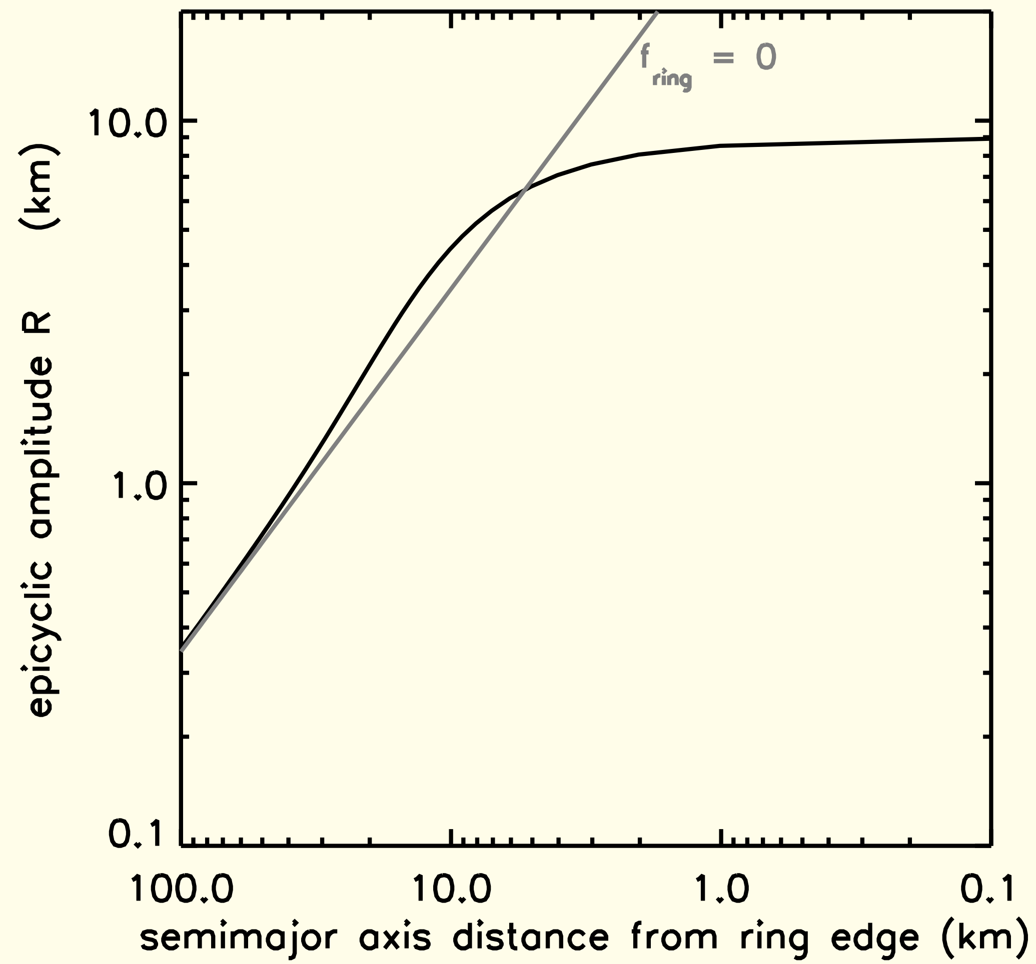
Note that the satellite's role is indirect—by perturbing the ring,
it tries to excite the viscous torque T_ν,
but the torque-balance requires $T_\nu = 0$,
which drives $\sigma_0(a) \to 0$ at the ring's edge.

An A Ring Simulation

Shown is a trial solution for the outer A ring, which is perturbed by $m = 7$ ILR with Janus/Epimetheus (treated here as a single satellite)

This simulation assumes:

- $\sigma_0 = 30$ gm/cm^2 (Spilker etal 2004)
- $\nu = 20$ cm^2/sec (Porco et al 2007)
- $c = 1$ mm/sec
  (eg, $h = 10$m, or $Q_{\text{Toomre}} = 2$)



Saturn is far to left, while Janus/Epimetheus are far right.

This simulated ring's epicyclic amplitude is $R = 9$km at its edge, which is comparable to Voyager measurement (Porco et al 1984) and recent Cassini measurements (Spitale et al 2008).

Model shows that increasing σ_0 decreases epicyclic amplitude R at the edge.

