Lecture Notes for AST 608
Selected Topics in Astronomy & Astrophysics

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February 23, 2004

revised March 15, 2004

Perturbed Motion

from Chapters 6 & 8 of Murray & Dermott's *Solar System Dynamics*

Lecture 1 examined the 2–body problem, where it was shown that m_2's motion $\mathbf{r}(t)$ can be described by a set of constant orbit elements $(a, e, i, \Omega, \omega, \tau)$.

So for any time t you can go from
$\mathbf{r}(t)$ & $\mathbf{v} = \dot{\mathbf{r}}(t) \leftrightarrow (a, e, i, ...)$ using the recipe on page 24.

Now add an additional perturbing acceleration $\mathbf{a}_p$ that alters m_2's motion (ie, the gravity of another planet, PR drag, aerodynamic drag, etc).

If we wish to continue to use orbit elements to describe m_2's motion, then the elements become functions of time, ie, $a = a(t), e = e(t), i = i(t)$, etc.

The elements $(a, e, i, ...)$ are referred to as m_2's *osculating* elements (from *osculare* = to kiss in Latin).

m_2's trajectory is no longer along a fixed keplerian ellipse.

In this case, its trajectory $\mathbf{r}(t)$ & $\dot{\mathbf{r}}(t)$ can be obtained from the $(a, e, i, ...)$ which describes an ellipse that 'kisses' m_2's trajectory at time t, but diverges from that trajectory at earlier/later times.

To get m_2's $\mathbf{r}(t)$ & $\dot{\mathbf{r}}(t)$ at other times, you need to know how the perturbation $\mathbf{a}_p$ causes the (a,e,i,...) to evolve over time.

1. First we will derive Gauss' form for the planetary equations that provide $\dot{a}, \dot{e}$, etc, due to the perturbation $\mathbf{a_p}$.

2. We will then derive an alternate form of these eqn's,
the Lagrange planetary eqn's.

3. Then use these equations to solve various dynamical problems, such as the motion of a small body orbiting near a resonance with a planet, the motion of a small body due to planetary oblateness, due to drag forces, etc.

Gauss' planetary equations

Calculate $\dot{a}$ = the rate at which the perturbing acceleration $\mathbf{a}_p$ alters m_2's semimajor axis (see Section 2.9 in text):

Do this by considering the work the perturbation does on m_2:

$$W = \int_{\mathbf{r}_0}^{\mathbf{r}(t)} m_2 \mathbf{a}_p \cdot d\mathbf{r}' = \int_{t_0}^{t} m_2 \mathbf{a}_p \cdot \frac{d\mathbf{r}}{dt'} dt' \tag{4.1}$$

$$\text{so} \quad \frac{dW}{dt} = m_2 \mathbf{a}_p \cdot \dot{\mathbf{r}} \tag{4.2}$$

= the rate at which the perturbation alters m_2's energy $m_2 E$ where

$$E = -\frac{G(m_1 + m_2)}{2a} = m_2\text{'s specific energy} \tag{4.3}$$

$$\text{so} \quad \frac{dE}{dt} = \frac{G(m_1 + m_2)}{2a^2}\dot{a} = \frac{1}{2}n^2 a\dot{a} = \frac{1}{m_2}\frac{dW}{dt} = \mathbf{a}_p \cdot \dot{\mathbf{r}} \tag{4.4}$$

$$\text{or} \quad \dot{a} = \frac{2\mathbf{a}_p \cdot \dot{\mathbf{r}}}{n^2 a} \tag{4.5}$$

$$\tag{4.6}$$

Adopt a cylindrical coordinate system with the x–y axes in the orbit plane and $\hat{\mathbf{x}}$ pointing to periapse:

$$\begin{aligned} \text{Let} \quad \mathbf{a}_p &= a_r\hat{\mathbf{r}} + a_\theta\hat{\theta} + a_z\hat{\mathbf{z}} && (4.7)\\ \text{also} \quad \dot{\mathbf{r}} &= \dot{r}\hat{\mathbf{r}} + r\dot{f}\hat{\theta} \quad \text{where } f = \text{true anomaly} && (4.8)\\ \text{Thus} \quad \dot{a} &= \frac{2(a_r\dot{r} + a_\theta r\dot{f})}{n^2 a} && (4.9) \end{aligned}$$

From eqn's 1.87–1.88:

$$\begin{aligned} \dot{r} &= \frac{nae\sin f}{\sqrt{1-e^2}} \quad \text{and} \quad r\dot{f} = \frac{na(1+e\cos f)}{\sqrt{1-e^2}} && (4.10)\\ \text{so} \quad \dot{a} &= \frac{2}{n\sqrt{1-e^2}}\left[a_r e\sin f + a_\theta(1+e\cos f)\right] && (4.11) \end{aligned}$$

which is the rate at which $\mathbf{a}_p$ alters m_2's a.

To get $\dot{e}$, start with m_2's specific angular momentum

$$\begin{aligned} \mathbf{h} &= \mathbf{r}\times\dot{\mathbf{r}} = r^2\dot{f}\hat{\mathbf{z}} = \sqrt{\mu a(1-e^2)}\hat{\mathbf{z}} && (4.12)\\ \text{so} \quad \dot{\mathbf{h}} &= \mathbf{r}\times\ddot{\mathbf{r}} && (4.13)\\ \text{where} \quad \ddot{\mathbf{r}} &= -\frac{G(m_1+m_2)}{r^3}\hat{\mathbf{r}} + \mathbf{a}_p = \text{ accelerations on } m_2 && (4.14)\\ \text{thus} \quad \dot{\mathbf{h}} &= \mathbf{r}\times\mathbf{a_p} = ra_\theta\hat{\mathbf{z}} - ra_z\hat{\theta} = \text{ torque on } m_2\text{'s orbit} && (4.15) \end{aligned}$$

Note that the torque along the $\hat{\theta}$ direction merely 'tips-over' m_2's orbit plane—it does not alter its total $\mathbf{h}$.

However the torque in the $\hat{\mathbf{z}}$ direction does alter $\mathbf{h}$ at the rate

$$\begin{aligned} \dot{h} &= ra_\theta = \frac{d}{dt}\sqrt{\mu a(1-e^2)} && (4.16)\\ &= \frac{1}{2}\sqrt{\frac{\mu}{a}(1-e^2)}\dot{a} + \frac{1}{2}\sqrt{\frac{\mu a}{1-e^2}}(-2e\dot{e}) && (4.17)\\ \text{so} \quad e\dot{e} &= -\sqrt{\frac{1-e^2}{\mu a}}ra_\theta + \frac{1-e^2}{2a}\frac{2}{n\sqrt{1-e^2}}\left[a_r e\sin f + a_\theta(1+e\cos f)\right] && (4.18)\\ &= \frac{\sqrt{1-e^2}}{an}\left[a_r e\sin f + a_\theta\left(1+e\cos f - \frac{r}{a}\right)\right] && (4.19) \end{aligned}$$

where $\mu = n^2a^3$ and $r = a(1 - e\cos E)$ where E = eccentric anomaly, so

$$\dot{e} = \frac{\sqrt{1-e^2}}{an}\left[a_r \sin f + a_\theta(\cos f + \cos E)\right] \tag{4.20}$$

The text also calculates

$$\frac{di}{dt} = \frac{\sqrt{1-e^2}}{an}\frac{\cos(\omega + f)}{1 + e\cos f}a_z \tag{4.21}$$

as well as $\dot{\omega}, \dot{\Omega}$, and $\dot{\tau}$.

Please confirm those derivations since we will use them later.

Application: orbit decay due to drag force

A satellite in a low–Earth orbit suffers an aerodynamic drag with to the Earth's upper atmosphere. Suppose the acceleration due to this drag has the form $\mathbf{a}_p = -\alpha n\dot{\mathbf{r}}$ where the drag coefficient $\alpha \ll 1$. Derive the time–averaged orbital evolution timescales $\tau_a \equiv |a/\dot{a}|$ and $\tau_e \equiv |e/\dot{e}|$ assuming the satellite has $e \ll 1$ and $i = 0$.

$$\text{Since} \quad \mathbf{a}_p = -\alpha n\dot{\mathbf{r}} = -\alpha n(\dot{r}\hat{\mathbf{r}} + r\dot{f}\hat{\theta}), \tag{4.22}$$

$$\Rightarrow a_r = -\alpha n\dot{r} = -\frac{\alpha n^2 ae\sin f}{\sqrt{1-e^2}} \tag{4.23}$$

$$\text{and} \quad a_\theta = -\alpha n r\dot{f} = -\frac{\alpha n^2 a(1 + e\cos f)}{\sqrt{1-e^2}} \tag{4.24}$$

$$\text{so} \quad \dot{a} = \frac{2}{n\sqrt{1-e^2}}\left[a_r e\sin f + a_\theta(1 + e\cos f)\right] \tag{4.25}$$

$$= -\frac{2\alpha na}{1-e^2}[e^2\sin^2 f + (1 + e\cos f)^2] \tag{4.26}$$

$$\simeq -2\alpha na \text{ to lowest order in small } e \tag{4.27}$$

Thus the satellite's orbit decays over a timescale

$$\tau_a = \left|\frac{a}{\dot{a}}\right| \simeq \frac{1}{2\alpha n} = \frac{T}{4\pi\alpha} \quad \text{where } T = 2\pi/n = \text{orbit period} \tag{4.28}$$

Similarly,

$$\begin{aligned}
\dot{e} &= -\alpha n\left[e\sin^2 f + (1+e\cos f)(\cos f + \cos E)\right] && (4.29)\\
\text{but}\quad \frac{r}{a} &= 1 - e\cos E = \frac{1-e^2}{1+e\cos f} && (4.30)\\
\text{so}\quad \cos E &= \frac{1}{e}\left(1 - \frac{1-e^2}{1+e\cos f}\right) && (4.31)\\
&= \frac{1}{e}\left(\frac{e\cos f + e^2}{1+e\cos f}\right) = \frac{\cos f + e}{1+e\cos f} && (4.32)\\
\text{so}\quad \dot{e} &= -\alpha n(e\sin^2 f + \cos f + e\cos^2 f + \cos f + e) && (4.33)\\
&= -\alpha n(2e + 2\cos f) && (4.34)
\end{aligned}$$

which varies with m_2's true anomaly f which is a function of time t.

However if the orbit decay is slow, ie, $\alpha \ll 1$, the instantaneous $\dot{e}$ is not interesting. Rather, we are interested in the long–term (ie, time–averaged) variation in $e(t)$:

$$< \dot{e} > \;\simeq\; -2\alpha n < e + \cos f(t) > \qquad (4.35)$$

where the <> is time–averaged over an orbital period.

