CHAPTER 1

REVIEW OF CLASSICAL MECHANICS

Newton's laws of motion provide the starting point for all of the physics that is discussed in this text. This brief review of elementary classical mechanics discusses those laws, and the useful conservation theorems that follow from them.

1.1 NEWTON'S LAWS OF MOTION

I: A body remains at rest or in uniform motion unless acted upon by a force . In other words, its velocity is constant when the force on that body is $\mathbf{F} = 0$.

II: A body acted upon by a force moves such that the time rate of change of its momentum equals that force, namely, $\dot{\mathbf{p}} = \mathbf{F}$, where $\mathbf{p} = m\dot{\mathbf{r}}$ is the body's linear momentum , m its mass, $\mathbf{r}$ its position vector, and its velocity $\dot{\mathbf{r}} = d\mathbf{r}/dt$ where the derivative is with respect to time t. This is the familiar $\mathbf{F} = m\ddot{\mathbf{r}}$ law.

III: If two bodies exert forces on each other, those forces are equal in magnitude and opposite in direction. Thus if $\mathbf{F_{12}}$ is the force on particle 1 that is exerted by particle 2, then $\mathbf{F_{21}} = -\mathbf{F_{12}}$.

1.2 REFERENCE FRAMES AND COORDINATE SYSTEMS

A reference frame is the coordinate grid that that is used to measure all particles' positions and velocities. Newton's laws are valid in an *inertial* reference frame, and law I indicates that an inertial reference frame is one that is stationary or moving with a constant velocity.

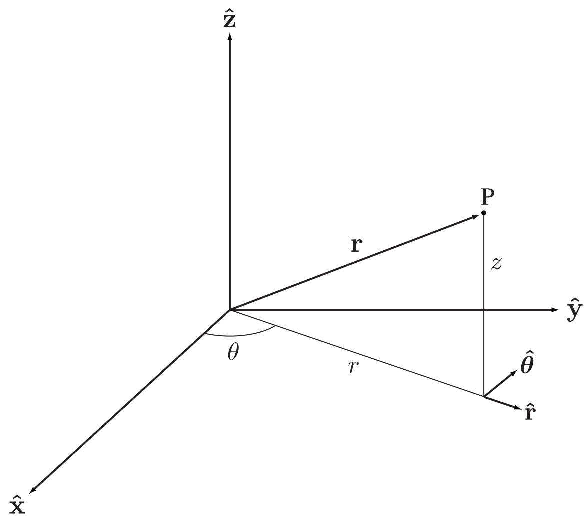

Figure 1.1 Position vector $\mathbf{r}$ for a particle at point P. Note that the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ lie in the $\hat{\mathbf{x}}$–$\hat{\mathbf{y}}$ plane.

Cartesian and cylindrical coordinate systems will be used in this text. In those coordinate systems, the position vector for particle at point P is (see Fig. 1.1)

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \qquad \text{in Cartesian coordinates} \tag{1.1}$$

$$= r\hat{\mathbf{r}} + z\hat{\mathbf{z}} \qquad \text{in cylindrical coordinates.} \tag{1.2}$$

In this cylindrical coordinate system, the unit vector $\hat{\mathbf{r}}$ is always confined to the $\hat{\mathbf{x}}$–$\hat{\mathbf{y}}$ plane. Note also the distinction in the lengths $r = \sqrt{x^2 + y^2}$ and $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. In these coordinate systems, the particle's velocity is

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}} + \dot{z}\hat{\mathbf{z}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{\mathbf{z}}, \tag{1.3}$$

and its acceleration is

$$\ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{x}\hat{\mathbf{x}} + \ddot{y}\hat{\mathbf{y}} + \ddot{z}\hat{\mathbf{z}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\theta} + \ddot{z}\hat{\mathbf{z}}. \tag{1.4}$$

1.3 LINEAR AND ANGULAR MOMENTA

Law II indicates that a particle's linear momentum $\mathbf{p} = m\dot{\mathbf{r}}$ is conserved (*i.e.*, a constant) when the total force on it is zero. That particle's angular momentum is $\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}}$, and rate at which $\mathbf{L}$ varies is the the torque on that particle, $\mathbf{T} = d\mathbf{L}/dt = m(\dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}}) = \mathbf{r} \times \mathbf{F}$. When the net torque on that particle is zero, its angular momentum is conserved.

1.4 WORK AND ENERGY

Suppose force $\mathbf{F}$ displaces a particle a small differential distance $\mathbf{dr}'$ during a short time interval dt; see Fig. 1.2. The small amount of work that that force performs on that particle

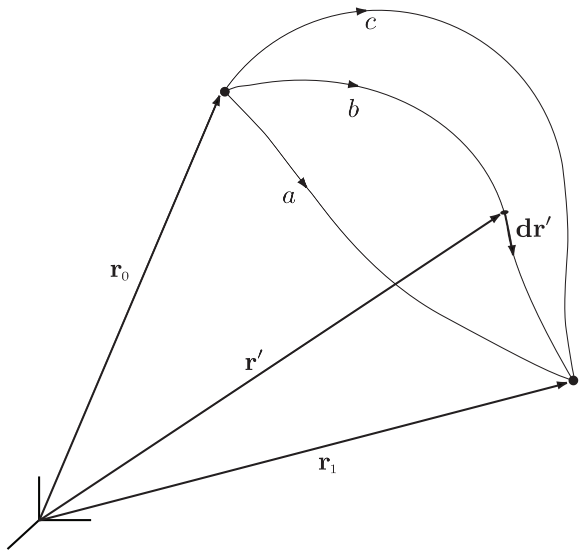

Figure 1.2 A particle is displaced from $\mathbf{r}_0$ to $\mathbf{r}_1$ by force $\mathbf{F}$ along three possible paths. The work done on the particle is $W = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot \mathbf{dr}'$, and if W is independent of the choice of path $(a, b$, or $c)$, then the force is said to be *conservative*.

is $dW = \mathbf{F} \cdot \mathbf{dr}'$, so the total work done on that particle as that force drives the particle from position $\mathbf{r}_0$ to $\mathbf{r}_1$ is the sum of all the contributions dW along that path, which is

$$W = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot \mathbf{dr}'. \tag{1.5}$$

Note also that

$$dW = m\ddot{\mathbf{r}} \cdot \mathbf{dr} = m\ddot{\mathbf{r}} \cdot \frac{\mathbf{dr}}{dt} dt = m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} dt = \frac{1}{2} m \frac{d}{dt}(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) = \frac{1}{2} m \frac{d(v^2)}{dt}, \tag{1.6}$$

where $v^2 = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$ is the square of the particle's velocity. The total work that $\mathbf{F}$ must do to drive the particle from $\mathbf{r}_0 \rightarrow \mathbf{r}_1$ is then

$$W = \frac{1}{2} m \int_{\mathbf{r}_0}^{\mathbf{r}_1} d(v^2) = \frac{1}{2} m (v_1^2 - v_0^2) = T_1 - T_0, \tag{1.7}$$

where $T_i = \frac{1}{2} m v_i^2$ is the particle's kinetic energy when at position $\mathbf{r}_i$. Thus the work done on the particle is simply its change in kinetic energy. Also note that the rate at which force $\mathbf{F}$ does work on the particle is

$$P = \frac{dW}{dt} = m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}}, \tag{1.8}$$

which is also known as the power delivered to that particle by force $\mathbf{F}$.

