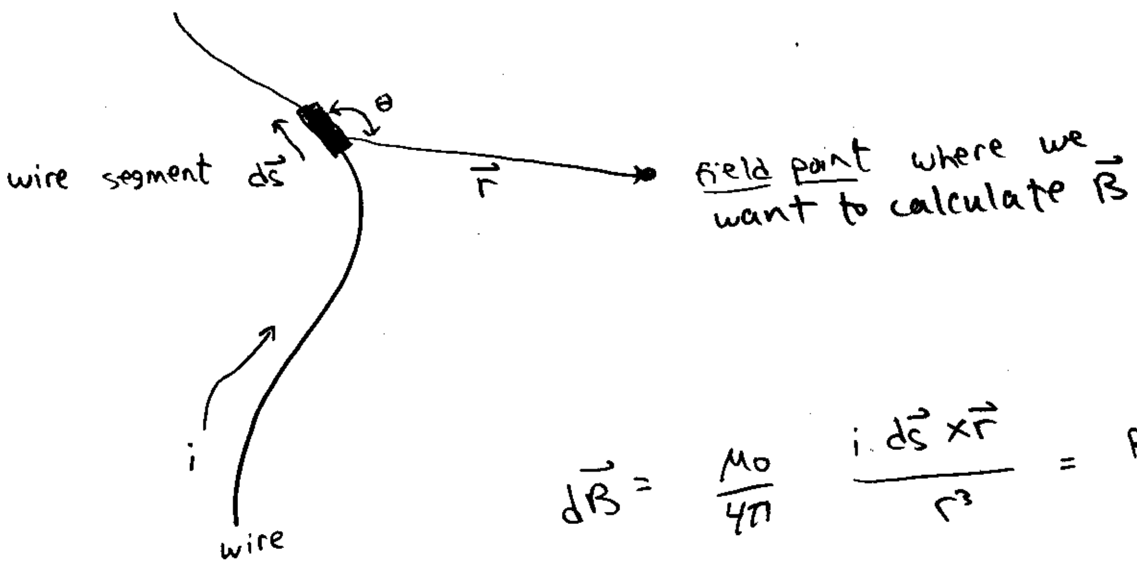
Chapter 30: Magnetic Fields Due to Currents

See Sections 1-5

The Bio-Sarvart Law is used to calculate $\vec{B}$ that is generated by some current i that flows thru a wire:



$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i\, d\vec{s} \times \vec{r}}{r^3} = \text{Biot-Savart Law}$$

= small field $d\vec{B}$ that is generated by wire segment $d\vec{s}$ that lies a distance $\vec{r}$ from the field point

The Bio-Savart Law was determined experimentally by B & S' in ~1800, where

$\mu_0 = 1.26 \times 10^{-6}$ T-m/A = permeability constant

The total $\vec{B}$-field is $\vec{B} = \int d\vec{B} = \frac{\mu_0}{4\pi} \int \frac{i\, d\vec{s} \times \vec{r}}{r^3}$

integrate along entire wire

Note that $|d\vec{s} \times \vec{r}| = r \sin\theta\, ds$ so

$$|d\vec{B}| = \frac{\mu_0}{4\pi} \frac{i \sin\theta\, ds}{r^2}$$ varies as $\frac{1}{r^2}$ (just like $d\vec{E}$)