Recall the elliptic expansion for f, Eqn. (1.151):

$$\begin{aligned}
f &\simeq M + 2e\sin M + \mathcal{O}(e^2) \text{ where } M = nt = \text{mean anomaly} && (4.36)\\
\text{so}\quad \cos f &\simeq \cos(M + 2e\sin M) && (4.37)\\
&= \cos M\cos(2e\sin M) - \sin M\sin(2e\sin M) && (4.38)\\
&\simeq \cos M - 2e\sin^2 M + \mathcal{O}(e^2) && (4.39)\\
\text{and}\quad < \dot{e} > &\simeq -2\alpha n < e + \cos M - 2e\sin^2 M + \mathcal{O}(e^2) > && (4.40)
\end{aligned}$$

Now do the time–averaging:

$$
\begin{aligned}
< \cos M > &= \frac{1}{T}\int_0^T \cos(nt)dt = 0 && (4.41)\\
< \sin^2 M > &= \frac{1}{T}\int_0^T \sin^2(nt)dt = \frac{1}{T}\int_0^T \frac{1}{2}[1-\cos(2nt)]dt = \frac{1}{2} && (4.42)\\
\Rightarrow < \dot{e} > &\simeq -2\alpha e n\left[e + 0 - e + \mathcal{O}(e^2)\right] && (4.43)\\
&= 0 \text{ to } \mathcal{O}(e^2) && (4.44)
\end{aligned}
$$

Actually, if you use use an exact expression for $\cos f$, you still get $< \dot{e} >= 0$. Nonetheless, this example shows you how to obtain time–averaged variations in the orbit elements, which you will use in the following assignment...

The Lagrange planetary equations

We have been considering a 2–body system composed of masses m_1, m_2, and a perturbing acceleration $\mathbf{a}_p$ that acts on m_2 and causes its orbit to deviate from a Keplerian ellipse. Newton's law of motion for m_2 is

$$\ddot{\mathbf{r}} = -\frac{G(m_1+m_2)}{r^3}\hat{\mathbf{r}} + \mathbf{a}_p = -\nabla\left(\frac{G(m_1+m_2)}{r} + \Phi_p\right) \qquad (4.45)$$

where Φ_p is the potential that generates the perturbation $\mathbf{a}_p$.

If we wish to describe m_2's motion using its osculating orbit elements $(a, e, i, ...)$, we can use Gauss' eqn's for $\dot{a}$, $\dot{e}$, etc, to track m_2's evolution.

However it will be desirable to rewrite this potential function $\Phi_p(\mathbf{r})$ in terms of the orbit elements $(a, e, i, ...)$, ie,

$$\Phi_p(\mathbf{r}) = -R(a, e, i, \Omega, \tilde{\omega}, \tau) \quad \text{where } R = \textit{disturbing function}, \qquad (4.46)$$

and then derive an alternate set of equations $\dot{a}$, $\dot{e}$, etc, that depend upon partial derivatives of R, analogous to Newton's Law.

These new equations are the Lagrange planetary equations,
and they will be derived from Gauss' planetary eqn's:

$$\dot{a} = \frac{2}{n\sqrt{1-e^2}}\left[a_r e \sin f + a_\theta(1 + e\cos f)\right] \tag{4.47}$$

$$\dot{e} = \frac{\sqrt{1-e^2}}{an}\left[a_r \sin f + a_\theta(\cos f + e\cos E)\right] \tag{4.48}$$

$$\frac{di}{dt} = \frac{\sqrt{1-e^2}}{an}\frac{\cos(\omega + f)}{1 + e\cos f}a_z \tag{4.49}$$

$$\dot{\Omega} = \frac{\sqrt{1-e^2}}{an\sin i}\frac{\sin(\omega + f)}{(1 + e\cos f)}a_z \tag{4.50}$$

$$\dot{\tilde{\omega}} = \frac{\sqrt{1-e^2}}{ane}\left[-a_r\cos f + a_\theta\left(\frac{2 + e\cos f}{1 + e\cos f}\right)\sin f + 2\sin^2\left(\frac{i}{2}\right)\right] \tag{4.51}$$

Although only $\dot{a}$ and $\dot{e}$ are derived in class,
the remaining equations are derived in Section 2.9 of the text.

The text also shows that Gauss' eqn' for $\dot{\tau}$ yields terms that are $\dot{\tau} \propto t a_r$ which become problematic at large times t. WHY?

However this is not really a problem since you can always reset your clock by some integral number of periods T, ie, $t \rightarrow t-$ integer $\times T$ without affecting the dynamics.

The text also shows that you can sidestep this difficulty altogether by inventing a new orbit element $\epsilon = \tilde{\omega} - n\tau =$ *mean longitude at epoch.*

The combination $\lambda = M + \tilde{\omega} = n(t-\tau) + \tilde{\omega} = nt + \epsilon$ is known as the *mean longitude* since $\lambda \simeq f + \tilde{\omega} =$ *true longitude* when $e \simeq 0$ and $M \simeq f$ (see eqn' 1.151).
Hence when $t = 0$, $\lambda = \epsilon =$ longitude at epoch.

The resulting Gauss' eqn' for $\dot{\epsilon}$ is

$$\dot{\epsilon} = -\frac{2(1-e^2)}{an(1 + e\cos f)}a_r + (1 - \sqrt{1-e^2})\dot{\tilde{\omega}} + 2\sqrt{1-e^2}\sin^2\left(\frac{i}{2}\right)\dot{\Omega} \tag{4.52}$$

Start with

$$\mathbf{a}_p \;=\; a_r\hat{\mathbf{r}} + a_\theta\hat{\theta} + a_z\hat{\mathbf{z}} = \nabla R \tag{4.53}$$

where again, $\hat{\mathbf{r}}$ and $\hat{\theta}$ lie in m_2's orbit plane and $\hat{\mathbf{z}}$ is perpendicular.

Suppose $\mathbf{a}_p$ displaces m_2 a small distance $d\mathbf{r} = dr\hat{\mathbf{r}} + rd\theta\hat{\theta} + dz\hat{\mathbf{z}}$.

The differential work dW done on m_2 by $\mathbf{a}_p$ is

$$dW \;=\; m_2\mathbf{a}_p \cdot d\mathbf{r} \tag{4.54}$$

$$\text{so}\quad \frac{dW}{m_2} \;=\; a_r dr + a_\theta r d\theta + a_z dz = \nabla R \cdot d\mathbf{r} \tag{4.55}$$

where $R = R(a, e, i, \Omega, \tilde{\omega}, \epsilon)$

Deriving the Lagrange planetary eqn's requires doing the following tasks:

1. Write $\nabla R \cdot d\mathbf{r}$ in terms of $\frac{\partial R}{\partial a}da$, $\frac{\partial R}{\partial e}de$, etc.

2. write dr, $d\theta$, dz in terms of da, de, di, etc.

3. Insert these results into eqn' (4.55) to relate $a_r, a_\theta, a_z \leftrightarrow \frac{\partial R}{\partial a}da, \frac{\partial R}{\partial e}de$, etc.

4. Insert those results into Gauss' eqn's to obtain Lagrange eqn's
that relate $\dot{a}, \dot{e}$, etc, to $\frac{\partial R}{\partial a}da, \frac{\partial R}{\partial e}de$, etc.

task 1.

Relate $\nabla R \cdot d\mathbf{r}$ to the $\frac{\partial R}{\partial a}da$, etc:

$$\begin{aligned}
\nabla R &= \sum_{i=1}^{3} \frac{\partial R}{\partial x_i}\hat{\mathbf{x}}_i && (4.56)\\
\text{which is shorthand for } &= \frac{\partial R}{\partial x}\hat{\mathbf{x}} + \frac{\partial R}{\partial y}\hat{\mathbf{y}} + \frac{\partial R}{\partial z}\hat{\mathbf{z}} && (4.57)
\end{aligned}$$

had I chosen to use Cartesian coordinates $x_i = x, y$, or z
(but I could have used spherical coords, cylindrical, etc).

Since $R(a, e, i, \Omega, \tilde{\omega}, \tau) = -\Phi_p(x, y, z) \Rightarrow a = a(x_i), e = e(x_i)$, etc, so

$$\begin{aligned}
\frac{\partial R}{\partial x_i} &= \frac{\partial R}{\partial a}\frac{\partial a}{\partial x_i} + \frac{\partial R}{\partial e}\frac{\partial e}{\partial x_i} + \cdots && (4.58)\\
&= \sum_{j=1}^{6} \frac{\partial R}{\partial c_j}\frac{\partial c_j}{\partial x_i} && (4.59)
\end{aligned}$$

where c_j is shorthand for the j^{th} orbit element $(a, e, i, ...)$, and

$$\nabla R = \sum_{i=1}^{3}\sum_{j=1}^{6} \frac{\partial R}{\partial c_j}\frac{\partial c_j}{\partial x_i}\hat{\mathbf{x}}_i \qquad (4.60)$$

And since

$$\begin{aligned}
d\mathbf{r} &= dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}} = \sum_{k=1}^{3} dx_k\hat{\mathbf{x}}_k, && (4.61)\\
\hat{\mathbf{x}}_i \cdot d\mathbf{r} &= dx_i && (4.62)\\
\text{so} \quad \nabla R \cdot d\mathbf{r} &= \sum_{i=1}^{3}\sum_{j=1}^{6} \frac{\partial R}{\partial c_j}\frac{\partial c_j}{\partial x_i}dx_i && (4.63)\\
&= \sum_{j=1}^{6} \frac{\partial R}{\partial c_j}\left[\sum_{i=1}^{3} \frac{\partial c_j}{\partial x_i}dx_i\right] && (4.64)
\end{aligned}$$

Since $c_j = c_j(x,y,z)$, the quantity in the [] is just dc_j:

$$dc_j = \frac{\partial c_j}{\partial x}dx + \frac{\partial c_j}{\partial y}dy + \frac{\partial c_j}{\partial z}dz = \sum_{i=1}^{3} \frac{\partial c_j}{\partial x_i}dx_i \tag{4.65}$$

$$\text{so} \quad d\mathbf{r} = \sum_{j=1}^{6} \frac{\partial R}{\partial c_j}dc_j \tag{4.66}$$

$$\text{and} \quad \nabla R \cdot d\mathbf{r} = \frac{\partial R}{\partial a}da + \frac{\partial R}{\partial e}de + \cdots = \mathbf{a}_p \cdot d\mathbf{r} \tag{4.67}$$

$$\Rightarrow a_r dr + a_\theta d\theta + a_z dz = \frac{\partial R}{\partial a}da + \frac{\partial R}{\partial e}de + \frac{\partial R}{\partial i}di + \frac{\partial R}{\partial \Omega}d\Omega \tag{4.68}$$

$$+ \frac{\partial R}{\partial \omega}d\omega + \frac{\partial R}{\partial \epsilon}d\epsilon \tag{4.69}$$

task 2.