The work done on the particle by force $\mathbf{F}$ is also related to changes in its potential energy U. This text is largely concerned with *conservative* forces, and a conservative force is one where the work W performed on a particle is independent of the particular path that takes the particle from $\mathbf{r}_0$ to $\mathbf{r}_1$; see Fig. 1.2 When that is the case, then the vector force $\mathbf{F}$ can always be written as the gradient of a scalar $U(\mathbf{r})$ that is a function of position $\mathbf{r}$ only:

$$\mathbf{F} = -\nabla U, \tag{1.9}$$

where U is the system's *potential energy*. The gradient of U in Cartesian and cylindrical coordinates is

$$\begin{aligned} \nabla U &= \frac{\partial U}{\partial x}\hat{\mathbf{x}} + \frac{\partial U}{\partial y}\hat{\mathbf{y}} + \frac{\partial U}{\partial z}\hat{\mathbf{z}} \\ &= \frac{\partial U}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{\partial U}{\partial z}\hat{\mathbf{z}} \end{aligned} \tag{1.10}$$

So for example, the component of force along the $\hat{\mathbf{x}}$ axis is $F_x = -\partial U/\partial x$, while the azimuthal force is $F_\theta = -(\partial U/\partial \theta)/r$. Then the work, Eqn. (1.5), becomes

$$W = -\int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla U \cdot \mathbf{dr}, \tag{1.11}$$

where $U(\mathbf{r})$ is a function of the particle's *trajectory* $\mathbf{r}(t)$, which is the path traced by the particle over time. Next, use the chain rule to calculate dU/dt in a Cartesian coordinate system:

$$\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt} = (\nabla U)\cdot\dot{\mathbf{r}}. \tag{1.12}$$

Thus $dU = (\nabla U)\cdot\mathbf{dr}$ is the small change in the particle's potential energy that occurs as it advances a small distance $\mathbf{dr}$ along its trajectory during the short time interval dt. The work done on the particle can now be written as

$$W = -\int_{\mathbf{r}_0}^{\mathbf{r}_1} dU = -(U_1 - U_0) \tag{1.13}$$

where $U_i = U(\mathbf{r}_i)$ is the potential energy of the particle when it is at position $\mathbf{r}_i$. Thus the work done on the particle is also -1 times its change in potential energy. And since $W = T_1 - T_0 = -(U_1 - U_0)$, this means that the particle's energy at the endpoints of the trajectory, $E_1 = T_1 + U_1 = T_0 + U_0 = E_0$, is a constant, which tells us that the particle's energy $E = T + U$ is conserved, provided of course that the force acting on the particle is conservative. Conservative systems are frictionless (*i.e.*. have no velocity–dependent forces), and any external forces, if present, do not have any time dependence.

1.4.1 potential energy, and the potential

According to Eqns. (1.5) and (1.13), the system's potential energy $U(\mathbf{r})$ is $-1\times$ the work done on the particle as it is moved from the reference position $\mathbf{r}_0$ to its present position $\mathbf{r}$, so

$$U(\mathbf{r}) = -W = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot \mathbf{dr}', \tag{1.14}$$

where $\mathbf{r}'$ is a dummy variable that runs along the integration path (see Fig. 1.2). Note also that the unimportant constant U_0 has been dropped from the above, which means that the system's energy scale has been calibrated such that $U(\mathbf{r}_0) = 0$.

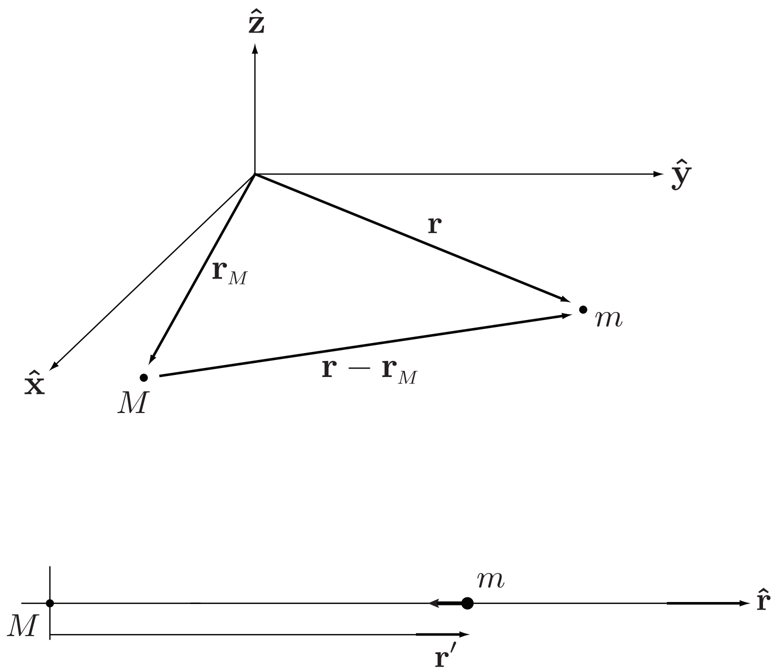

Figure 1.3 Two gravitating particles, one of mass m at $\mathbf{r}$, and the other of mass M at $\mathbf{r}_M$. The lower figure illustrates how this system's potential energy U is calculated at particle m is drawn from the reference site $\mathbf{r}_0$ to its final position at $\mathbf{r}$.