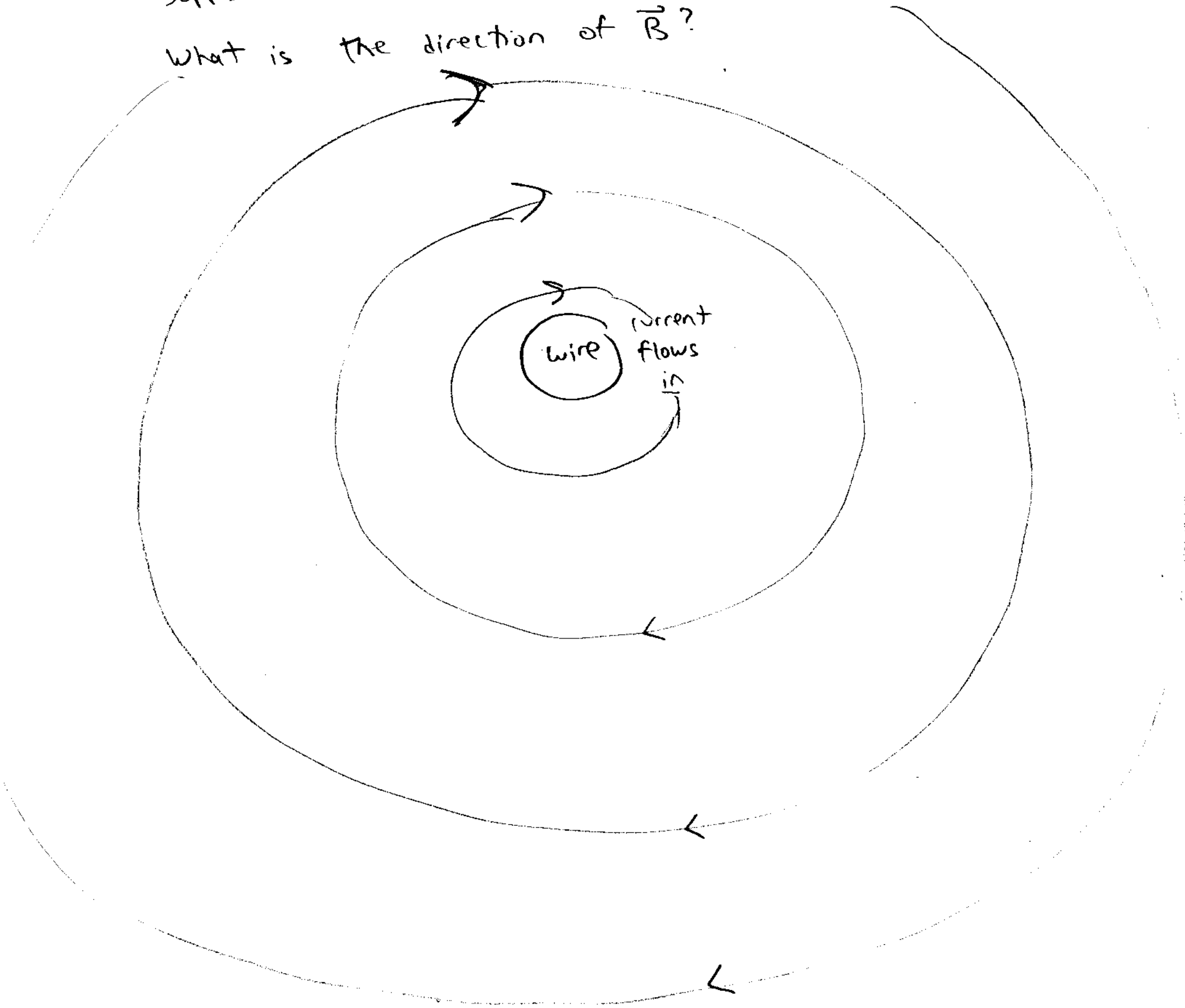
The direction of $d\vec{B}$ obeys the right-hand-rule:

