A Model of Saturn's Interacting Ring–Satellite System

Saint Mary's University
Halifax, Nova Scotia, Canada

Joseph M. Hahn
Saint Mary's University
Institute for Computational Astrophysics

Institute for Computational Astrophysics ICA

Introduction

Saturn's rings represents one of the Solar System's greatest mysteries (Fig. 1). The origin of these rings, as well as their past and present evolution, are all poorly understood. The youthful appearance of these rings is particularly puzzling; the purity of these water–ice rings due to the lack of contamination by dark interplanetary dust suggest an 'exposure' age that is of order 100 million years (Doyle *et al.* 1989). Similarly, the small satellites orbiting just beyond the main A ring seem even younger, ∼10 million years, due to their gravitational interactions with the rings (see below). The main challenge then is to understand why these rings appear to be so much younger than the Solar System. The following describes a model that will track the dynamical evolution of Saturn's coupled ring-satellite system. The goal of this modeling effort will be to determine the past and future histories of this system, and to infer the origin of Saturn's rings.

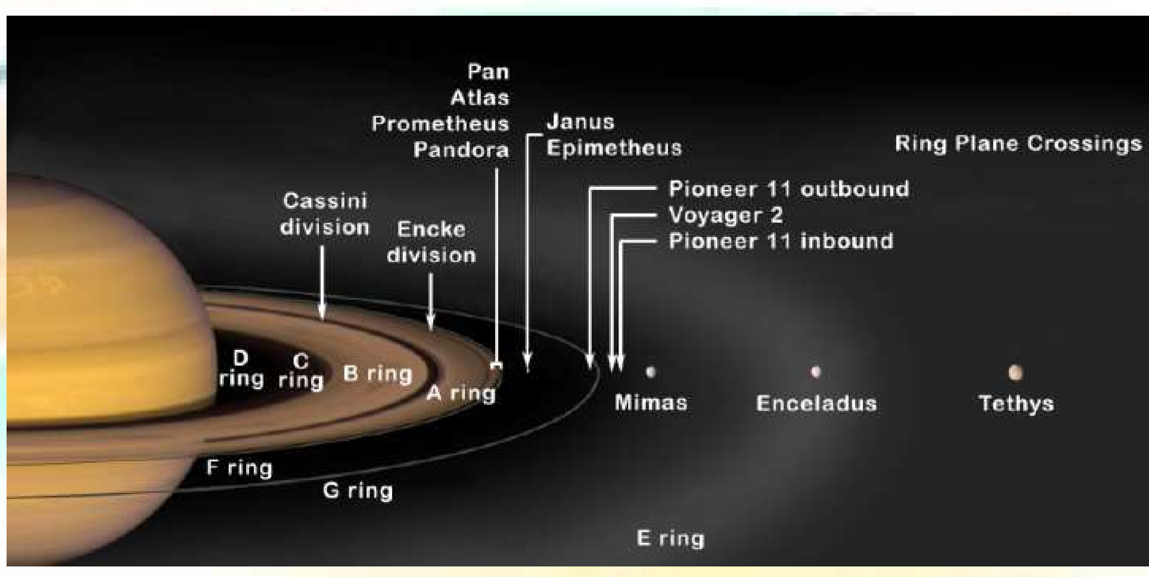

Figure 1: Saturn's rings and satellites, with other major satellites, like Dione, Rhea, and Titan, off to the right. Art by David Seal (JPL).

The ring's radial evolution

Saturn's rings evolve radially due to the viscous torque that results from the ring particles' frequent collisions, but also due to torques exerted by Saturn's satellites. The rings' evolution is described by two continuity equations, one for the ring's surface mass density $\sigma(r,t)$, and another for the ring's angular momentum surface density $\ell(r,t)=\sigma r^2\Omega$:

$$\frac{\partial\sigma}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\sigma v_r)=0 \quad\text{and}\quad \frac{\partial\ell}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\ell v_r)=g \qquad (1)$$

where $\Omega(r)=\sqrt{GM_{\text{Saturn}}/r^3}$ is the ring's angular velocity, $v_r(r,t)$ is the ring's radial velocity, and the source term on the right is the torque density $g(r,t)$. These two equations can then combined into the single equation

$$\frac{\partial\ell}{\partial t}=-\frac{2}{\sqrt{r}}\frac{\partial}{\partial r}(r^{3/2}g). \qquad (2)$$

This is the standard equation for the evolution of a Keplerian disk, but written in terms of the ring's angular momentum density $\ell\propto\sigma\sqrt{r}$ (see Pringle 1981). The rings' evolution is driven by the torque density $g=g_\nu+g_{\text{res}}$ that has two parts: the viscous torque g_ν plus the resonant torque g_{res} that is due interactions with the satellites.