■ EXAMPLE 1.1

To illustrate the calculation of U, consider a simple gravitating two–particle system. The *field* particle, which is the particle of interest, has a mass m and a position vector $\mathbf{r}$, and it is free to move about the system, while the *source* mass M, which is the source of the force that disturbs m, remains at a fixed position $\mathbf{r}_M$. According to Newton's law of gravity, the force on m due to M is

$$\mathbf{F} = -\frac{GMm(\mathbf{r} - \mathbf{r}_M)}{|\mathbf{r} - \mathbf{r}_M|^3}, \tag{1.15}$$

and this force law is written so that it is evident that force $\mathbf{F}$ draws the field mass m towards the source mass M. Put the origin on M so that $\mathbf{r}_M = 0$, and recall that U is $-1\times$ the work done on the particle as M's force delivers particle m from the reference site $\mathbf{r}_0$ to its present position $\mathbf{r}$. Thus the force on m when it is at an intermediate distance $r' = |\mathbf{r} - \mathbf{r}_M|$ away from M is $\mathbf{F} = -(GMm/r'^2)\hat{\mathbf{r}}$ where $\hat{\mathbf{r}}$ is the usual unit radial vector; see the lower part of Fig. 1.3. The potential energy of this two–particle system is then

$$U(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' = \int_{\infty}^{r} \frac{GmM}{r'^2} dr' = -\frac{GmM}{r}, \tag{1.16}$$

where the arbitrary reference distance r_0 has been set to infinity, as is required since Eqn. (1.14) has set $U(\mathbf{r}_0) = 0$ at the reference site.

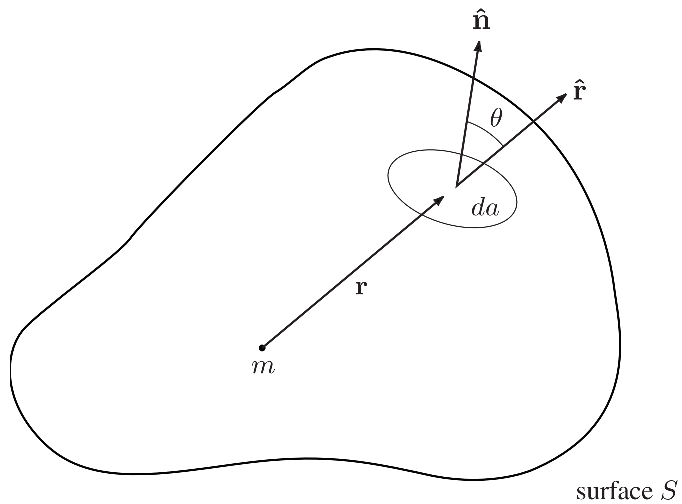

Figure 1.4 A Gaussian surface S surrounds a volume that contains mass m. The position vector $\mathbf{r}$ also points to the small area element $\mathbf{da} = \hat{\mathbf{n}}da$ on surface S.

Another useful quantity is the potential energy per unit mass, $\Phi(\mathbf{r}) = U/m$, also known as the system's *potential*. From Eqn. (1.16), the gravitational potential that the target mass m experiences due to the source mass M is

$$\Phi = \frac{U}{m} = -\frac{GM}{r}. \tag{1.17}$$

Newton's 2nd law, which is the principal equation of motion for this text, now becomes

$$\ddot{\mathbf{r}} = -\nabla\Phi. \tag{1.18}$$

1.4.2 Gauss' law

Next, derive Gauss' law, by placing an imaginary *Gaussian surface* S around a gravitating point mass m. The gravitational acceleration $\mathbf{g}$ at some point on the surface is

$$\mathbf{g} = -\nabla\Phi = -\frac{Gm}{r^2}\hat{\mathbf{r}}. \tag{1.19}$$

Let $\mathbf{da} = da\hat{\mathbf{n}}$ represent a small patch on S of area da whose orientation is described by a unit vector $\hat{\mathbf{n}}$ that is normal to da; see Fig. 1.4. The gravitational flux that passes through that area is $\mathbf{g} \cdot \mathbf{da}$, and in analogy with electrostatics, that flux can be thought of as a measure of the number of 'lines of force' that pass through $\mathbf{da}$. The total gravitational flux Ψ that passes through surface S is then

$$\Psi = \int_S \mathbf{g} \cdot \hat{\mathbf{n}}da. \tag{1.20}$$

Since $\mathbf{g} \cdot \hat{\mathbf{n}} = -Gm\cos\theta/r^2$, the total flux through surface S is $\Psi = -Gm\int_S \cos\theta da/r^2$. Figure 1.5 shows that $\cos\theta da$ is the projected area of da as seen by an observer sitting on

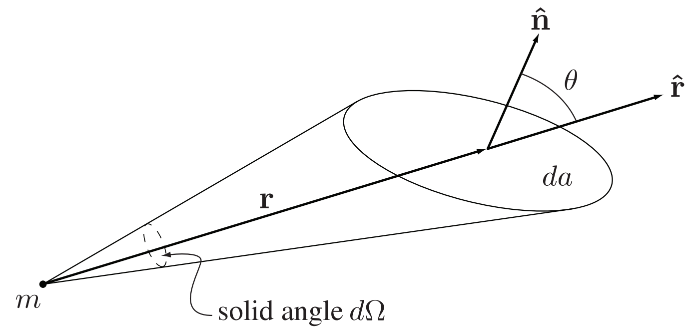

Figure 1.5 The area element da subtends a solid angle $d\Omega = \cos\theta da/r^2$.

m, so $d\Omega = \cos\theta da/r^2$ is the solid angle that da subtends, as seen from mass m. Thus the flux through S is $\Psi = -Gm\int_S d\Omega = -4\pi Gm$ since $\int_S d\Omega = 4\pi$ is also the solid angle of a sphere. And if surface S contains multiple masses m_i, then $M_{enc} = \sum m_i$ is the total mass enclosed by surface S, and the gravitational flux becomes

$$\Psi = \int_S \mathbf{g}\cdot\hat{\mathbf{n}}da = -4\pi GM_{enc}. \tag{1.21}$$

This is the integral form of Gauss' law, and it is very useful for problems that have a high degree of symmetry such that the area integral is easily evaluated.

■ **EXAMPLE 1.2**

Use Gauss's law to calculate the gravitational acceleration $\mathbf{g}(\mathbf{r})$ inside and outside a sphere of radius R and a constant density ρ. Then calculate the sphere's gravitational potential $\Phi(\mathbf{r})$.