Let your thumb point along current i,

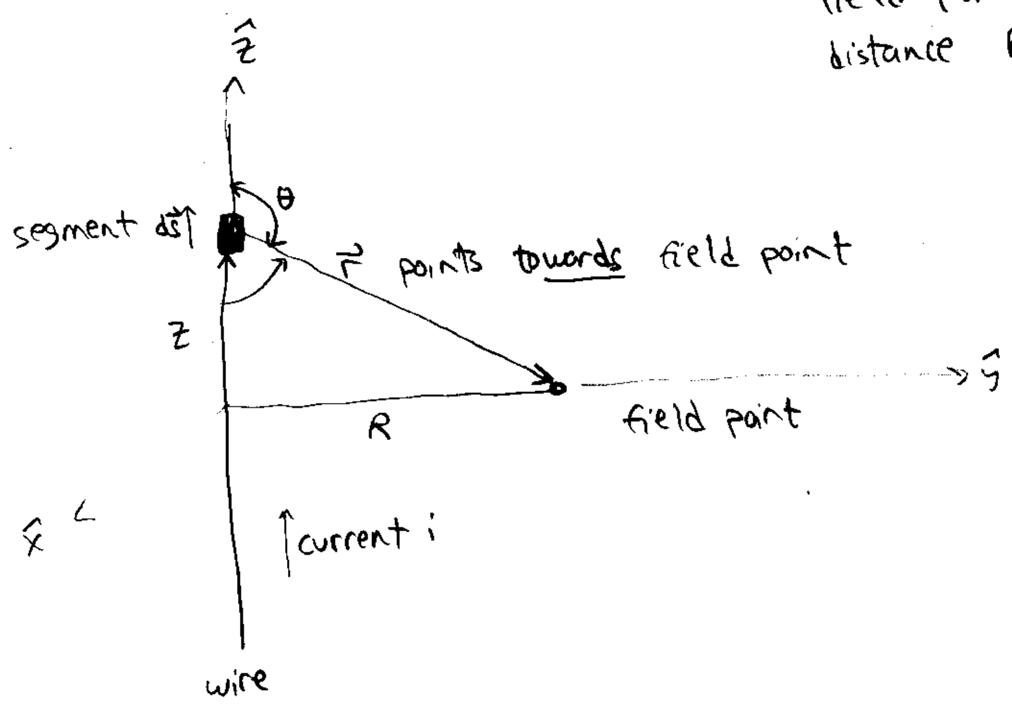
and your fingers will curl along the direction of $\vec{B}$

Suppose a current flows into this page along a wire.

What is the direction of $\vec{B}$?



$\vec{B}$ due to a long straight wire: calculate $\vec{B}$ at the field point that lies a distance R away from wire



use Bio-Savart law: $d\vec{B} = \frac{\mu_0 i}{4\pi} \frac{d\vec{s} \times \vec{r}}{r^3}$

which way does $d\vec{B}$ point? Into page along $-\hat{x}$.

Does the direction of $d\vec{B}$ change as segment $d\vec{s}$ runs up the length of the wire?

with $d\vec{s} = dz\,\hat{z}$ and $d\vec{s} \times \vec{r} = dz\,\underbrace{\hat{z} \times \vec{r}}_{r\sin\theta\,(-\hat{x})}$

$$= -r\sin\theta\, dz\, \hat{x}$$

so $\vec{B} = \int d\vec{B} = -\frac{\mu_0 i}{4\pi} \int \frac{\sin\theta\, dz}{r^2}\, \hat{x}$ where $r = \sqrt{R^2 + z^2}$

Note also that $\sin\theta = \dfrac{R}{r}$

so $B = |\vec{B}| = \dfrac{\mu_0 i}{4\pi} \displaystyle\int \dfrac{R}{r^3}\, dz$

what are my integration limits ?