The viscous torque density

The viscous torque density is

$$g_\nu(r,t)=-\frac{3}{2r}\frac{\partial}{\partial r}(\ell\nu) \quad \text{(Pringle 1981).} \qquad (3)$$

where ν is the ring's viscosity. A ring of colliding particles has a viscosity $\nu\simeq v_d^2\tau/2\Omega=e_d^2K\ell/2$, where $v_d\simeq e_d r\Omega$ is the ring particles dispersion velocity due to their eccentricities e_d (Goldreich and Tremaine 1982), and $\tau=K\sigma$ is the ring optical depth where K = ring opacity.

The resonant torque density

The resonant torque g_{res} is due to the spiral density waves that the satellites can launch at their various inner Lindblad resonances, or ILRs (*aka,* mean–motion resonances). If satellite j has a semimajor axis a_j, then its m_j^{th} ILR is at $r_{m_j}=(1-1/m_j)^{2/3}a_j$. A satellite that launches a wave in the ring will be gravitationally attracted to the resulting spiral density pattern (see Fig. 2) and exert the torque (Goldreich & Tremaine 1978)

$$T_{m_j}=-f_{m_j}\mu_j^2\sigma_{m_j}r_{m_j}^4\Omega_{m_j}^2=-f_{m_j}\mu_j^2\ell_{m_j}r_{m_j}^2\Omega_{m_j} \qquad (4)$$

where μ_j is the satellite's mass in Saturn units, all quantities are evaluated that the m_j^{th} resonance, f_{m_j} is a numerical coefficient, and the sign indicates that the satellite is withdrawing angular momentum from the rings. For simplicity it will be assumed that the rings' viscosity damps the wave uniformly over a radial distance Δ_ν, so the resulting torque density at this resonance is $g_{m_j}=T_{m_j}/2\pi r_{m_j}\Delta_\nu$.

The total torque density within the rings is thus

$$g(r,t)=g_\nu+\sum_j\sum_{m_j}g_{m_j} \qquad (5)$$

where the sums are over all satellites j having resonances m_j in the rings.

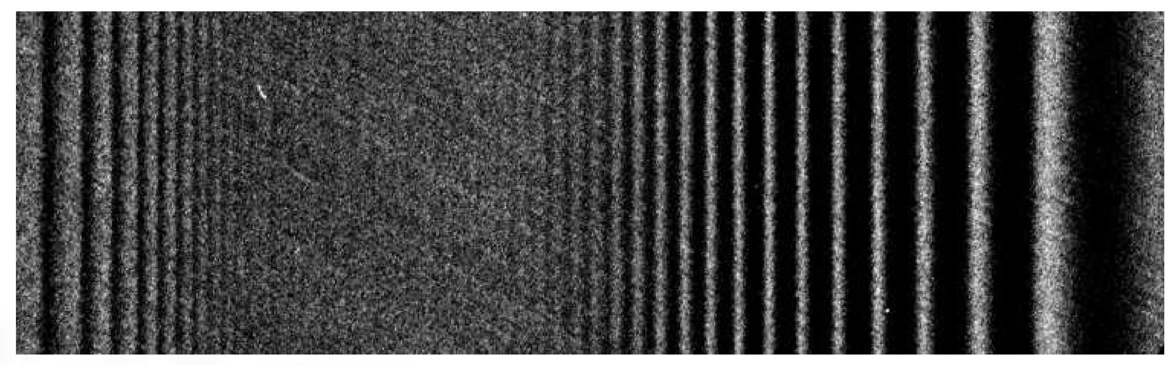

Figure 2: A 200×50 km snapshot of spiral waves in Saturn's A ring, with Saturn far off to the left. Shown are *spiral density waves* (left) launched at Prometheus' $m=11$ ILR, and *spiral bending waves* launched by Mimas at its 5:3. Image from the Cassini/CICLOPS website.