The body's spherical symmetry means that $\mathbf{g}(\mathbf{r}) = g(r)\hat{\mathbf{r}}$, and that a spherical Gaussian surface of radius r would be in order; see Fig. 1.6. The normal to surface S always points radially, so $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ and the gravitational flux through surface S is $\Psi = \int_S \mathbf{g}\cdot\hat{\mathbf{r}}da = \int_S g(r)da = g(r)4\pi r^2 = -4\pi GM_{enc}$ where the mass enclosed by S is

$$M_{enc}(r) = \begin{cases} \frac{4\pi}{3}\rho r^3 & r < R \\ \frac{4\pi}{3}\rho R^3 \equiv M & r \geq R. \end{cases} \tag{1.22}$$

The acceleration due to the sphere is then

$$g(r) = -\frac{GM_{enc}(r)}{r^2} = \begin{cases} -\frac{4\pi}{3}G\rho r & r < R \\ -\frac{GM}{r^2} & r \geq R. \end{cases} \tag{1.23}$$

The sphere's potential Φ is obtained by using Eqns. (1.14) and (1.17) to calculate Φ for a particle of mass m, and noting that the force that the sphere exerts on this particle is $\mathbf{F} = m\mathbf{g}$. This then yields

$$\Phi(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{g}(\mathbf{r}')\cdot d\mathbf{r}', \tag{1.24}$$

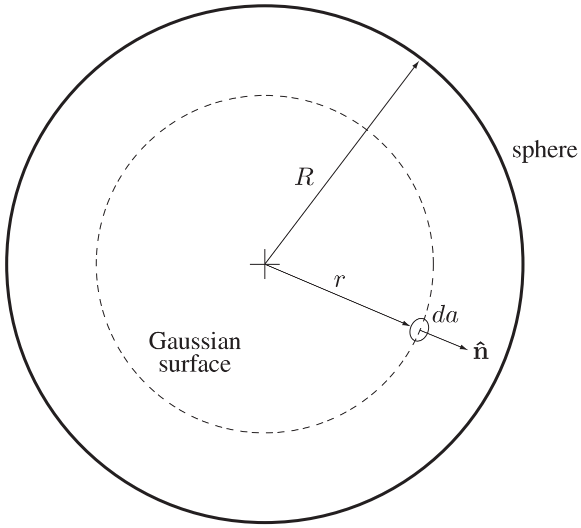

Figure 1.6 A uniform sphere of radius R and density ρ also contains a spherical Gaussian surface of radius r. A small area element da on the Gaussian surface having a unit normal $\hat{\mathbf{n}}$ is also indicated.

where $\mathbf{r}_0$ is again an arbitrary reference point. Taking $\mathbf{r}_0$ to be at infinity, the potential at any site exterior to the sphere is

$$\Phi(r > R) = -\int_{\infty}^{r} g(r' > R)dr' = -\frac{GM}{r}, \tag{1.25}$$

which is the potential of a point mass, as expected. The potential inside the sphere is

$$\begin{aligned}\Phi(r < R) &= -\int_{\infty}^{r} g(r')dr' = -\int_{\infty}^{R} g(r' > R)dr' - \int_{R}^{r} g(r' < R)dr' \\ &= -\frac{GM}{R} + \frac{2\pi}{3}G\rho(r^2 - R^2) = -\frac{2\pi}{3}G\rho(3R^2 - r^2).\end{aligned} \tag{1.26}$$

Be sure to use the same reference point $\mathbf{r_0}$ when calculating $\Phi(\mathbf{r})$ in both the interior and exterior zones.

1.4.3 Poisson's equation

A differential form of Gauss' law is obtained from the definition of the gravitational flux

$$\Psi = \int_S \mathbf{g}\cdot\hat{\mathbf{n}}da = -4\pi GM_{enc} = -4\pi G\int_V \rho(\mathbf{r})dV \tag{1.27}$$

where the enclosed mass is $M_{enc} = \int_V \rho(\mathbf{r})dV$ where $\rho(\mathbf{r})$ is the density of matter in volume V that is enclosed by surface S (see Fig. 1.4). Next, invoke the divergence theorem of vector calculus, Eqn. (A.24a), which allows the surface integral to be recast as a volume integral,

$$\Psi = \int_S \mathbf{g}\cdot\hat{\mathbf{n}}da = \int_V \nabla\cdot\mathbf{g}dV, \tag{1.28}$$

noting that $\nabla \cdot \mathbf{g} = -\nabla^2\Phi$ by Eqn. (1.19). Thus $\Psi = -\int_V \nabla\Phi^2 dV = -4\pi G \int_V \rho dV$, so

$$\int_V (\nabla^2\Phi - 4\pi G\rho)dV = 0. \tag{1.29}$$

This result must hold for any arbitrary volume V, which means that the integrand itself must be zero, so

$$\nabla^2\Phi = 4\pi G\rho. \tag{1.30}$$

This is *Poisson's equation*, which is the differential form of Gauss' law, and its relates the mass distribution $\rho(\mathbf{r})$ to its gravitational potential $\Phi(\mathbf{r})$. This equation is of fundamental importance to hydrodynamic studies of gravitating fluids, and it is used to study the formation of galaxies and stars. This equation will also be used later in this text to study gravitational instabilities, and also spiral wave theory.

Lastly, note that free space, where $\rho = 0$, obeys the *Laplace equation*,

$$\nabla^2\Phi = 0 \tag{1.31}$$

1.4.4 the gravitational stress tensor

The density of the gravitational force that is exerted on a distribution of matter is $\mathbf{f} = \rho\mathbf{g}$ where the gravitational acceleration is $\mathbf{g} = -\nabla\Phi$ and the matter density is $\rho = \nabla^2\Phi/4\pi G$ from Poisson's equation (1.30), so the force density can be written $\mathbf{f} = -\mathbf{g}(\nabla \cdot \mathbf{g})/4\pi G$. Also recall that the curl of the gradient of a scalar is zero (e.g. Eqn. A.16), so $\nabla \times \mathbf{g} = 0$. So one can also write $\mathbf{f} = -[\mathbf{g}(\nabla\cdot\mathbf{g}) - \mathbf{g}\times(\nabla\times\mathbf{g})]/4\pi G$, but $\mathbf{g}\times(\nabla\times\mathbf{g}) = \frac{1}{2}\nabla\mathbf{g}^2 - (\mathbf{g}\cdot\nabla)\mathbf{g}$ by Eqn. (A.15) so $\mathbf{f} = -[(\mathbf{g}\cdot\nabla)\mathbf{g} + \mathbf{g}(\nabla\cdot\mathbf{g}) - \frac{1}{2}\nabla\mathbf{g}^2]/4\pi G$, noting that the first term is the convective operator, Eqn. (A.22). Now let $\hat{\mathbf{x}}_i$ refers to the three cartesian coordinates $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, or $\hat{\mathbf{z}}$ when $i = 1, 2$, or 3. Then the i^{th} component of the force density f_i is