$$\frac{\mu_0 i}{4\pi} \int_{-\infty}^{\infty} \frac{R\, dz}{(R^2+z^2)^{3/2}}$$

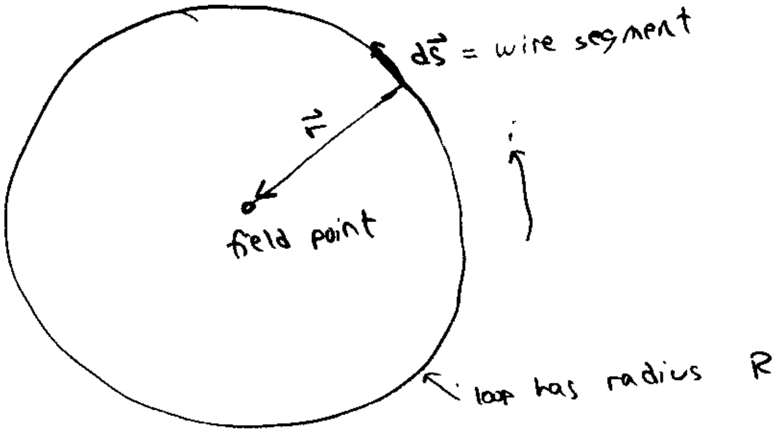
note that upper & lower halfs of wire contribute the same

$$= \frac{\mu_0 i R}{4\pi} \underbrace{\frac{z}{R^2\sqrt{R^2+z^2}} \Bigg|_{-\infty}^{\infty}}_{\frac{1}{R^2} - \left(\frac{-1}{R^2}\right) = \frac{2}{R^2}}$$

see Appendix E

$\Rightarrow B = \dfrac{\mu_0 i}{2\pi R}$ = B field due to infinite wire.

Calculate $\vec{B}$ at the center of a current loop:



Bio-Savart Law: $d\vec{B} = \frac{\mu_0}{4\pi} \frac{i\, d\vec{s} \times \vec{r}}{r^3}$

which way does $d\vec{B}$ point? Out of this page

what about for other segments $d\vec{s}$ on this loop?

Since $d\vec{s}$ is perpendicular to $\vec{r}$, $|d\vec{s} \times \vec{r}| = R\, ds$

so $$B = \int dB = \frac{\mu_0 i}{4\pi} \int \frac{R\, ds}{R^3}$$

$$= \frac{\mu_0 i}{4\pi R^2} \underbrace{\int ds}_{\text{what is this?}} \quad 2\pi R$$

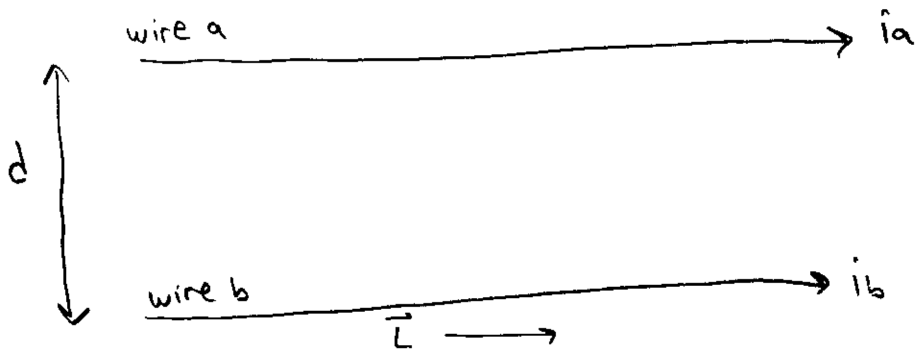
so $B = \frac{\mu_0 i}{2R}$ = field at center of a current loop.

Force exerted by Two parallel Currents

Two wires carry currents i_a and i_b are separated by distance d. These currents flow in the same direction.

Both wires generate fields B_a & B_b.

Since they carry moving charges, they also exert a magnetic force on each other:



Let $\vec{B_a}$ = field generated by wire a, evaluated at wire b.

which way does $\vec{B_a}$ point? $\vec{B_b}$?
into page   out

Recall that $B_a = \frac{\mu_0 i_a}{2\pi d}$ = field at wire b due to wire a.

Also recall that $\vec{F_{ba}}$ = force on wire b due to a
$= i_b \vec{L} \times \vec{B_a}$ which points up, towards a.
(Eqn 29-26)

$\Rightarrow$ so $F_{ba} = \frac{\mu_0 L i_a i_b}{2\pi d}$ = force on b due to a.