Now write displacements $dr, d\theta, dz$ in terms of the da, de, di, etc.

To do this, use the following 2-body formulae 1.97, 1.98, 1.120, 1.122, 1.124, & 1.125 to show that the radial displacement dr can be written

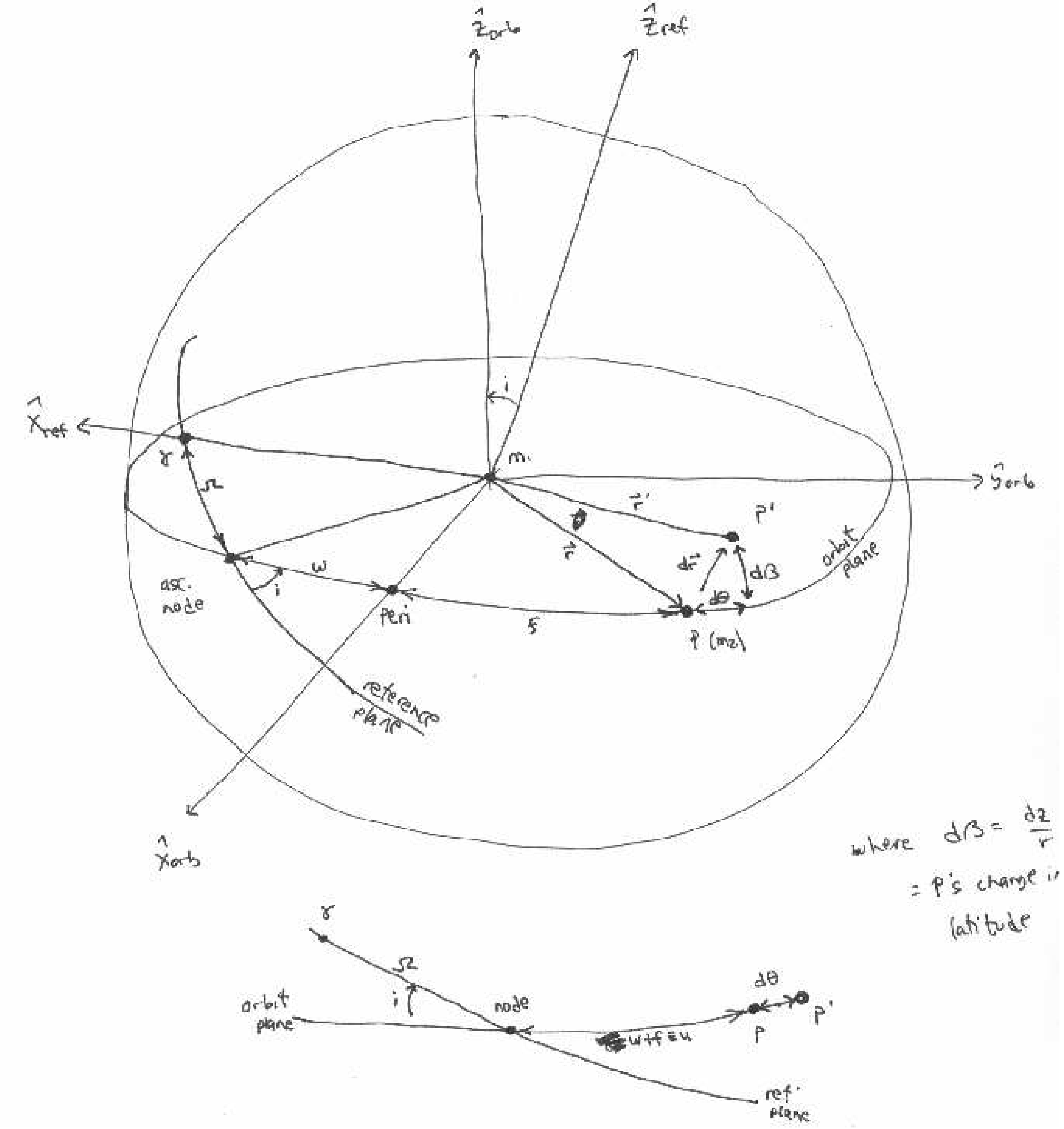
$$dr = \frac{r}{a}da - a\cos f de + \frac{ae\sin f}{\sqrt{1-e^2}}dM \tag{4.70}$$

(which you should confirm for yourself)
where $M = nt + \epsilon - \tilde{\omega}$ so $dM = d\epsilon - d\tilde{\omega}$
(remember we are considering orbital displacements $d\tilde{\omega}$, $d\epsilon$, etc,
that occur at constant t),
so

$$dr = \frac{r}{a}da - a\cos f de + \frac{ae\sin f}{\sqrt{1-e^2}}(d\epsilon - d\tilde{\omega}) \tag{4.71}$$

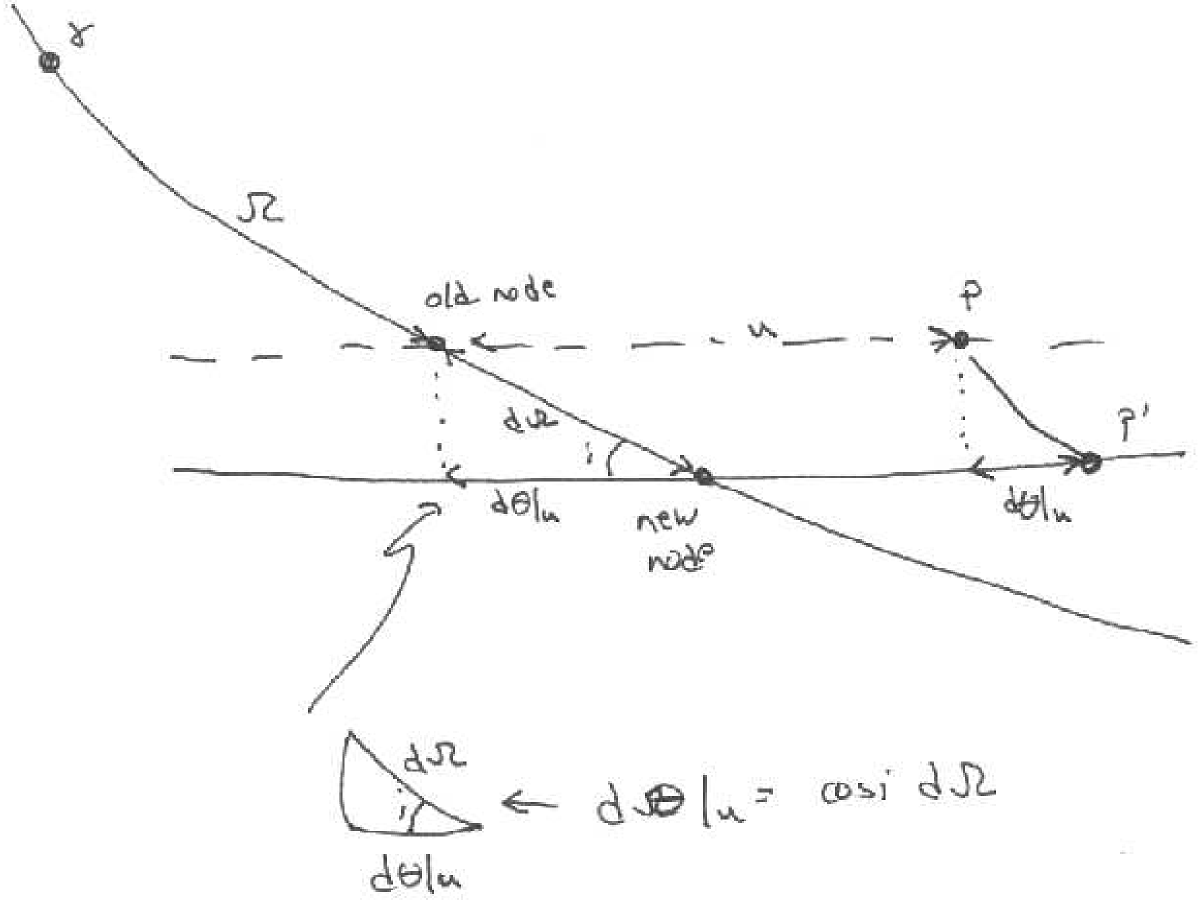
Next project m_2's position vector $\mathbf{r}$ onto the unit celestial sphere at spot P:

Recall that the perturbation $\mathbf{a}_p$ has displaced $\mathbf{r} \rightarrow \mathbf{r} + d\mathbf{r}$, which moves m_2 from $P \rightarrow P$:



Now get m_2's azimuthal displacement $d\theta$ in terms of $d\Omega$ $du = d\omega + df$

First calculate $d\theta|_u$, which due to a change in Ω by amount $d\Omega$, while $u = \omega + f$ is kept constant.



The figures shows that $d\theta|_u = \cos i d\Omega$.

If we instead let $u = \omega + f$ vary by amount du with Ω kept constant, then

$$\begin{aligned} d\theta|_\Omega &= d\omega + df && (4.72) \\ \text{and the total} \quad d\theta &= d\theta|_u + d\theta|_\Omega = \cos i d\Omega + d\omega + df && (4.73) \\ &= (\cos i - 1)d\Omega + d\tilde{\omega} + df && (4.74) \end{aligned}$$

since $\tilde{\omega} = \omega + \Omega$.

To finish this off, we need to write df in terms of $d\tilde{\omega}$, etc.