$$f_i = -\frac{1}{4\pi G}\left[\sum_{j=1}^{3} g_j \frac{\partial g_i}{\partial x_j} + \sum_{j=1}^{3} g_i \frac{\partial g_j}{\partial x_j} - \frac{1}{2}\frac{\partial g^2}{\partial x_i}\right] \tag{1.32}$$

$$= -\frac{1}{4\pi G}\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(g_i g_j - \frac{1}{2}\delta_{ij}g^2\right) \tag{1.33}$$

$$= -\frac{1}{4\pi G}\sum_{j=1}^{3}\frac{\partial \sigma_{ij}}{\partial x_j} \tag{1.34}$$

where the

$$\sigma_{ij} = \frac{1}{4\pi G}(g_i g_j - \frac{1}{2}\delta_{ij}g^2) \tag{1.35}$$

are the elements of a 3×3 matrix that is called the the gravitational stress tensor. Note that the stress tensor is symmetric, $\sigma_{ij} = \sigma_{ji}$. Now form the vector $\boldsymbol{\sigma}_i = \sum_j \sigma_{ij}\,\hat{\mathbf{x}}_j$, which allows one to write f_i more compactly as

$$f_i = -\nabla \cdot \boldsymbol{\sigma}_i. \tag{1.36}$$

To demonstrate the meaning of the vector $\boldsymbol{\sigma}_i$, consider some volume V and calculate the total force on that volume along direction $\hat{\mathbf{x}}_i$, which is

$$F_i = \int_V f_i dV = -\int_V \nabla \cdot \boldsymbol{\sigma}_i dV = -\int_A \boldsymbol{\sigma}_i \cdot \mathbf{da} \tag{1.37}$$

where the right hand side is obtained using the divergence theorem (A.24a), and the A in the above is the area that bounds volume V. So the volume's i^{th} momentum component evolves at the rate $dp_i/dt = F_i = -\int_A \boldsymbol{\sigma}_i \cdot \mathbf{da}$ due to force F_i. Evidently $\boldsymbol{\sigma}_i$ is the flux of the material's i^{th} component of linear momentum due to gravity. Note that the minus sign in Eqn. (1.37) accounts for the fact that as momentum flows out of volume V through area $\mathbf{da}$ at the rate $\boldsymbol{\sigma}_i \cdot \mathbf{da}$, volume V reacts as if its momentum is reduced at the rate $-\boldsymbol{\sigma}_i \cdot \mathbf{da}$.

1.4.4.1 gravitational transport of angular momentum The material's angular momentum density is $\boldsymbol{\ell} = \rho \mathbf{r} \times \dot{\mathbf{r}}$, and that quantity evolves at the rate $d\boldsymbol{\ell}/dt = \mathbf{r} \times d(\rho \dot{\mathbf{r}})/dt$. The following is interested in the evolution that is due solely to the system's gravity, which is a conservative force, so $d(\rho \dot{\mathbf{r}})/dt = \mathbf{f}$ and $d\boldsymbol{\ell}/dt = \mathbf{r} \times \mathbf{f}$ as expected. The integrated torque across some volume V of this material is

$$\begin{aligned}
\mathbf{T} = \int_V \frac{d\boldsymbol{\ell}}{dt} dV = \int_V (\mathbf{r} \times \mathbf{f}) dV = \int_V &[(x_3 \nabla \cdot \boldsymbol{\sigma}_2 - x_2 \nabla \cdot \boldsymbol{\sigma}_3)\, \hat{\mathbf{x}}_1 \\
&+ (x_1 \nabla \cdot \boldsymbol{\sigma}_3 - x_3 \nabla \cdot \boldsymbol{\sigma}_1)\, \hat{\mathbf{x}}_2 + (x_2 \nabla \cdot \boldsymbol{\sigma}_1 - x_1 \nabla \cdot \boldsymbol{\sigma}_2)\, \hat{\mathbf{x}}_3] dV \\
= \int_V &\{[\nabla \cdot (x_3 \boldsymbol{\sigma}_2 - x_2 \boldsymbol{\sigma}_3) - \boldsymbol{\sigma}_2 \cdot \nabla x_3 + \boldsymbol{\sigma}_3 \cdot \nabla x_2]\, \hat{\mathbf{x}}_1 \\
&+ [\nabla \cdot (x_1 \boldsymbol{\sigma}_3 - x_3 \boldsymbol{\sigma}_1) - \boldsymbol{\sigma}_3 \cdot \nabla x_1 + \boldsymbol{\sigma}_1 \cdot \nabla x_3]\, \hat{\mathbf{x}}_2 \\
&+ [\nabla \cdot (x_2 \boldsymbol{\sigma}_1 - x_1 \boldsymbol{\sigma}_2) - \boldsymbol{\sigma}_1 \cdot \nabla x_2 + \boldsymbol{\sigma}_2 \cdot \nabla x_1]\, \hat{\mathbf{x}}_3\} dV.
\end{aligned} \tag{1.38}$$

Now note that $-\boldsymbol{\sigma}_2 \cdot \nabla x_3 + \boldsymbol{\sigma}_3 \cdot \nabla x_2 = -\sigma_{23} + \sigma_{32} = 0$ since the gravitational stress tensor is symmetric. The other similar terms in Eqn. (1.38) also sum to zero, so the total torque on volume V has components $T_i = -\int_V \nabla \cdot \mathbf{F}_i dV = -\int_A \mathbf{F}_i \cdot \mathbf{da}$ where $\mathbf{F}_i$ is the flux of angular momentum about axis $\hat{\mathbf{x}}_i$ such that

$$\mathbf{F}_1 = x_2 \boldsymbol{\sigma}_3 - x_3 \boldsymbol{\sigma}_2 \tag{1.39}$$

$$\mathbf{F}_2 = x_3 \boldsymbol{\sigma}_1 - x_1 \boldsymbol{\sigma}_3 \tag{1.40}$$

$$\mathbf{F}_3 = x_1 \boldsymbol{\sigma}_2 - x_2 \boldsymbol{\sigma}_1. \tag{1.41}$$

Chapter 12 will use this result to calcuate the rate at which a gravitating disk will transport angular momentum via density waves. In that system the quantity of interest is the angular momentum flux about the disk normal $\hat{\mathbf{x}}_3 = \hat{\mathbf{z}}$, and the preferred coordinate system is cyclindrical coordinates. And in problem 1.7 you will show that the disk's angular momentum flux is

$$\mathbf{F}_z = \frac{r}{4\pi G} \left[g_r g_\theta\, \hat{\mathbf{r}} - \frac{1}{2}(g_r^2 - g_\theta^2)\, \hat{\boldsymbol{\theta}} \right] \tag{1.42}$$

in cylindrical coordinates, where g_r and g_θ are the radial and azimuthal components of the disk's gravitational acceleration.