What is the force on wire a due to b?

by Newton's 3rd Law: same magnitude, opposite direction.

what if the currents were anti-parallel, ie, they ran in opposite directions?

→ flip the sign on either i_a OR i_b reverses the direction of the force

⇒ The magnetic force due to antiparallel currents is repulsive,

while parallel currents attract.

Ampere's Law

The brute-force way to calculate $\vec{B}$ is to use the Bio-Savart Law: $\vec{B} = \int \frac{\mu_0}{4\pi} \frac{i\, d\vec{s} \times \vec{r}}{r^3}$

which can be very laborious...

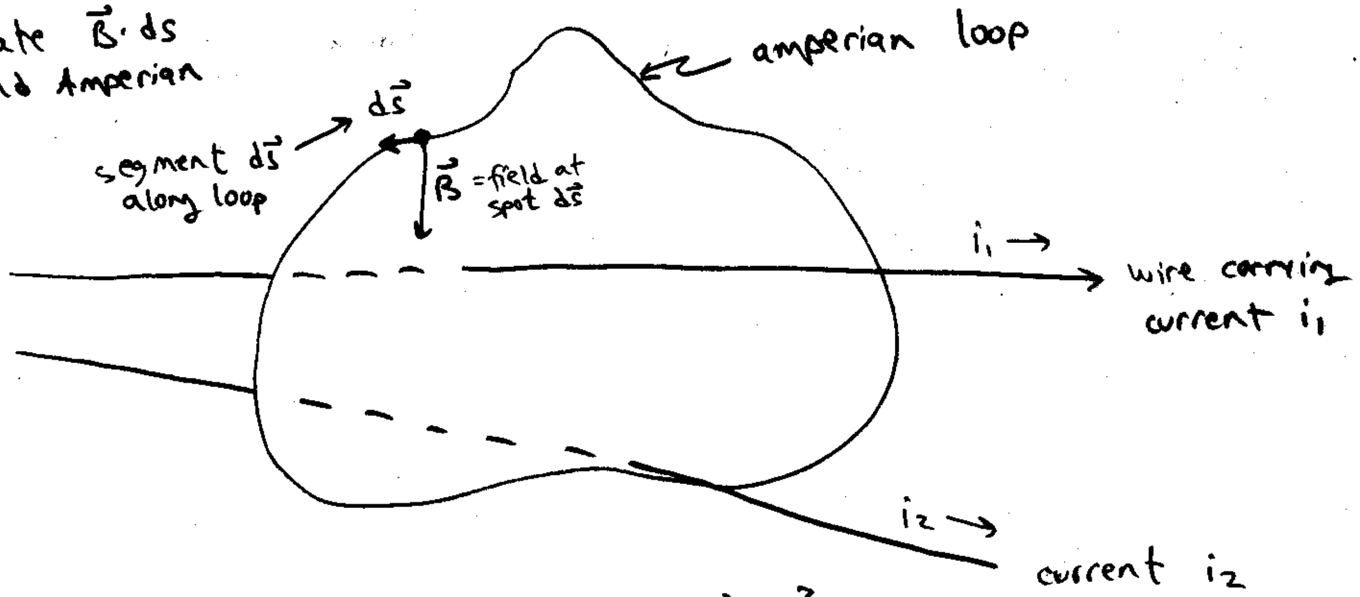
However Ampere's Law can, <u>in some circumstances,</u> simply the calculation of $\vec{B}$ considerably:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc}$$ Ampere's Law