Now take the logarithm of eqn' (1.123) and differentiate it to show that

$$\frac{df}{\sin f} = \frac{de}{1-e^2} + \frac{dE}{\sin E} = \left(\frac{1}{1-e^2} + \frac{a}{r}\right) de + \frac{a dM}{r \sin E} \tag{4.75}$$

Since $M = nt + \epsilon - \tilde{\omega}$, $dM = d\epsilon - d\tilde{\omega}$ (t is kept constant in this development), and with $\sin f = \frac{a}{r}\sqrt{1-e^2}\sin E$ (which you used in problem 5),

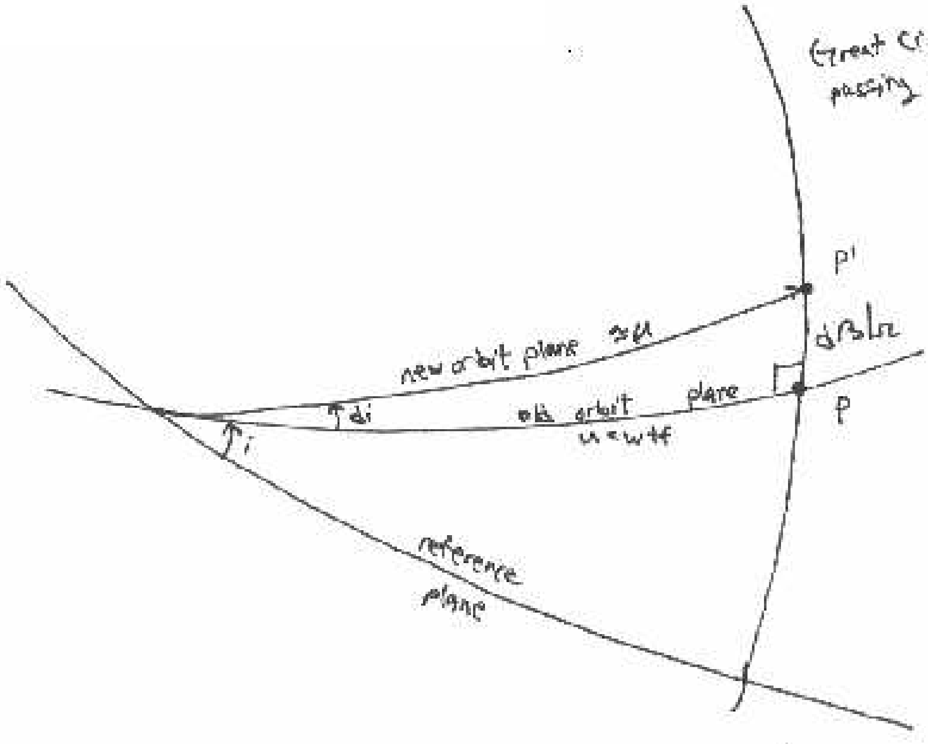
$$df = \sin f \left(\frac{1}{1-e^2} + \frac{a}{r}\right) de + \left(\frac{a}{r}\right)^2 \sqrt{1-e^2}(d\epsilon - d\tilde{\omega}) \tag{4.76}$$

$$\text{so} \quad d\theta = -(1-\cos i)d\Omega + \left[1 - \left(\frac{a}{r}\right)^2 \sqrt{1-e^2}\right] d\tilde{\omega} \tag{4.77}$$

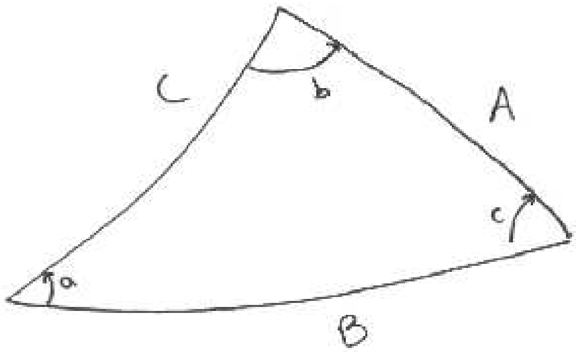
$$+\sin f \left(\frac{2 + e\cos f}{1-e^2}\right) de + \left(\frac{a}{r}\right)^2 \sqrt{1-e^2} d\epsilon \tag{4.78}$$

Lastly, get latitude change $d\beta$ in terms of the changes $d\Omega$ and di.

From the diagram, $d\beta = d\beta|_\Omega + d\beta|_i$



Further progress requires the use of *spherical trigonometry*, which has two important formulae:



spherical triangle formed by 3 arcs on the unit celestial sphere

The *fundamental formula of spherical trigonometry*:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \tag{4.79}$$

and the *sine–formula*:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \tag{4.80}$$

All of these are derived in W. Smart's text *Spherical Astronomy.*

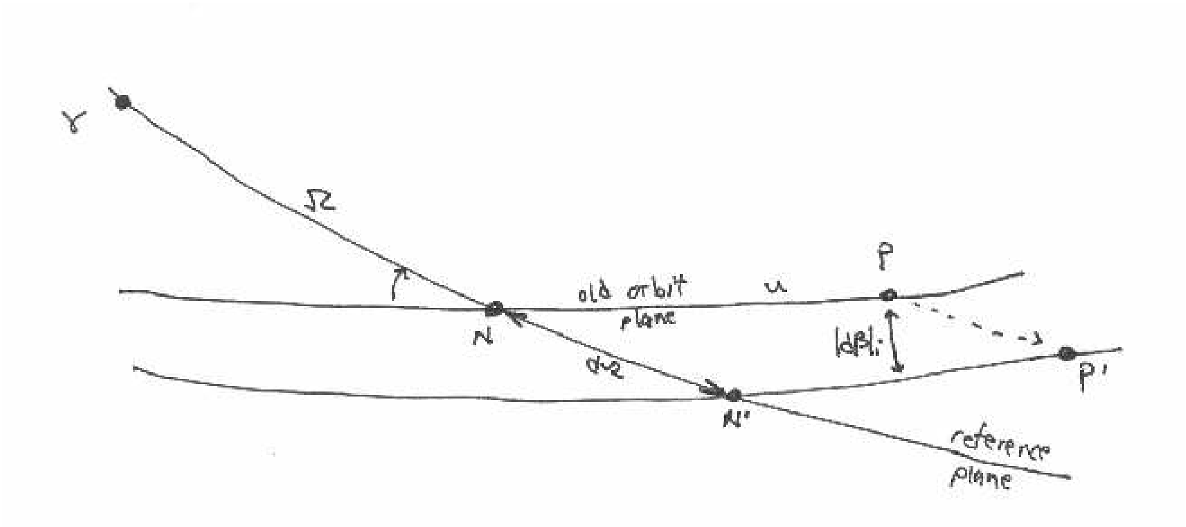
The sine–formula then says that

$$\begin{aligned} \frac{\sin \pi/2}{\sin u} &= \frac{\sin d\beta|_\Omega}{\sin di} && (4.81) \\ \Rightarrow d\beta|_\Omega &= \sin u di \quad \text{since } d\beta \text{ and } d\Omega \text{ are small} && (4.82) \end{aligned}$$

Also note that the fundamental formula recovers the *law of cosines* when the angles $|a, b, c| \ll 1$:

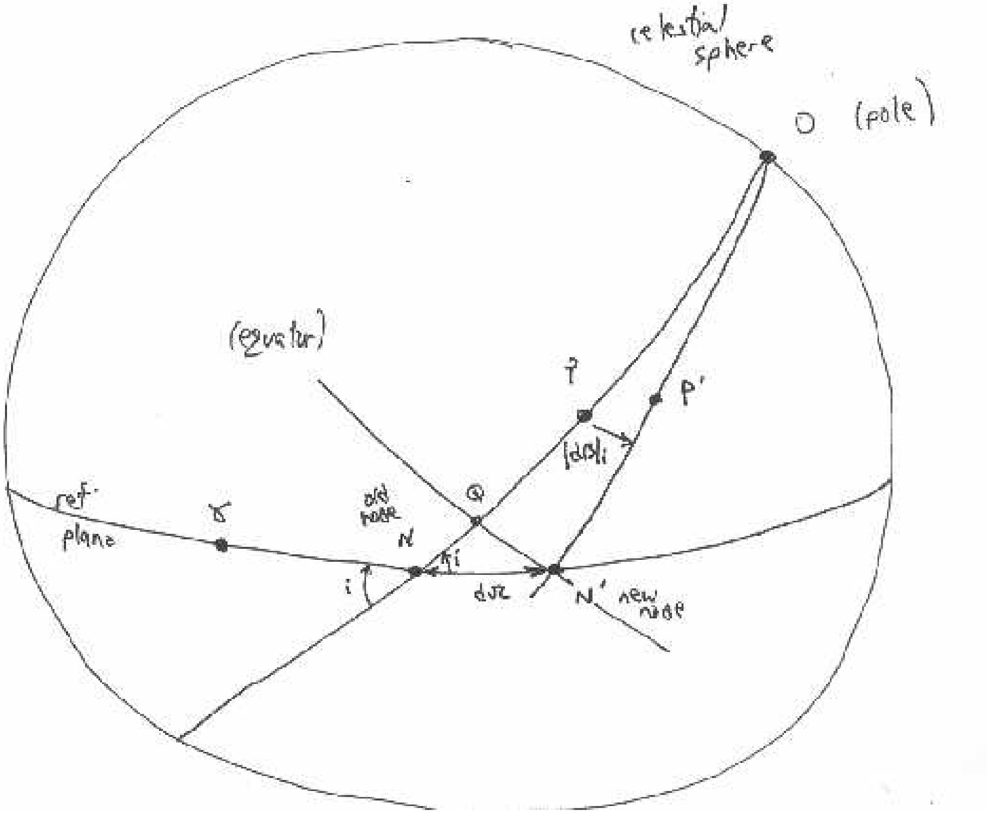
$$\begin{aligned} 1 - \frac{1}{2}a^2 &\simeq \left(1 - \frac{1}{2}b^2\right)\left(1 - \frac{1}{2}c^2\right) + bc \cos A && (4.83) \\ \text{so} \quad a^2 &\simeq b^2 + c^2 - 2bc \cos A && (4.84) \end{aligned}$$

Now get $d\beta|_i$ that is due to the change $d\Omega$:

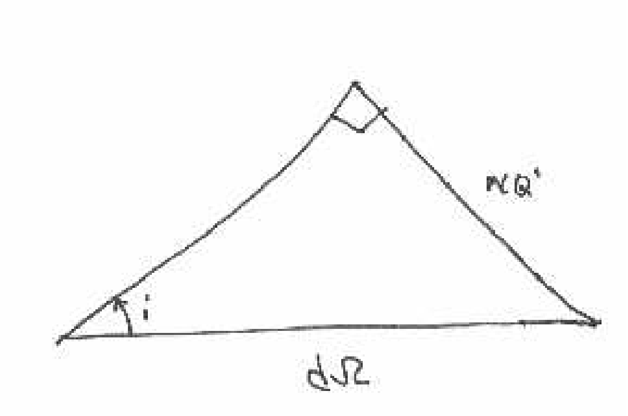


Note that $d\beta|_i < 0$ when $d\Omega > 0$

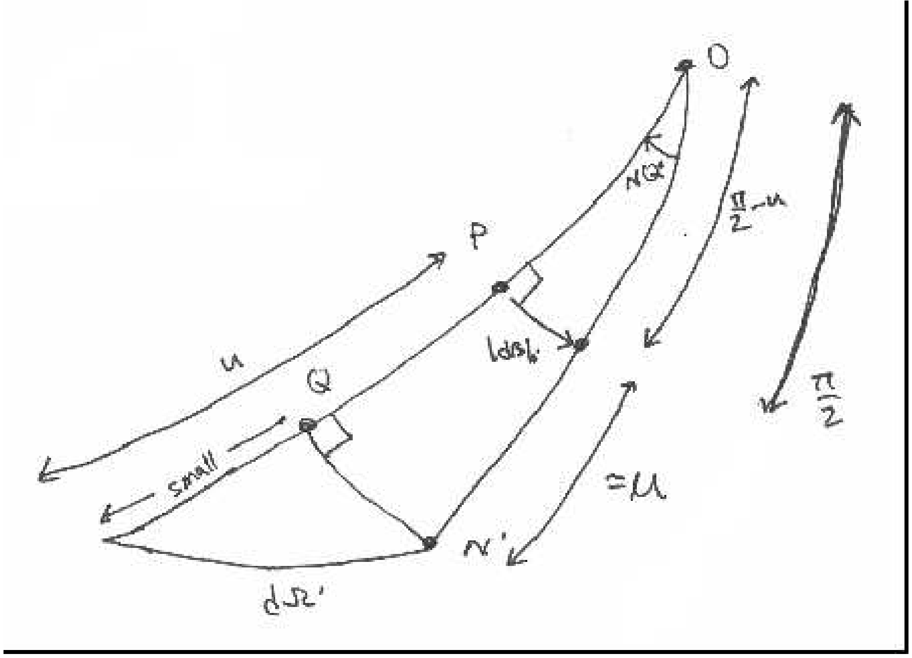
Displacement $d\Omega$ causes $P \rightarrow P'$ and node $N \rightarrow N'$.
Let arcs NP and $N'P'$ lie on two great circles that meet at point O (which can be regarded as a pole).