1.5 ROTATING REFERENCE FRAMES

At times it will be convenient to work in a rotating reference frame. Rotations are described by a vector $\boldsymbol{\omega}$ whose magnitude and direction indicate the angular rate rotation about the rotation axis. A particle's velocity in the rotating reference frame, $\dot{\mathbf{r}}_\mathbf{r}$ is related to its velocity in the stationary frame, $\dot{\mathbf{r}}_\mathbf{s}$, via

$$\dot{\mathbf{r}}_\mathbf{r} = \dot{\mathbf{r}}_\mathbf{s} - \boldsymbol{\omega}\times\mathbf{r}, \tag{1.43}$$

where $\mathbf{r}$ is the particle's position relative to the rotating origin. If the reference frame's rotation is steady, $\boldsymbol{\omega}$ is a constant, and the particle's equation of motion becomes

$$\ddot{\mathbf{r}}_\mathbf{r} = -\nabla\Phi - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - 2\boldsymbol{\omega}\times\dot{\mathbf{r}}_\mathbf{r}. \tag{1.44}$$

The two new terms are of course the centrifugal and Coriolis accelerations, respectively, that appear when Newton's second law is adapted for use in a non–inertial reference frame.

1.6 SYSTEMS OF PARTICLES

The following summarizes the conservation theorems that result when Newton's laws are applied to a system composed of N discrete particles. These are offered here without proof, since their derivations can be found in any text on classical mechanics.

1.6.1 linear momentum

The system is composed of N particles, where m_j is the mass of the j^{th} particle that has a position vector $\mathbf{r}_j$. The system's center of mass is

$$\mathbf{R} = \frac{1}{M}\sum_{j=1}^{N} m_k\mathbf{r}_j \tag{1.45}$$

where $M = \sum_j m_j$ is the total mass of the system, and its total linear momentum momentum is $\mathbf{P} = \sum_j \mathbf{p}_j$ where $\mathbf{p}_j = m_j\dot{\mathbf{r}}_j$ is the momentum of particle j. Note also that $\dot{\mathbf{P}} = M\ddot{\mathbf{R}}$.

The N particles can be mutually interacting, so the force $\mathbf{F}_j$ on any one particle j can be written

$$\mathbf{F}_j = \sum_{k\neq j}\mathbf{f}_{jk} + \mathbf{F}_j^e = m_j\ddot{\mathbf{r}}_j, \tag{1.46}$$

where $\mathbf{f}_{jk}$ is the force on particle j due to body k, and $\mathbf{F}_j^e$ represents any additional force on j that is *external* to the system. For instance, if the system were a star cluster that inhabits a galaxy, then $\mathbf{f}_{jk}$ would be the force on star j due to star k, while the external force $\mathbf{F}_j^e$ would represent the galactic tide, which is the acceleration that the galaxy exerts on star j relative to the cluster's center of mass. The total force on the system is the above summed over all particles, but the sum of all the internal forces, $\sum\sum_{j\neq k}\mathbf{f}_{jk}$, is zero by Newton's 3$^{\text{rd}}$ law, so the total force on the system is

$$\sum_{j=1}^{N}\mathbf{F}_j = \sum_{j=1}^{N}\mathbf{F}_j^e = \mathbf{F}^e, \tag{1.47}$$

where $\mathbf{F}^e$ is the sum of all the external forces. It then follows that

$$\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \sum_{j=1}^{N} m_k\ddot{\mathbf{r}}_j = \sum_{j=1}^{N}\mathbf{F}_j^e = \mathbf{F}^e, \tag{1.48}$$

which means that the system's center of mass evolves as if it were a single particle of mass M under the influence of the total external force $\mathbf{F}^e$. The system's total linear momentum $\mathbf{P}$ is also conserved when the total external force is zero.

1.6.2 angular momentum

The system's total angular momentum about the origin is $\mathbf{L} = \sum_j \mathbf{L}_j$, where $\mathbf{L}_j = m_j \mathbf{r}_j \times \dot{\mathbf{r}}_j$ is the angular momentum of particle j about the same origin. Note that the origin need not coincide with the system's center of mass, so write a particle's position vector as $\mathbf{r}_j = \mathbf{R} + \mathbf{r}'_j$, where $\mathbf{r}'_j$ is its position relative to the center of mass $\mathbf{R}$. It can then be shown that the system's total angular momentum is

$$\mathbf{L} = \mathbf{R} \times \mathbf{P} + \sum_{j=1}^{N} \mathbf{L}'_j, \tag{1.49}$$

where $\mathbf{L}'_j = m_j \mathbf{r}'_j \times \dot{\mathbf{r}}'_j$ is the angular momentum of particle j about the center of mass. Thus the system's total angular momentum is the sum of two parts: the angular momentum due to the center of mass's motion about the origin, $\mathbf{R} \times \mathbf{P}$, plus the angular momentum of the system about its center of mass, $\sum_j \mathbf{L}'_j$.

All of the systems considered in this text obey the *strong form* of Newton's 3rd law, which assumes that the internal force exerted between any two particles, $\mathbf{f}_{jk}$, is directed along the line joining particles j and k. It can then be shown that the internal forces do not alter the system's total angular momentum—only external forces alter $\mathbf{L}$, and at the rate

$$\dot{\mathbf{L}} = \sum_{j=1}^{N} \mathbf{T}^e_j = \mathbf{T}^e \tag{1.50}$$

where $\mathbf{T}^e_j = \mathbf{r}_j \times \mathbf{F}^e_j$ is the external torque exerted on particle j, and $\mathbf{T}^e$ is the total external torque on system. So when the strong form of Newton's 3rd applies, which is usually the case, then the system's total angular momentum is conserved when there are no external torques.