Satellite Migration

The torque on satellite j due to the waves it excites at its ILRs is $-\sum_{m_j}T_{m_j}>0$, which causes the satellite' semimajor axis to expand at the rate

$$\frac{da_j}{dt}=\frac{2|\sum_{m_j}T_{m_j}|}{\mu_jM_{\text{Saturn}}a_j\Omega_j} \qquad (6)$$

where the sum is again over all of j's ILR's in the rings.

A simple 1–satellite model

The following will consider a very simple scenario where the ring is sculpted by a single satellite's m^{th} ILR. For numerical work it will be convenient to use dimensionless equations of motion. Let $\ell_1=\sigma_1r_1^2\Omega_1$ be the ring's initial angular momentum density at some reference radius r_1 having an orbital period $t_1=2\pi/\Omega(r_1)$. The relevant dimensionless quantities are thus $\ell'=\ell/\ell_1$, $x=r/r_1$, with $t'=\mu_j^2t/t_1$ being a convenient dynamical time–unit, and $g'=g/(\mu_j^2\ell_1/t_1)$ the dimensionless torque density. The dimensionless dynamical equations (2–6) then become

$$\frac{\partial\ell'}{\partial t'}=-\frac{2}{\sqrt{x}}\frac{\partial}{\partial x}(x^{3/2}g') \quad \text{where}$$

$$g'=g'_\nu+g'_{\text{res}}=\nu_0\frac{\ell'}{x}\frac{\partial\ell'}{\partial x}-\frac{T'_m}{x_m\Delta'_\nu}=\text{ring torque density}$$

where $T'_m=f_m\sqrt{x_m}\ell'_m=$ resonant torque

and $\frac{dx_j}{dt'}=4\mu_{rj}\sqrt{x_j}T'_m=$ satellite's orbit expansion rate,

where the constants $\nu_0=3\pi e_d^2\tau_1/\mu_j^2$ is a dimensionless measure of the ring's viscosity, and $\mu_{rj}=\pi\sigma_1r_1^2/M_j$ is roughly the ring's mass in units of the satellite's mass M_j. Evidently the evolution of this system is governed by three parameters: the viscosity parameter ν_0, the ring–satellite mass ratio μ_{rj}, and the wave damping scale–length Δ'_ν.

Sculpting the rings at Mimas' 2:1

Mimas is one of the major ring perturbers due to its mass and proximity (see Fig. 1). Mimas' mass is $\mu_j=6.8\times10^{-8}$ in Saturn units, while the A ring surface density is $\sigma\sim50$ gm/cm^2. Thus the disk–satellite mass ratio at $r_1=2$ Saturn radii R_{Saturn} is $\mu_{rj}\sim0.6$. Because Mimas' mass is comparable to the rings itself, its resonant torque is powerful enough to carve open a gap at its 2:1 resonance (*i.e.,* its $m=2$ ILR) in the rings, which also sustains the inner edge of the Cassini Division (e.g., Goldreich and Tremaine 1978); see Fig. 3.