If O is a pole, the arc $N'Q$ lies on another great circle that is perpendicular to the pole direction O thus arc $N'Q$ lies along an equatorial great circle and arc $N'O = \pi/2$



The diagram above indicates that $NQ' = \sin i d\Omega$



And that

$$\frac{\sin(\pi/2-u)}{\sin\pi/2} = \frac{\sin|d\beta|_i}{\sin N'Q} = \frac{|d\beta|_i}{N'Q} \tag{4.85}$$

$$\Rightarrow |d\beta|_i = \cos u \sin i d\Omega \tag{4.86}$$

$$\text{so} \quad d\beta|_i = -\cos u \sin i d\Omega \tag{4.87}$$

where the sign is a consequence of the fact that $d\Omega > 0$ must yield $d\beta|_i < 0$

Thus $d\beta \;=\; d\beta|_\Omega + d\beta|_i = \sin(\omega+f)di - \cos(\omega+f)\sin i d\Omega = \frac{dz}{r}$ (4.88)

while dr and $d\theta$ are given by Eqn's (4.71) and (4.74).
This completes task 2...

task 3.

Recall that the differential work that the perturbation $\mathbf{a}_p$ does on m_2 is

$$
\begin{aligned}
\frac{dW}{m_2} &= a_r dr + a_\theta r d\theta + a_z dz \\
&= a_r\left[\frac{r}{a}da - a\cos f de + \frac{ae\sin f}{\sqrt{1-e^2}}(d\epsilon - d\tilde{\omega})\right] \\
&\quad + a_\theta r\left\{-(1-\cos i)d\Omega + \left[1-\left(\frac{a}{r}\right)^2\sqrt{1-e^2}\right]d\tilde{\omega}\right. \\
&\quad \left. + \sin f\left(\frac{2+e\cos f}{1-e^2}\right)de + \left(\frac{a}{r}\right)^2\sqrt{1-e^2}d\epsilon\right\} \\
&\quad + a_z r\left[\sin(\omega+f)di - \cos(\omega+f)\sin i d\Omega\right] \\
&= \frac{\partial R}{\partial a}da + \frac{\partial R}{\partial e}de + \frac{\partial R}{\partial i}di + \frac{\partial R}{\partial \Omega}d\Omega + \frac{\partial R}{\partial \tilde{\omega}}d\tilde{\omega} + \frac{\partial R}{\partial \epsilon}d\epsilon
\end{aligned}
$$

Since these equations hold for any arbitrary displacements da, de, etc, their coefficients to da, de, etc, must be equal, which implies

$$\frac{\partial R}{\partial a} = a_r\left(\frac{r}{a}\right) \tag{4.89}$$

$$\frac{\partial R}{\partial e} = -a_r a\cos f + a_\theta r\left(\frac{2+e\cos f}{1-e^2}\right)\sin f \tag{4.90}$$

$$\frac{\partial R}{\partial i} = a_z r\sin(\omega+f) \tag{4.91}$$

$$\frac{\partial R}{\partial \Omega} = -a_\theta r(1-\cos i) - a_z r\cos(\omega+f)\sin i \tag{4.92}$$

$$= -2a_\theta r\sin^2\left(\frac{i}{2}\right) - a_z r\cos(\omega+f)\sin i \tag{4.93}$$

$$\frac{\partial R}{\partial \tilde{\omega}} = -a_r\frac{ae\sin f}{\sqrt{1-e^2}} + a_\theta r\left[1-\left(\frac{a}{r}\right)^2\sqrt{1-e^2}\right] \tag{4.94}$$

$$\frac{\partial R}{\partial \epsilon} = a_r\frac{ae\sin f}{\sqrt{1-e^2}} + a_\theta\frac{a^2}{r}\sqrt{1-e^2} \tag{4.95}$$

task 4.

You can then use eqn's 4.89–4.95 to replace the a_r, a_θ, a_z in Gauss' eqn's to obtain the *Lagrange planetary eqn's:*

$$
\begin{aligned}
\dot{a} &= \frac{2}{an}\frac{\partial R}{\partial \epsilon} && (4.96)\\
\dot{e} &= -\frac{\sqrt{1-e^2}}{a^2ne}(1-\sqrt{1-e^2})\frac{\partial R}{\partial \epsilon} - \frac{\sqrt{1-e^2}}{a^2ne}\frac{\partial R}{\partial \tilde{\omega}} && (4.97)\\
\frac{di}{dt} &= -\frac{\tan\frac{i}{2}}{a^2n\sqrt{1-e^2}}\left(\frac{\partial R}{\partial \epsilon} + \frac{\partial R}{\partial \tilde{\omega}}\right) - \frac{1}{a^2n\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \Omega} && (4.98)\\
\dot{\Omega} &= \frac{1}{a^2n\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} && (4.99)\\
\dot{\tilde{\omega}} &= \frac{\sqrt{1-e^2}}{a^2ne}\frac{\partial R}{\partial e} + \frac{\tan\frac{i}{2}}{a^2n\sqrt{1-e^2}}\frac{\partial R}{\partial i} && (4.100)\\
\dot{\epsilon} &= -\frac{2}{an}\frac{\partial R}{\partial a} + \frac{\sqrt{1-e^2}}{a^2ne}(1-\sqrt{1-e^2})\frac{\partial R}{\partial e} + \frac{\tan\frac{i}{2}}{a^2n\sqrt{1-e^2}}\frac{\partial R}{\partial i} && (4.101)
\end{aligned}
$$

It is recommended that you confirm that these Lagrange planetary eqn's do indeed follow from eqn's 4.89–4.95 and Gauss' eqn's.

This completes the derivation of the Lagrange planetary eqn's...phew!

However a much simpler set of equations results when the particle has a small $e \ll 1$ and $i \ll 1$:

$$
\begin{aligned}
\dot{a} &= \frac{2}{an}\frac{\partial R}{\partial \epsilon} && (4.102)\\
\dot{e} &\simeq -\frac{\partial R/\partial \tilde{\omega}}{a^2ne} && (4.103)\\
\frac{di}{dt} &\simeq -\frac{\partial R/\partial \Omega}{a^2ni} && (4.104)\\
\dot{\Omega} &\simeq \frac{\partial R/\partial i}{a^2ni} && (4.105)\\
\dot{\tilde{\omega}} &\simeq \frac{\partial R/\partial e}{a^2ne} && (4.106)\\
\dot{\epsilon} &\simeq -\frac{2}{an}\frac{\partial R}{\partial a} && (4.107)
\end{aligned}
$$

The disturbing function R

Use of the Lagrange planetary eqn's requires writing the disturbing function R.

Lets consider the simplest case of 3 bodies:
mass m_c = mass of central body at $\mathbf{R_c}$,
mass m at $\mathbf{R}$,
and mass m' at $\mathbf{R}'$:

where the relative coordinates are

$$\begin{aligned} \mathbf{r} &= \mathbf{R} - \mathbf{R_c} && (4.108) \\ \text{and} \quad \mathbf{r}' &= \mathbf{R}' - \mathbf{R_c} && (4.109) \end{aligned}$$

The EOM for these bodies are:

$$\ddot{\mathbf{R}} = -\frac{Gm_c}{r^3}\mathbf{r} + \frac{Gm'}{|\mathbf{r}' - \mathbf{r}|^3}(\mathbf{r}' - \mathbf{r}) \tag{4.110}$$

$$\text{and} \quad \ddot{\mathbf{R}}_{\mathbf{c}} = \frac{Gm}{r^3}\mathbf{r} + \frac{Gm'}{\mathbf{r}'^3}\mathbf{r}' \tag{4.111}$$

$$\text{so} \quad \ddot{\mathbf{r}} = \ddot{\mathbf{R}} - \ddot{\mathbf{R}}_{\mathbf{c}} \tag{4.112}$$

$$= -\frac{G(m_c + m)}{r^3}\mathbf{r} + Gm'\left(\frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} - \frac{\mathbf{r}'}{\mathbf{r}'^3}\right) \tag{4.113}$$

$$= \nabla(U + R) \tag{4.114}$$

$$\text{where} \quad U \equiv -\frac{G(m_c + m)}{r} = -1\times \text{ usual 2–body potential} \tag{4.115}$$

$$\text{and} \quad R \equiv Gm'\left(\frac{1}{|\mathbf{r}' - \mathbf{r}|} - \frac{\mathbf{r}\cdot\mathbf{r}'}{\mathbf{r}'^3}\right) \tag{4.116}$$

is the disturbing function that m feels due to perturber m'.

However we need to confirm that ∇R does indeed recover the perturbing acceleration, ie, the right term in eqn' (4.113).

In cartesian coordinates,

$$\nabla R = Gm'\left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial x}\hat{\mathbf{y}} + \frac{\partial}{\partial x}\hat{\mathbf{z}}\right)\times \tag{4.117}$$

$$\left[\frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} - \frac{xx' + yy' + zz'}{r'^3}\right] \tag{4.118}$$

$$= Gm'\left\{-\frac{2(x - x')\hat{\mathbf{x}} + 2(y - y')\hat{\mathbf{y}} + 2(z - z')\hat{\mathbf{z}}}{2[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} - \frac{x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}}}{r'^3}\right\} \tag{4.119}$$

$$= Gm'\left(\frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} - \frac{\mathbf{r}'}{\mathbf{r}'^3}\right) \tag{4.120}$$

check.

But what if there are multiple perturbing masses m'?