1.6.3 energy

Arguments similar to that given in Section 1.4 will show that the total energy of a conservative system, $E = T + U$, is conserved. The system's total kinetic energy is

$$T = \frac{1}{2} M \dot{\mathbf{R}}^2 + \sum_{j=1}^{N} \frac{1}{2} m_j \dot{\mathbf{r}}'^2_j, \tag{1.51}$$

which is the kinetic energy due to the motion of the center of mass plus that due to internal motions about the center of mass. The system's total potential energy is

$$U = \sum_{k=1}^{N} \sum_{\ell=k+1}^{N} U^i_{k\ell} + \sum_{k=1}^{N} U^e_k \tag{1.52}$$

where $U^i_{k\ell}$ is the potential energy associated with the internal force $\mathbf{f}_{k\ell}$ exerted between particles k and ℓ, and U^e_k is the potential energy due to the external force $\mathbf{F}^e_k$ on particle

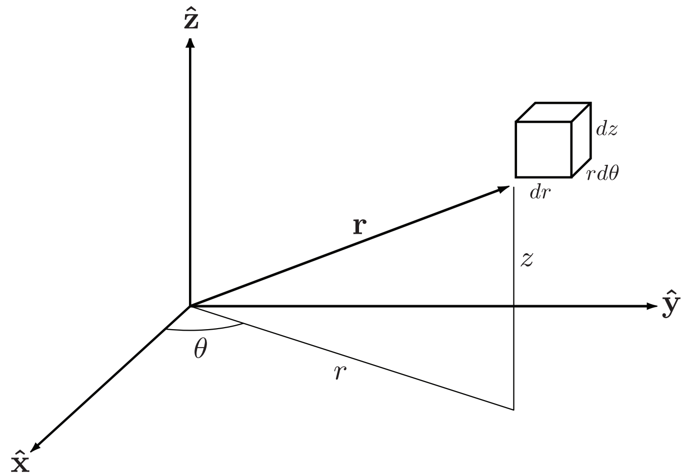

Figure 1.7 The small volume element $dV = dxdydz$ in Cartesian coordinates and $dV = rdrd\theta dz$ in cylindrical coordinates.

k. The double sum in the above is constructed to avoid any overcounting of energies or forces. The force on any one particle j is then

$$\mathbf{F}_j = -\nabla_j U = -\nabla_j \sum_{k \neq j} U_{jk}^i - \nabla_j U_j^e, \tag{1.53}$$

where $\nabla_j U$ indicates the gradient of U calculated with respect to particle j's spatial coordinates. This of course is identical to Eqn. (1.46), since the internal force on j due to k is $\mathbf{f}_{jk} = -\nabla_j U_{jk}^i$, and the external force on j is $\mathbf{F}_j^e = -\nabla_j U_j^e$.

1.6.4 continuous systems

The preceding results obtained above for a system of discrete particles is easily generalized for a continuous distribution of matter that has a volume density $\rho(\mathbf{r})$. To account for this, replace the mass m_j appearing in any of the above sums with $\rho(\mathbf{r})dV$, where dV is a small volume element, and replace the summation symbol with an integral. So for example, the center of mass for some cloud of matter would be (from Eqn. 1.45)

$$\mathbf{R} = \frac{1}{M}\int_V \rho(\mathbf{r}')\mathbf{r}' dV', \tag{1.54}$$

where the integration proceeds over the cloud's volume V, with $M = \int_V \rho(\mathbf{r}')dV'$ being the cloud's total mass. If the integration is performed in a Cartesian coordinate system, then the differential volume element is $dV' = dx'dy'dz'$. But if cylindrical coordinates are used, then $dV' = r'dr'd\theta' dz'$ (see Fig. 1.7).

But if the matter distribution is two–dimensional, such as plane or a shell, then replace the small mass element $\rho(\mathbf{r})dV$ with $\sigma(\mathbf{r})dA$, where $\sigma(\mathbf{r})$ is the mass surface density, and dA is the small area element, and integrate over the body's entire area A, e.g., $M = \int_A \sigma(\mathbf{r}')dA'$. So for example, if the surface is a plane, then $dA' = dx'dy'$ or $dA' = r'dr'd\theta'$ in Cartesian or polar coordinates.

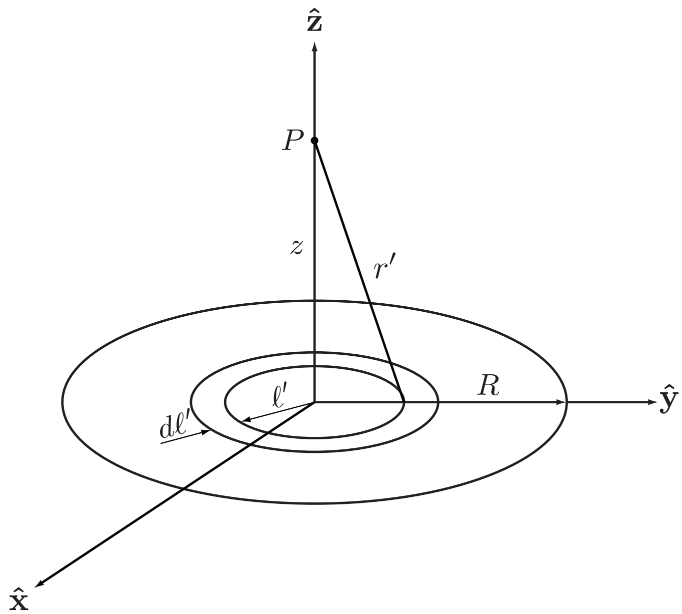

Figure 1.8 A disk has radius R and surface density σ, and its gravitational potential Φ is to be evaluated at the field point P that lies a vertical distance z from the disk's center. To calculate Φ, treat the disk as concentric annuli having radii ℓ', radial width $d\ell'$, and differential area $dA' = 2\pi\ell' d\ell'$, and sum the contributions from all annuli.

■ EXAMPLE 1.3

A thin flat disk has a radius R, a constant surface density σ, and is in keplerian rotation about its center. The disk's angular velocity at its outer edge is Ω. Calculate the disk's total angular momentum.