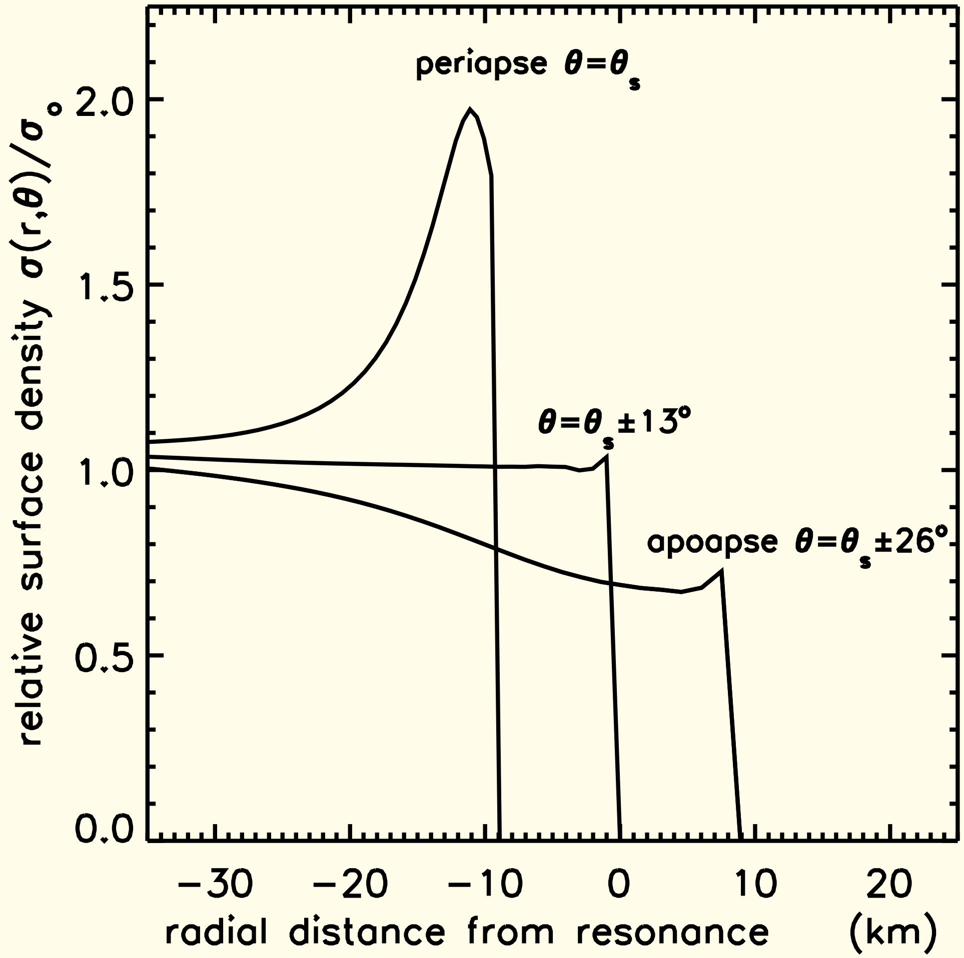
σ variations at the Outer A

Shown are radial cuts of the A ring's surface density, obtained from

$$\frac{\sigma(r,\theta)}{\sigma_0} = \left[1 - \frac{de}{dx}\cos[m(\theta - \theta_s) - \tilde{\omega}]\right]^{-1}$$

Along the satellite's long' ($\theta = \theta_s$), the satellite's perturbation shoves ring matter inwards, increasing σ at the edge.

At $\theta - \theta_s = 180°/m$ (apoapse), ring matter is drawn outwards, reducing σ there.



Saturn is far left, Janus/Epimetheus are far right.

$\Rightarrow$Large variations is σ are expected w/in the A ring's outermost ~ 10km, due to satellite's compression & rarefaction of the ring-edge.

Similar σ-variations should also occur in the B ring over a distance comparable to its epicyclic amplitude, $R \sim 50$km (Spitale & Porco 2006).