In this case, let $m \rightarrow m_i$ and $\mathbf{r} \rightarrow \mathbf{r_i}$,
while $m' \rightarrow m_j$ and $\mathbf{r}' \rightarrow \mathbf{r_j}$,
and sum of the contributions from all j perturbers:

$$R \rightarrow R_i \ = \ \sum_{j \neq i} G m_j \left(\frac{1}{|\mathbf{r_j} - \mathbf{r_i}|} - \frac{\mathbf{r_i} \cdot \mathbf{r_j}}{\mathbf{r_j}^3} \right) \tag{4.121}$$

which is the disturbing function for particle i due to all other perturbers j.

The first term in the parentheses is the *direct term*, while the second is the *indirect* term that is a consequence of the moving origin that rides on m_c.

Had we placed the origin on the center–of-mass, there would be no indirect term.

In the remainder of this discussion we will be content to consider the simple problem of a single low–mass particle that is being perturbed by a planet of mass m' in a fixed circular orbit—the R3BP again!

$$R \ \equiv \ G m' \left(\frac{1}{|\mathbf{r}' - \mathbf{r}|} - \frac{\mathbf{r} \cdot \mathbf{r}'}{\mathbf{r}'^3} \right) \tag{4.122}$$

In particular we will concentrate on the motion of a particle orbiting near a planet's *mean–motion* resonances.

The best way to examine the effects of a resonance is to *Fourier expand* the particle's disturbing function R.

Fourier expansion of R

$$
\begin{aligned}
\text{Let} \quad \Delta &= |\mathbf{r}' - \mathbf{r}| = \sqrt{r'^2 + r^2 - 2rr'\cos\psi} && (4.123)\\
&= r'(1+\beta^2 - 2\beta\cos\psi)^{1/2} \equiv r'\Gamma && (4.124)\\
\text{where} \quad \beta &\equiv \frac{r}{r'} \qquad \text{and } \Gamma(\beta,\psi) = (1+\beta^2-2\beta\cos\psi)^{1/2} && (4.125)\\
\text{Also} \quad \frac{\mathbf{r}\cdot\mathbf{r}'}{\mathbf{r}'^3} &= \frac{rr'\cos\psi}{r'^3} = \frac{\beta\cos\psi}{r'} && (4.126)\\
\text{so} \quad R &= \frac{Gm'}{r'}\left(\frac{1}{\Gamma} - \beta\cos\psi\right) && (4.127)
\end{aligned}
$$

which is periodic in ψ.

Fourier expand $1/\Gamma(\beta,\psi)$:

$$
\begin{aligned}
\frac{1}{\Gamma} &= \frac{1}{(1+\beta^2-2\beta\cos\psi)^{1/2}} && (4.128)\\
&= \frac{1}{2}b_{1/2}^{(0)}(\beta) + \sum_{m=1}^{\infty} b_{1/2}^{(m)}(\beta)\cos(m\psi) && (4.129)
\end{aligned}
$$

where the $b_{1/2}^{(m)}(\beta)$ coefficients
are the amplitudes of each term in the Fourier series.

Note that Γ is even in ψ, so there are no odd $\sin(m\psi)$ terms in this series.

To obtain the coefficients $b_{1/2}^{(m)}$,
multiply the above by $\cos(m'\psi)$ and integrate over all ψ:

$$\int_{-\pi}^{\pi} \frac{\cos(m'\psi)d\psi}{(1+\beta^2-2\beta\cos\psi)^{1/2}} = \int_{-\pi}^{\pi} \cos(m'\psi) \times \tag{4.130}$$

$$\left[\frac{1}{2}b_{1/2}^{(0)} + \sum_{m=1}^{\infty} b_{1/2}^{(m)} \cos(m\psi)\right] d\psi \tag{4.131}$$

Note that the contribution from the $m \neq m'$ terms is zero,
while the $m = m'$ term integrates to $\pi b_{1/2}^{(m)}$, so the Fourier coefficient is

$$\Rightarrow b_{1/2}^{(m)}(\beta) = \frac{2}{\pi}\int_{0}^{\pi} \frac{\cos(m'\psi)d\psi}{(1+\beta^2-2\beta\cos\psi)^{1/2}} \tag{4.132}$$

However we will soon need the slightly more general *Laplace coefficients*

$$b_{s}^{(m)}(\beta) = \frac{2}{\pi}\int_{0}^{\pi} \frac{\cos(m'\psi)d\psi}{(1+\beta^2-2\beta\cos\psi)^{s}} \tag{4.133}$$

The Fourier expansion for R then becomes

$$R = \frac{Gm'}{r'}\left(\frac{1}{2}b_{1/2}^{(0)} + \sum_{m=1}^{\infty} b_{1/2}^{(m)} \cos(m\psi) - \beta\cos\psi\right) \tag{4.134}$$

$$= \frac{Gm'}{r'}\left[\frac{1}{2}b_{1/2}^{(0)} + \sum_{m=1}^{\infty} \hat{b}_{1/2}^{(m)}(\beta) \cos(m\psi)\right] \tag{4.135}$$

$$\text{where} \quad \hat{b}_{1/2}^{(m)}(\beta) \equiv b_{1/2}^{(m)}(\beta) - \beta\delta_{m1} \tag{4.136}$$

$$\text{or} \quad R = \frac{Gm'}{r'}\sum_{m=0}^{\infty} \tilde{b}_{1/2}^{(m)}(\beta) \cos(m\psi) \tag{4.137}$$

$$\text{where} \quad \tilde{b}_{1/2}^{(m)}(\beta) \equiv f_m \hat{b}_{1/2}^{(m)}(\beta) \quad \text{and} \quad f_m \equiv \begin{cases} 1/2 & m = 0 \\ 1 & m \neq 0 \end{cases} \tag{4.138}$$

Recall that the disturbing function is to be written in terms of the small particle's orbit elements $(a, e, i, ...)$
as well as that of the perturber's orbit elements $(a', e', i', ...)$.

That then is the last step in this analysis of R...

R in terms of orbit elements

Consider the simple case of a perturbing planet of mass m' in a circular orbit of radius $r' = a'$ since $e' = 0$.

The planet also has a longitude $\phi' = \tilde{\omega}' + f' \equiv \lambda'$,
while the particle has a longitude $\phi = \tilde{\omega} + f$

We will also assume the system is coplanar, $i = 0 = i'$
(nonzero inclinations will be considered later in the homework...).

We will also assume that the particle is in a low–eccentricity orbit, $e \ll 1$.

The elliptic expansions of the particle's orbit, eqn's (1.142–1.151) are

$$\begin{aligned} \frac{r}{a} &\simeq 1 - e\cos M + \mathcal{O}(e^2) && (4.139)\\ \text{and} \quad f &\simeq M + 2e\sin M + \mathcal{O}(e^2) && (4.140)\\ \text{so} \quad \phi &= \tilde{\omega} + f \simeq \tilde{\omega} + M + 2e\sin M = \lambda + 2e\sin M && (4.141) \end{aligned}$$

where $M = n(t - \tau)$ = mean anomaly,
and $\lambda = \tilde{\omega} + M = nt + \epsilon$ = the particle's mean longitude,
with $\phi' = \tilde{\omega}' + M' = \lambda'$ being the planet's longitude.

Note that we are only preserving terms to an accuracy of $\mathcal{O}(e)$, while the derivation in Chapter 6 is much more rigorous, and keeps higher–order terms in its series expansions.

Now we can insert these expressions for r/a and ϕ into the $b_s^{(m)}(\beta)$ and $\cos\psi$ that appear in the disturbing function R, and Taylor expand to order $\mathcal{O}(e)$. Taylor expand $\tilde{b}_s^{(m)}(\beta)$:

$$\beta = \frac{r}{r'} \simeq \frac{a(1-e\cos M)}{a'} \equiv \alpha(1-e\cos M) \quad \text{where } \alpha \equiv \frac{a}{a'} \tag{4.142}$$

$$\text{thus} \quad \tilde{b}_s^{(m)}(\beta) = \tilde{b}_s^{(m)}(\alpha - e\alpha\cos M) \tag{4.143}$$

$$\simeq \tilde{b}_s^{(m)}(\alpha) - e\alpha\frac{d\tilde{b}_s^{(m)}}{d\beta}\bigg|_{\beta=\alpha}\cos M \tag{4.144}$$

$$= \tilde{b}_s^{(m)}(\alpha) - e\alpha D\tilde{b}_s^{(m)}(\alpha)\cos M + \mathcal{O}(e^2) \tag{4.145}$$

where D is shorthand for $\partial/\partial\alpha$.

We will also need the Taylor series expansion of $\cos m\psi$:

$$\cos m\psi = \cos m(\phi' - \phi) \simeq \cos m(\lambda' - \lambda - 2e\sin M) \tag{4.146}$$

$$\simeq \cos(m\Delta\lambda) + 2me\sin M\sin(m\Delta\lambda) + \mathcal{O}(e^2) \tag{4.147}$$

where $\Delta\lambda \equiv \lambda' - \lambda$.

Insert these expansions into eqn' (4.137):

$$R \simeq \frac{Gm'}{a'}\sum_{m=0}^{\infty}[\tilde{b}_{1/2}^{(m)} - e\alpha D\tilde{b}_s^{(m)}\cos M] \times \tag{4.148}$$

$$[\cos(m\Delta\lambda) + 2me\sin M\sin(m\Delta\lambda)] \tag{4.149}$$

$$\simeq \frac{Gm'}{a'}\sum_{m=0}^{\infty}[\tilde{b}_{1/2}^{(m)}\cos(m\Delta\lambda) - e\alpha D\tilde{b}_{1/2}^{(m)}\cos M\cos(m\Delta\lambda) \tag{4.150}$$

$$+2me\tilde{b}_{1/2}^{(m)}\sin M\sin(m\Delta\lambda)] \tag{4.151}$$

Next make use of the following trig identities:

$$\cos A \cos B \;=\; \frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right] \tag{4.152}$$

$$\sin A \sin B \;=\; \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right] \tag{4.153}$$