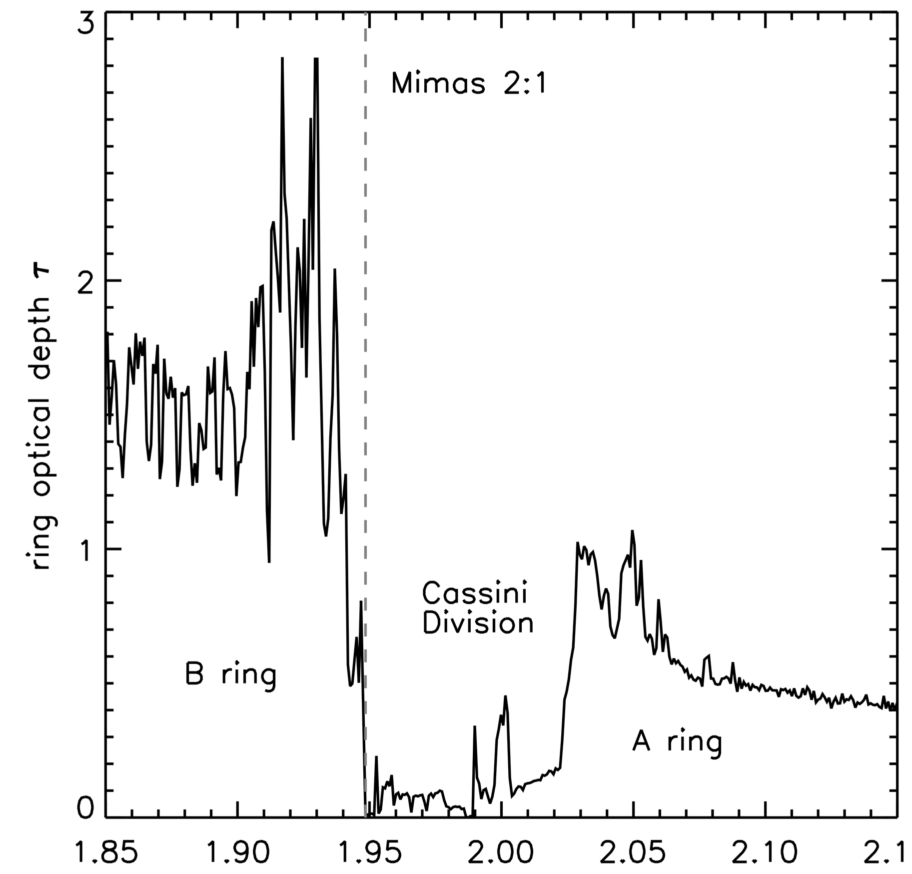

Figure 3: Saturn's rings' normal optical depth τ versus distance r, provided by Mark Showalter/PDS Rings Node.

Note also that the main rings' optical depth is $\tau_1\sim1$ (Fig. 3), and the ring particles have eccentricities $e_d\sim6\times10^{-8}$ (Burns 2005), so the viscous parameter $\nu_0\sim7$ is not small, and thus the ring's viscosity also plays a role in the formation of this gap.

Another interesting curiosity is the radial width of the Cassini Division, which is $\Delta_\nu\sim0.07R_{\text{Saturn}}\sim4000$ km (Fig. 3). This is considerably wider than the usual wave–damping scale–length of $\Delta_\nu\sim500$ km that is observed elsewhere in the rings. As Franklin *et al.* (1984) point out, the mechanism that sustains this wide gap is a puzzle.

Figure 4 shows a first attempt at a numerical integration of the equations of motion for Mimas' interaction with the ring at its $m=2$ IRL. Initially the ring is undisturbed, but the figure shows that Mimas will have dug out an appreciable gap in the disk after a dimensionless time $t'=1.5\times10^{-4}$, which corresponds to a real time of $t=t't_1/\mu_j^2\sim50$ million years. Unfortunately, the current version of the numerical code is rather fragile—it is only reliable when the ring viscosity is too low by $\times20$ and when the wave–damping length is long by $\times50$, and the code does not yet allow the satellite's orbit to expand as it draws angular momentum out of the ring. A more robust algorithm is still in development...