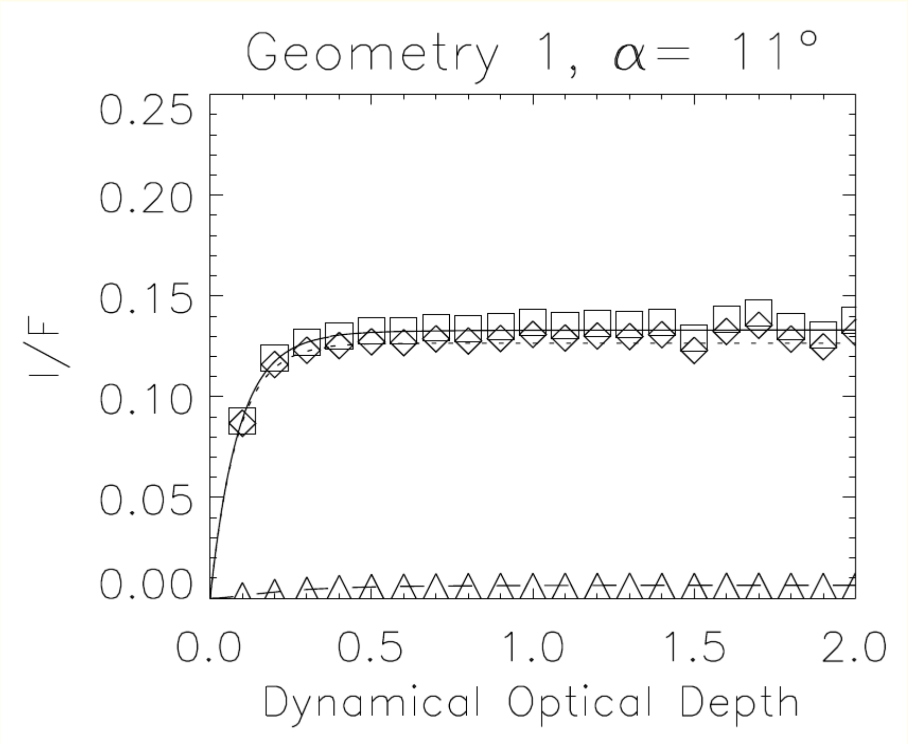
Why You Might Not See These σ Variations...

Note that the ring optical depth $\tau \propto \sigma$

And that a ring observer measures I/F (ring's relative surface brightness)

However I/F is NOT $\propto \tau$ or σ

Recent photometric model of ring particles (Porco et al, submitted) shows that I/F *saturates* at $\tau \gtrsim 0.3$



from Porco et al (2008), submitted

- note that the outer edges of the A and B rings have $\tau_A \sim 0.6$ and $\tau_B \sim 1.5$

$\Rightarrow$The large increases in $\sigma \propto \tau$ anticipated at periapse might not be visible, due to I/F *saturation*.

However, the reduced σ that occurs are apoapse might be visible.

Might be more likely to observe the periapse peaks in σ at radio wavelengths, where τ is lower.

Summary of Results

- We have used the streamline formalism of BGT to develop a model of a broad planetary ring that is confined by a satellite's m^{th} LR.

- The model accounts for the satellite's forcing, as well as the ring's internal forces (g, p, and ν) that are excited by the satellite's perturbations.

- The model provides a useful probe that will extract the ring's physical properties (σ_0, c, and ν).

- The model rings also exhibits sharp edges, as expected (BGT82,85).

- Simulated A ring's epicyclic amplitude is consistent with $\sigma_0 \sim 30$ gm/cm^2, similar to that inferred from density waves in the outer A (Spilker et al 2004).

- The B ring is still being examined, but preliminary results suggests $20 \lesssim \sigma_0 \lesssim 30$ gm/cm^2.

- The ring's internal forces can shift the resonance location,
  but only by a few meters!
  - so we still haven't explained why the B ring edge lies 24km exterior to the ILR, but we are working on that...

- Large increases in the ring-edge's surface density are expected at periapse, where the satellite has shoved ring material inwards.
  - this might not be observable at optical wavelengths, due to I/F saturation
  - instead, radio occultations might have greater success at detecting this effect.

Future Activities

Once the outer edges of the A and B rings are 'solved', we then intend to:

- apply the model to Saturn's many narrow, eccentric ringlets
  (eg, Huygens, Columbo, Maxwell, etc.)

- revise the streamline model in the tight-winding approximation,
  and use it to investigate the many NL density waves seen in Saturn's rings.