With $\Delta\lambda = \lambda' - \lambda = M' - M + \Delta\tilde{\omega}$ where $\Delta\tilde{\omega} = \tilde{\omega}' - \tilde{\omega}$, then

$$\cos(m\Delta\lambda)\cos M = \cos m(M' - M + \Delta\tilde{\omega})\cos M \tag{4.154}$$

$$= \frac{1}{2}\cos[mM' - (m-1)M + m\Delta\tilde{\omega}] + \frac{1}{2}\cos[mM' - (m+1)M + m\Delta\tilde{\omega}] \tag{4.155}$$

$$\text{but}\quad m\Delta\tilde{\omega} = m\tilde{\omega}' - m\tilde{\omega} = m\tilde{\omega}' - (m-1)\tilde{\omega} - \tilde{\omega} = m\tilde{\omega}' - (m+1)\tilde{\omega} + \tilde{\omega} \tag{4.156}$$

$$\text{so}\quad \cos(m\Delta\lambda)\cos M = \frac{1}{2}\cos[m\lambda' - (m-1)\lambda - \tilde{\omega}] \tag{4.157}$$

$$+\frac{1}{2}\cos[m\lambda' - (m+1)\lambda + \tilde{\omega}] \tag{4.158}$$

Similarly, you can show that

$$\sin(m\Delta\lambda)\sin M \;=\; \frac{1}{2}\cos[m\lambda' - (m+1)\lambda + \tilde{\omega}] \tag{4.159}$$

$$-\frac{1}{2}\cos[m\lambda' - (m-1)\lambda - \tilde{\omega}] \tag{4.160}$$

Now insert these results into eqn' (4.150):

$$R \;\simeq\; \frac{Gm'}{a'}\sum_{m=0}^{\infty}\Big\{\tilde{b}_{1/2}^{(m)}\cos(m\lambda' - m\lambda) \tag{4.161}$$

$$-\frac{1}{2}e\left(2m\tilde{b}_{1/2}^{(m)} + \alpha\frac{\partial\tilde{b}_{1/2}^{(m)}}{\partial\alpha}\right)\cos[m\lambda' - (m-1)\lambda - \tilde{\omega}] \tag{4.162}$$

$$+\frac{1}{2}e\left(2m\tilde{b}_{1/2}^{(m)} - \alpha\frac{\partial\tilde{b}_{1/2}^{(m)}}{\partial\alpha}\right)\cos[m\lambda' - (m+1)\lambda + \tilde{\omega}]\Bigg\} \tag{4.163}$$

Evidently, the disturbing function R that describes the planet's perturbations is a sum of terms having the form

$$\begin{aligned} R_m &= \frac{Gm'}{a'} C_m(\alpha) e \cos\varphi & (4.164) \\ \text{where} \quad C_m(\alpha) &= \text{ some function of } \alpha = a/a' & (4.165) \\ \text{and the } \textit{resonance angle} \quad \phi &= j_1\lambda' + j_2\lambda + j_4\tilde{\omega} & (4.166) \end{aligned}$$

Also note that

$$\sum_i j_i = 0 \qquad (4.167)$$

This is the *d'Alembert relation*, which is a consequence of the system's rotational invariance. Since we can rotate our coordinate system by any angle δ without altering the physics, φ must be unchanged, so $\sum j_i = 0$.

A far more general derivation of the expansion of R is given in Chapter 6, That derivation allows for both the particle and planet having a nonzero e, e', i, i', and it expands R to a much higher–order accuracy in these orbit elements. This more general derivation shows that R has the form

$$\begin{aligned} R_m &= \frac{Gm'}{a'} C_m(\alpha) e'^{|j_3|} e^{|j_4|} i'^{|j_5|} i^{|j_6|} \cos\varphi & (4.168) \\ \text{where} \quad \phi &= j_1\lambda' + j_2\lambda + j_3\tilde{\omega}' + j_4\tilde{\omega} + j_5\tilde{\Omega}' + j_6\tilde{\Omega} & (4.169) \end{aligned}$$

which also satisfies the d'Alembert relation. Evidently, our assumption that $e' = 0$ and that $i = i' = 0$, plus our Taylor expansion to $\mathcal{O}(e)$ only allowed us to capture the $j_4 = \pm 1$ terms in eqn' (4.168).

Resonances

Consider a particle perturbed by planet m' on a circular orbit of radius a'.
Again, assume the particle has $e \ll 1$ and the system is coplanar,
so the particle's disturbing function is eqn' (4.162).

Suppose the particle orbits at a site where the arguments of one of the cosine terms varies slowly with time, ie, where

$$\phi \;=\; j\lambda' - (j+k)\lambda + k\tilde{\omega} \quad \text{where } k = \pm 1 \tag{4.170}$$

Since φ varies slowly with time,

$$\dot{\varphi} \;\simeq\; jn' - (j+k)n \simeq 0 \quad (\text{this assumes } |\dot{\tilde{\omega}}| \text{ and } |\dot{\epsilon}| \ll 1) \tag{4.171}$$

$$\text{so} \quad \frac{n'}{n} \;=\; \left(\frac{a}{a'}\right)^{3/2} \simeq \frac{j+k}{j} = \frac{T}{T'} \quad \text{where } T = \text{orbit period} \tag{4.172}$$

$$\Rightarrow a \;\simeq\; \left(\frac{j+k}{j}\right)^{2/3} a' \tag{4.173}$$

is the approximate location of a $j : j+k$ mean–motion resonance,
also known a commensurability resonances since the orbit periods are commensurate, or ratios of whole numbers.

Note that the sign of k determines whether the resonance lie interior or exterior to the planet's orbit.

Lagrange Planetary eqn's

Now use the approximate form of the Lagrange planetary eqn's, (4.102), to show that large eccentricities are excited at resonances.

Lets assume the particle is an asteroid orbiting near Jupiter's inner 2:1 mean–motion resonance, so $j = 2 = m$, $k = -1$,
the resonance angle is $\varphi = 2\lambda' - \lambda - \tilde{\omega} = 2n't - nt - \epsilon - \tilde{\omega}$,
and the resonance location is $a \simeq a'/2^{2/3} \simeq 0.63a'$.

If we are near a resonance, then the disturbing function R, eqn (4.150),
is the sum of one 'slow' term plus numerous 'fast' terms:

$$R = \frac{Gm'}{a'}C_m(\alpha)e\cos\varphi + \text{ numerous fast terms} \quad (4.174)$$

$$\text{where} \quad C_m(\alpha) = -\frac{1}{2}\left[4b_{1/2}^{(2)}(\alpha) + \alpha\frac{\partial b_{1/2}^{(2)}}{\partial\alpha}\right] \quad \text{and } \alpha = a/a' \quad (4.175)$$

and the 'fast' terms are all the other terms that cos(coefficient × nt) which vary rapidly with time during the particle's orbit.
Next, invoke the 'averaging principal', which says that the average effect of the fast term tends to sum to $\simeq 0$ when we consider the particle's long–term motion.

The averaging principal says that we can ignore the fast terms in R since they are assumed to have no long–term effect on the particle's motion, and that the one slow term is eqn' (4.174) is representative of the particle's disturbing function.

Surprisingly, this hand–waving assumption works pretty well in celestial mechanics.

The Lagrange planetary equation for the massless particle's e is

$$\begin{aligned} \dot{e} &\simeq -\frac{\partial R/\partial\tilde{\omega}}{a^2ne} = -\frac{Gm'}{a'a^2n}C_m\sin\varphi && (4.176) \\ \text{and since}\quad Gm' &= GM_c(m'/m_c) = n^2a^3(m'/m_c), && (4.177) \\ \dot{e} &= -\alpha C_m\left(\frac{m'}{m_c}\right)n\sin(\dot{\varphi}t) && (4.178) \end{aligned}$$

We shall soon see that the $C_m(\alpha)$, which is a function of the Laplace coefficients, is or order $\mathcal{O}(C_m) \sim 1$.

The planet's mass is also small compared to the central mass m_c, so $|\dot{e}| \sim \mathcal{O}(nm'/m_c) \ll n \Rightarrow$ the particle's orbit varies on a timescale that is much slower than its orbit period.

This suggests that the orbit elements that appear on the RHS of the planetary eqn's can be treated as constants. Then eqn' (4.178) is the equation for simple harmonic motion, which has the solution

$$\begin{aligned} e(t) &\simeq \frac{\alpha C_m n}{\dot{\varphi}}\left(\frac{m'}{m_c}\right)\cos(\dot{\varphi}t) + e(0) && (4.179) \\ &= \frac{n}{2n' - n - \dot{\epsilon} - \dot{\tilde{\omega}}}\left(\frac{m'}{m_c}\right)\alpha C_m\cos\varphi + e(0) && (4.180) \end{aligned}$$

Evidently, large eccentricities can get excited at a mean–motion resonance, since that is the site where the denominator $\dot{\varphi} \simeq 0$.

Note that a precise determination of the resonance location and $e(t)$ requires using the planetary equations to obtain $\dot{\epsilon}$ and $\dot{\tilde{\omega}}$. However if you are not doing precision work, it is usually OK to simply assume these small rates=0.

The particle's variations in a are also small:

$$\dot{a} = \frac{2}{an}\frac{\partial R}{\partial\epsilon} = \frac{2Gm'}{ana'}C_m e\sin\phi \qquad (4.181)$$

which is much smaller than the $\dot{e}$ variation due to the factor of e;
In fact, the variations $\Delta a/a \sim 10^{-6}$ for the particle whose orbit is plotted in the text's Fig. 6.5.

Quantitative use of the Lagrange planetary equations requires determining the C_m coefficient in $e(t)$. Usually this must be done numerically.

However a numerical evaluation of the $b_s^{(m)}(\alpha)$, is trivial with math software like MAPLE, MATHEMATICA, MATLAB, etc.

You will also show in Assignment #5 that

$$\frac{db_s^{(m)}}{d\alpha} = s\left(b_{s+1}^{(m-1)} - 2\alpha b_{s+1}^{(m)} + b_{s+1}^{(m+1)}\right) \tag{4.182}$$

Evaluating the Laplace coefficients

Consequently,

$$\begin{aligned} C_m(\alpha) &= -2b_{1/2}^{(2)} - \frac{1}{2}\alpha D b_{1/2}^{(2)} && (4.183) \\ &= -2b_{1/2}^{(2)} - \frac{1}{4}\alpha\left(b_{3/2}^{(1)} - 2\alpha b_{3/2}^{(2)} + b_{3/2}^{(3)}\right) && (4.184) \end{aligned}$$

Then use symbolic math software such as MAPLE to evaluate these Laplace coefficients numerically:

```
crux:  /home/jhahn >xmaple
b:=(2/Pi)*Int(cos(m*phi)/(1+alpha^2-2*alpha*cos(phi))^s,phi=0..Pi);
```

$$b \ := \ \frac{2}{\pi}\int_0^{\pi} \frac{\cos(m\phi)d\phi}{(1+\alpha^2-2\alpha\cos(\phi))^s}$$

```
alpha:=(1/2)^(2/3);
```

$$\alpha \ := \ \frac{2^{1/3}}{2}$$

```
m:=2;
```

$$m \ := \ 2$$

```
s:=3/2;
```

$$s \ := \ \frac{3}{2}$$

```
evalf(b);
```

$$3.655435392$$

A numerical evaluation of all of the Laplace coefficients ultimately yields $C_m(\alpha) = -1.190493697$.

The origin of Pluto's orbit

Pluto orbits at Neptune's $3:2$ outer mean–motion resonance where $e = 0.25$ and $a = 40$ AU.