As we shall see in Chapter 2, Keplerian rotation means that the disk's angular velocity $\dot{\theta}$ varies with radius r in the disk as $r^{-3/2}$, so $\dot{\theta} = \Omega(r/R)^{-3/2}$, and the tangential velocity of any disk parcel is $\dot{\mathbf{r}} = r\dot{\theta}\hat{\boldsymbol{\theta}}$ (Eqn. 1.3). A small parcel of the disk will have mass $dm = \sigma dA$, where dA is the area of that disk element, so its angular momentum content is $\mathbf{dL} = dm\mathbf{r} \times \dot{\mathbf{r}}$ where $\mathbf{r} = r\hat{\mathbf{r}}$ is the position vector of that area element. The disk's angular momentum *density* is thus $\boldsymbol{\ell} = \mathbf{dL}/dA = \sigma r^2\dot{\theta}\hat{\mathbf{z}}$, and the disk's total angular momentum is $\mathbf{L} = \int_A \boldsymbol{\ell} dA$ where $dA = 2\pi r dr$ is the area of an annulus in the disk. The disk's total angular momentum is then

$$\mathbf{L} = \int_0^a \sigma r^2 \Omega \left(\frac{r}{R}\right)^{-3/2} \hat{\mathbf{z}} 2\pi r dr = 2\pi\sigma\Omega R^4 \hat{\mathbf{z}} \int_0^1 x^{3/2} dx = \frac{4}{5}\pi\sigma\Omega R^4 \hat{\mathbf{z}}. \qquad (1.55)$$

■ EXAMPLE 1.4

A disk has radius R and a constant surface density σ. Calculate the disk's gravitational potential Φ at a perpendicular distance z away from the disk's center. What is Φ in the limit that $R \gg |z|$ (*i.e.*, in the limit that the disk is a sheet having an infinite extent)? Then place a massless test particle in this system—what is its acceleration due to the sheet's gravity?

The gravitational potential due to a small mass element is $d\Phi = -Gdm'/r'$ (from Eqn. 1.17), so the total potential is $\Phi = \int_A d\Phi$, where the integration proceeds across the disk's area A, and r' is the distance from the source mass dm' to the so-called field point P where Φ will be calculated; see Fig. 1.8. Divide the disk into concentric annuli of radii ℓ', radial width $d\ell'$, and area $dA' = 2\pi\ell' d\ell'$, and note that all parts of a given annulus are equidistant from the field point, so $r' = \sqrt{\ell'^2 + z^2}$ and $d\Phi = -2\pi G\sigma\ell' d\ell'/\sqrt{\ell'^2 + z^2}$ is the potential due to a narrow ring. The disk's total potential is then

$$\Phi = \int_A d\Phi = -2\pi G\sigma \int_0^R \frac{\ell' d\ell'}{\sqrt{\ell'^2 + z^2}} = -2\pi G\sigma \int_{|z|}^{\sqrt{R^2+z^2}} du \tag{1.56}$$

upon the substitution $u = \sqrt{\ell'^2 + z^2}$, so

$$\Phi(z) = -2\pi G\sigma[\sqrt{R^2 + z^2} - |z|]. \tag{1.57}$$

To get Φ in the $R \gg |z|$ limit, it is convenient to rewrite Φ as

$$\Phi = -2\pi G\sigma R \left[\sqrt{1 + \left(\frac{z}{R}\right)^2} - \frac{|z|}{R}\right] \tag{1.58}$$

so we can invoke the binomial expansion, Eqn. (A.1):

$$\sqrt{1 + \left(\frac{z}{R}\right)^2} \simeq 1 + \frac{1}{2}\left(\frac{z}{R}\right)^2 + \mathcal{O}\left(\frac{z}{R}\right)^4, \tag{1.59}$$

so the potential of an infinite sheet with $|z|/R \ll 1$ is

$$\Phi = 2\pi G\sigma|z|, \tag{1.60}$$

upon dropping the unimportant constant.

To get the acceleration of a test particle, write $\Phi = s_z 2\pi G\sigma z$ where $s_z = \mathrm{sgn}(z)$, so the acceleration is $\ddot{\mathbf{r}} = -\nabla\Phi = -(\partial\Phi/\partial z)\hat{\mathbf{z}} = -s_z 2\pi G\sigma\hat{\mathbf{z}}$. Note that keeping proper track of the sign of z in Φ is key to getting the direction of the acceleration $\ddot{\mathbf{r}}$ correct, which as we see draws the test particle towards the sheet, as expected.

Problems

1.1 A thin uniform rod has mass M and length L. Show that its gravitational potential evaluated at the field point $\mathbf{r}$ is

$$\Phi(\mathbf{r}) = -\frac{GM}{L}\ln\frac{\sqrt{1 + \alpha^2 - 2\alpha\cos\theta} + \alpha - \cos\theta}{\sqrt{1 + \alpha^2 + 2\alpha\cos\theta} - \alpha - \cos\theta} \tag{1.61}$$

where r is the field point's distance from the rod center, $\alpha = L/2r$, and θ is the angle between $\mathbf{r}$ and the near part of the rod's long axis.

1.2 Use Gauss' law to obtain Eqn. (1.60), which is the gravitational potential of a thin, infinite sheet that has a constant surface mass density σ.

1.3 A flat, gaseous slab that has an infinite horizontal extent, a vertical thickness ℓ, and a constant volume density ρ. Show that the acceleration due to the slab is

$$g(z) = -4\pi G\rho z \tag{1.62}$$

at sites *inside* the slab, where z is the vertical distance from the slab's midplane. Then use Gauss' law to calculate g exterior to the slab, and show that your result agrees with Eqn. (1.60), namely, that $g = -\partial\Phi/\partial z$.

1.4 A massless test-particle is released from rest at a distance $z = 2\ell$ from the midplane of the slab described in Problem 1.3. Describe qualitatively the particle's subsequent motion. What is the particle's velocity when at the slab's midplane?

1.5 The test-particle from Problem 1.4 is instead released at a height $z < \ell$. What is the frequency of that particle's oscillations about the slab's midplane?

1.6 A constant density cylinder has an infinite length and a radius R. Calculate its gravitational acceleration $\mathbf{g}$ and potential Φ, both interior and exterior to the cylinder.

1.7 Show that Eqn. (1.41) can be written

$$\mathbf{F}_3 = \frac{1}{4\pi G}\left[(x_1 g_2 - x_2 g_1)\mathbf{g} + \frac{1}{2}(x_2\,\hat{\mathbf{x}}_1 - x_1\,\hat{\mathbf{x}}_2)g^2\right] = \frac{1}{4\pi G}\left[(\mathbf{r}\times\mathbf{g})\cdot\hat{\mathbf{z}}\,\mathbf{g} + \frac{1}{2}(\mathbf{r}\times\hat{\mathbf{z}})g^2\right]. \tag{1.63}$$

Then show that the above leads to Eqn. (1.42).

REFERENCES

1. S. T. Thornton & J. B. Marion, *Classical Dynamics of Particles and Systems*, Brooks/Cole, Belmont, CA, 2004. Formal derivations of the equations and conservation theorems quoted in Sections 1.5 and 1.6 can be found in Chapter 9 and 10 of this excellent undergraduate textbook on classical mechanics.