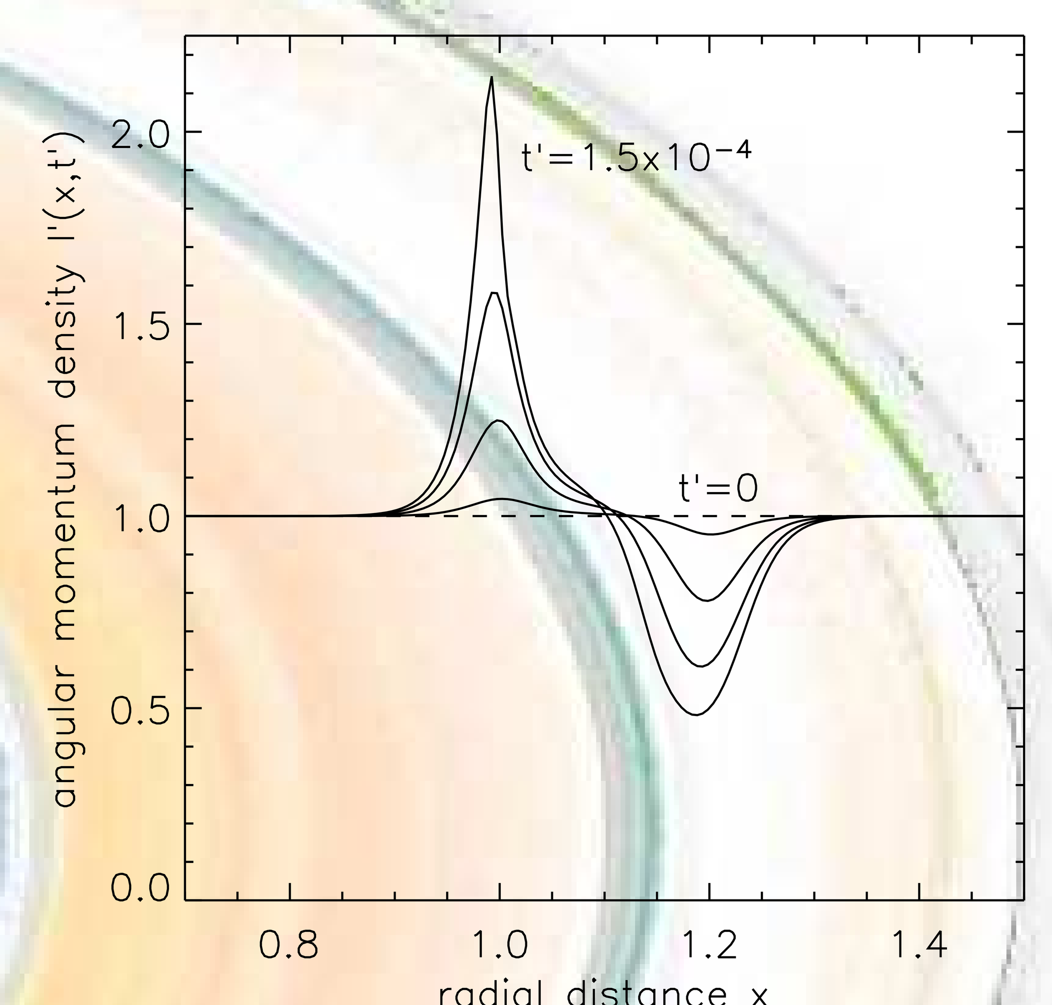

Figure 4: The ring's angular momentum density ℓ' is plotted versus distance x at selected times t'. The satellite's $m=2$ ILR lies at $x_m=1$, the ring's viscous parameter is $\nu_0=0.4$, the ring–satellite mass ratio is $\mu_{rj}=1$, and the wave–damping scale length is $\Delta_\nu=0.2$. Note that plots of the ring's surface density will be similar since $\sigma\propto\ell'/\sqrt{x}$.

Prometheus' orbital migration

Prometheus is one of the small shepherd satellites that straddle the narrow F ring (Fig. 1); Fig. 2 shows the spiral density waves it launches at its $m=11$ IRL in the A ring. This tiny satellite has a mass $\mu_j=2.5\times10^{-10}$, which boosts the ring–satellite mass ratio to $\mu_{rj}\sim200$ and the viscous parameter to $\nu_0=3\pi e_d^2\tau_1/\mu_j^2\sim5\times10^5$. Unlike Mimas, this tiny satellite is *not* going to carve open a gap at its resonances in the rings. Rather, the strong resonant torques is going to rapidly expand Prometheus' orbit faster than it can open a gap. Indeed, the resonant torques are so powerful that Prometheus, as well as its neighbors Pandora and perhaps the recently–discovered Methone, will all crash into Mimas in ~20 million years (Poulet and Sicardy 2001).

Parallels with planet migration

Lastly, it is worth noting the similarities between these satellites orbital evolution and type I & II planet migration. Lower–mass planets tend to suffer type I migration, which is the very rapid orbital evolution that is due to the exchange of angular momentum that occurs as the planet launches spiral density waves in the solar nebula gas disk (Ward 1997). This migration is usually inwards when there is disk material on both sides of the planet's orbit. Prometheus' orbital evolution is type I migration, but its orbit evolves outwards since it is repulsed by the interior rings.

Type II migration occurs when the planet is massive enough to open an annular gap in the solar nebula that is concentric with its orbit. The planet's orbit now co–evolves with the viscous disk, and migrates on the viscous timescale, usually inwards (Ward 1997). Mimas' opening of the Cassini Division is analogous of type II behavior, and its orbit may be evolving outwards on the rings' viscous timescale.