Use the Lagrange planetary equations to examine resonance trapping and the early history of Pluto's orbit.

Pluto is unlikely to have formed in this very eccentric orbit:

Suppose Pluto formed via the merger of 2 smaller proto–Pluto's.
Since $r \simeq a(1 - e\cos nt)$, $\dot{r} \simeq -ean\sin nt$, and the proto–Pluto's will collide with a relative velocity $v_{col} \sim \mathcal{O}(ean) \sim e\sqrt{GM_{\odot}/a} \sim 1$ km/sec.

This is comparable to Pluto's surface escape velocity $v_{esc} \simeq \sqrt{2Gm_P/R_P} \sim$ 1 km/sec, so collisions will tend to be *erosive* as impact–generated derbris escapes Pluto.

Rather, Pluto likely formed from a swarm of numerous planetesimals initially having small e, and then was later *trapped* in high–e orbit at Neptune's 3:2 resonance.

Models of the formation of the outer planets show that Neptune's orbit tends to slowly expand outwards as it gravitationally scatters the nearby planetesimals.

This outward expansion of Neptune's orbit can easily capture Pluto (plus numerous other Kuiper Belt Objects, or KBOs), at Neptune's 3:2 resonance.

If this did indeed happen, then Pluto will have a disturbing function

$$R = \frac{Gm_N}{a_N} C_m e \cos\varphi = \left(\frac{m_N}{m_\odot}\right) n^2 a^2 \alpha C_m(\alpha) e \cos\varphi \quad (4.185)$$

$$\text{where} \quad \varphi = j\lambda_N - (j+k)\lambda + k\tilde{\omega} \quad (4.186)$$

$$\text{and} \quad a_{res} = \left(\frac{j+k}{j}\right)^{2/3} a_N \quad (4.187)$$

where the N subscript refers to Neptune's quantities,
and all others refer to Pluto.

At the 3:2 outer mean–motion resonance, $j = 2$ and $k = 1$.

The Lagrange planetary eqn's are

$$\dot{a} = \frac{2}{an}\frac{\partial R}{\partial \epsilon} = 2(j+1)\left(\frac{m_N}{m_\odot}\right) an\alpha C_m \sin\varphi \quad (4.188)$$

$$\text{and} \quad \dot{e} = \frac{1}{a^2 ne}\frac{\partial R}{\partial \epsilon} = k\left(\frac{m_N}{m_\odot}\right) n\alpha C_m \sin\varphi \quad (4.189)$$

$$\text{thus} \quad e\dot{e} = \frac{k}{2(j+1)}\frac{\dot{a}}{a} \quad (4.190)$$

which is easily integrated:
$$\int_0^e e' de' = \int_{a_0}^a \frac{k}{2(j+1)}\frac{da'}{a'} \quad (4.191)$$

$$\text{so} \quad e^2 = \frac{k}{j+1}\ln\left(\frac{a}{a_0}\right) \quad (4.192)$$

assuming Pluto formed in an initially circular orbit $e_0 = 0$ with an initial semimajor axis a_0.

With $j = 2$, $k = 1$, $e = 0.25$, and $a = 40$ AU, Pluto's current orbit implies that it was initially orbiting at $a_0 = 33$ AU when it was initially trapped at Neptune's migrating 3:2 resonance.

And since $a_{res} = ((j+k)/j)^{2/3}a_N = a_0$ at the moment that Pluto was trapped in the 3:2 resonance, $\Rightarrow a_N = (2/3)^{2/3}a_0 = 25$ AU, which is Neptune's semimajor axis at the moment Pluto entered the 3:2 resonance.

Since Neptune currently orbits at $a_N = 30$ AU, this interpretation of Pluto's orbit implies that Neptune's orbit had expanded by $\Delta a_N = 5$ AU, or by 20% early in the Solar System's history.

This resonance trapping phenomena is also used to explain the Laplace resonance, which is the suite of 2:1 resonances inhabited by Jupiter's Galilean satellites Io, Europa, & Ganymede.

For many years it has been thought that Io's outward orbital migration due to tides was responsible for inserting Europa into the 2:1 with Io,
with Ganymede subsequently entering a 2:1 with Europa.

However people are now examining whether it was the *inward* orbit migration of Ganymede & Europa (due to their interactions with an early satellite–forming disk about Jupiter) might have established the Laplace resonance.

Precession of orbits

The precession of orbits is the steady rotation of longitudes $\tilde{\omega}$ and Ω that is a frequent result of non–Keplerian perturbations.

In planetary problems, the non–Keplerian perturbations are usually weak and precession is slow, ie, $|\dot{\tilde{\omega}}|$ and $|\dot{\Omega}|$ are $\ll n$.

A star orbiting in a galaxy will also precess due to the galaxy's gravitational potential. However that potential is usually very non–Keplerian, ie, Φ does not vary as r^{-1}. This results in fast precession, ie, $|\dot{\tilde{\omega}}|$ and $|\dot{\Omega}|$ are $\mathcal{O}(n)$.

The orbits of satellites tend to precess when orbiting a rotating planet that is slightly non–spherical. This is due to centrifugal forces in the planet when tend to flatten it slightly at the poles and fatten it at the equator.

Section 4.5 derives a rotating planet's potential:

$$\Phi(r,\theta) = -\frac{Gm_p}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R_p}{r}\right)^n P_n(\cos\theta)\right] \tag{4.193}$$

where R_p and m_p are the planet's radius and mass,
J_n = the planet's multipole moments (these are dimensionless numbers reported in Table A.4), and (r,θ) = the satellite's spherical coordinates.

The J_n are always smaller for larger n, and $R_p < r$, so

$$\Phi(r,\theta) \simeq -\frac{Gm_p}{r} + J_2\frac{Gm_p}{R_p}\left(\frac{R_p}{r}\right)^3 P_2(\cos\theta) \tag{4.194}$$

$$\text{and} \quad R = -J_2\frac{Gm_p}{R_p}\left(\frac{R_p}{r}\right)^3 P_2(\cos\theta) \tag{4.195}$$

is the disturbing function associated with planetary oblateness.

To use this in the Lagrange planetary eqn's, we need to write R in terms of orbit elements.

Consider a satellite in a low–e orbit with $i = 0$, so

$$\frac{r}{a} \simeq 1 - e\cos M + \frac{1}{2}e^2(1 - \cos 2M) \equiv 1 + x \quad \text{(see eqn' 1.142)} \tag{4.196}$$

$$\text{where } M = nt, \qquad \theta = \frac{\pi}{2}, \qquad \text{and} \qquad P_2(\cos\theta)) = -\frac{1}{2} \tag{4.197}$$

$$\text{so} \quad R \simeq \frac{J_2}{2}\frac{Gm_p}{R_p}\left(\frac{R_p}{a}\right)^3(1 + x)^{-3} \tag{4.198}$$

$$\text{where} \quad (1 + x)^\alpha \simeq 1 + \alpha x + \frac{1}{2}\alpha(\alpha - 1)x^2 + ... \tag{4.199}$$

$$\text{so} \quad (1 + x)^{-3} \simeq 1 - 3x + 6x^2 \tag{4.200}$$

$$= 1 + 3e\cos M - \frac{3}{2}e^2(1 - \cos 2M) + 6e^2\cos^2 M \tag{4.201}$$

$$\text{and} \quad R \simeq \frac{J_2}{2}\frac{Gm_p}{R_p}\left(\frac{R_p}{a}\right)^3 [1 + 3e\cos M \tag{4.202}$$

$$+ e^2(-3/2 + 3) + e^2\cos 2M(3/2 + 3)] \tag{4.203}$$

The precession rate for the satellite's longitude of periapse is

$$\dot{\tilde{\omega}} \simeq \frac{1}{a^2ne}\frac{\partial R}{\partial e} = \frac{J_2}{2}\frac{Gm_p}{R_pa^2n}\left(\frac{R_p}{a}\right)^3(3\cos M + 3 + 3\cos 2M) \tag{4.204}$$

where $n^2 \simeq Gm_p/a^3$.

However we are interested in the *time–averaged* periapse precession rate,

$$< \dot{\tilde{\omega}} > \simeq \frac{3}{2}J_2\left(\frac{R_p}{a}\right)^2 n \tag{4.205}$$

$$\text{so} \quad T_{\tilde{\omega}} = \frac{2\pi}{< \dot{\tilde{\omega}} >} \simeq \frac{2}{3J_2}\left(\frac{a}{R_p}\right)^2 T_{orb} \tag{4.206}$$

is the precession period $T_{\tilde{\omega}}$, where T_{orb} = satellite's orbital period.

For the satellite Prometheus, which orbits Saturn ($J_2 = 0.016$) at $a/R_p = 2.31$ in $T_{orb} = 0.61$ days, $T_{\tilde{\omega}} \simeq 220 T_{orb} = 130$ days.

Precession due to planetary oblateness clearly plays a role in the detailed location of a resonance, since

$$\dot{\varphi}(a) \; = \; jn' - (j+k)n(a) + k\dot{\tilde{\omega}}(a) = 0 \qquad (4.207)$$

defines the location of the resonance $a = a_{res}$.

Assignment # 5

1. Use Gauss' eqn's to calculate the inclination–damping timescale $\tau_i = |i/(di/dt)|$ to lowest order in i for a satellite in a circular, low–i orbit that suffers the drag acceleration $\mathbf{a_p} = -\alpha n \dot{\mathbf{r}}$ where α is a small dimensionless drag coefficient.

2. Use the Gauss' planetary equations to rederive eqn. (3.131), which is the rate at which a secondary's a evolves due to tides. Obtain the perturbing acceleration that the primary's tidal bulge exerts on the secondary, $\mathbf{a}_p$, from eqn. (3.104), taking care to interpret θ correctly.

3. Show that

$$\frac{db_s^{(m)}}{d\alpha} = s\left(b_{s+1}^{(m-1)} - 2\alpha b_{s+1}^{(m)} + b_{s+1}^{(m+1)}\right) \qquad (4.208)$$

4. What is the 'slow' term in the disturbing function for a particle orbiting near a planet's 4:3 outer mean motion resonance? What is the forcing function C_m and the resonance angle φ for this resonance? Where is this resonance located approximately (ie, ignore the effects of precession). Solve the Lagrange planetary eqn's to obtain an approximate expression for the particle's eccentricity $e(\alpha)$ where α is the ratio of semimajor axes, again ignoring the effects of precession. Plot on a log–log axis the particle's *forced eccentricity* $e(x)$ where x =particle's fractional distance from resonance, and indicate where you think your plot likely gives an unreliable result. *Note*—this last calculation requires a numerical evaluation of Laplace coefficients, which is easily done in MAPLE, MATLAB, & MATHEMATICA. I'll soon append a MAPLE script that will illustrate this type of calculation.

Due date: Tuesday April 6 in class.