This integral proceeds along a <u>closed</u> "Amperian" loop

i_{enc} = current enclosed by Amperian loop

we need to integrate $\vec{B} \cdot ds$ around Amperian loop

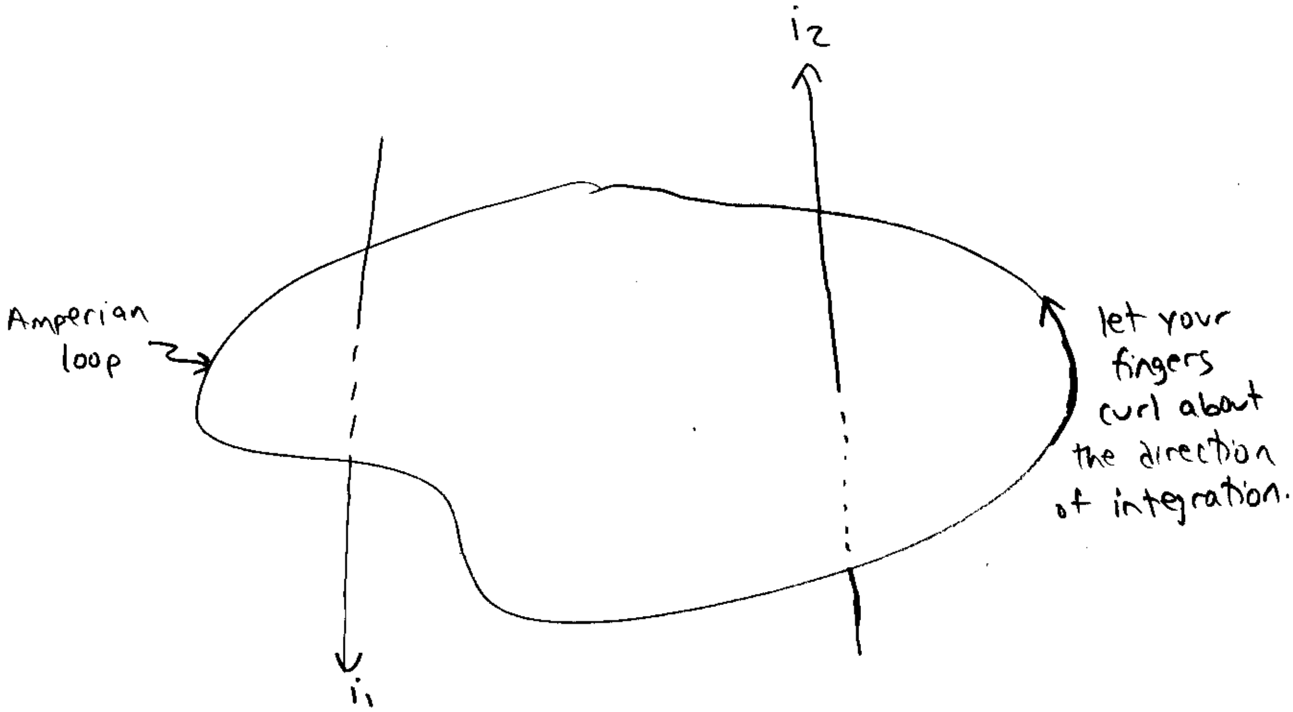


What is the enclosed current i_{enc}?

$i_{enc} = i_1 + i_2$

What if i_2 flowed in ~~the~~ opposite direction? $i_{enc} = i_1 - i_2$

Note: you must use the right-hand rule to determine the signs of any enclosed currents:

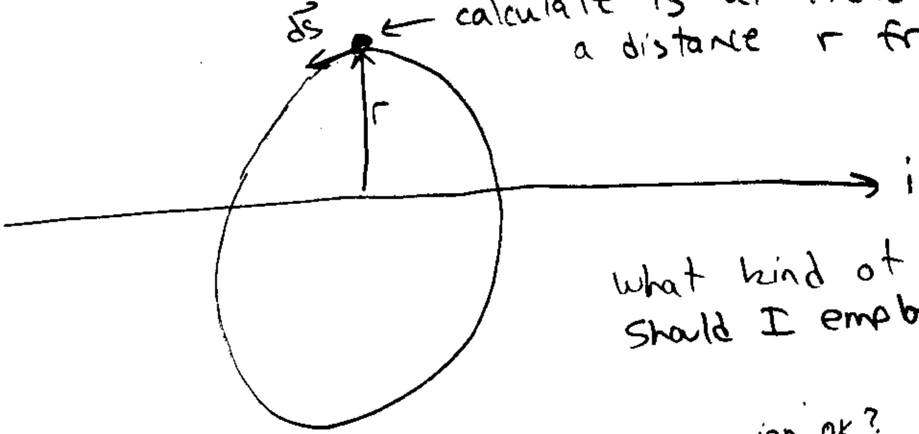


Positive currents point thumbwards, negative currents point anti-thumbwards

What is i_{enc} for this picture

$i_{enc} = -i_1 + i_2$

Current outside a long thin wire:



← calculate B at field point, a distance r from wire.

What kind of Amperian loop should I employ?

sign OK?

Ampere's Law: $$\oint \vec{B}\cdot d\vec{s} = \mu_0 i$$

Which way is $\vec{B}$ pointing? Use the R-H rule

Note that $\vec{B}$ is parallel to $d\vec{s}$,

so $\vec{B}\cdot d\vec{s} = B\cos\theta\, ds = B\,ds$

Does $|\vec{B}|$ vary about this loop?

So Ampere's Law becomes $\oint \vec{B}\cdot ds = B\int ds = \mu_0 i$

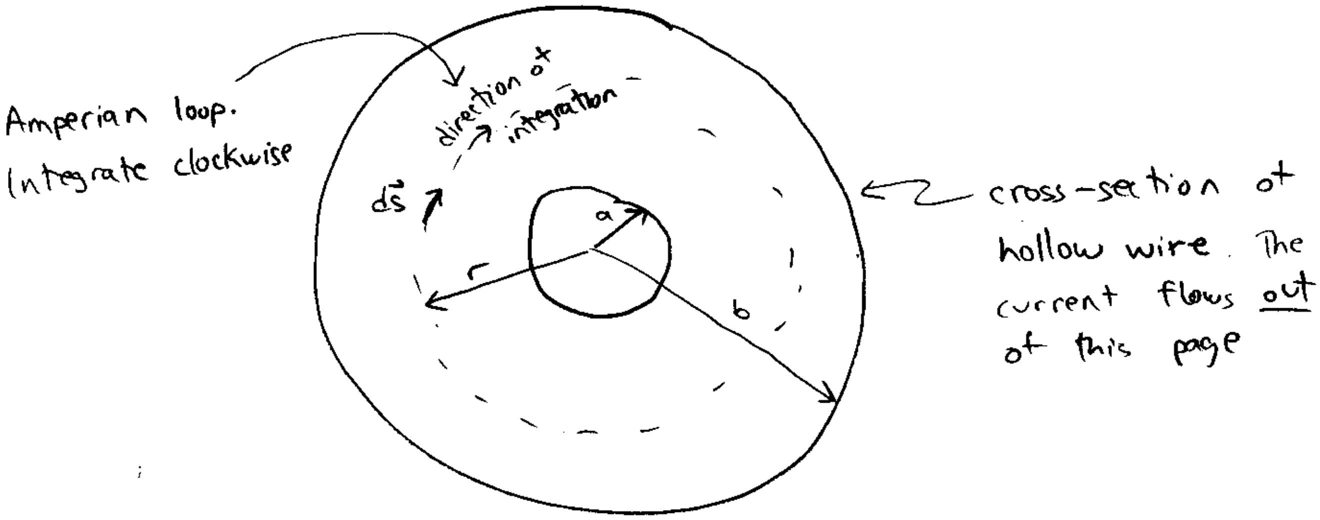
What is $\int ds = ?$

$$\Rightarrow \quad B = \frac{\mu_0 i}{2\pi r} \quad \text{as expected.}$$

⇒ Ampere's Law can simplify the calculation of $\vec{B}$ considerably...

Sample Problem 30-3

A hollowed-out wire carries a current density of $J(r) = cr^2$ = current-per-area:



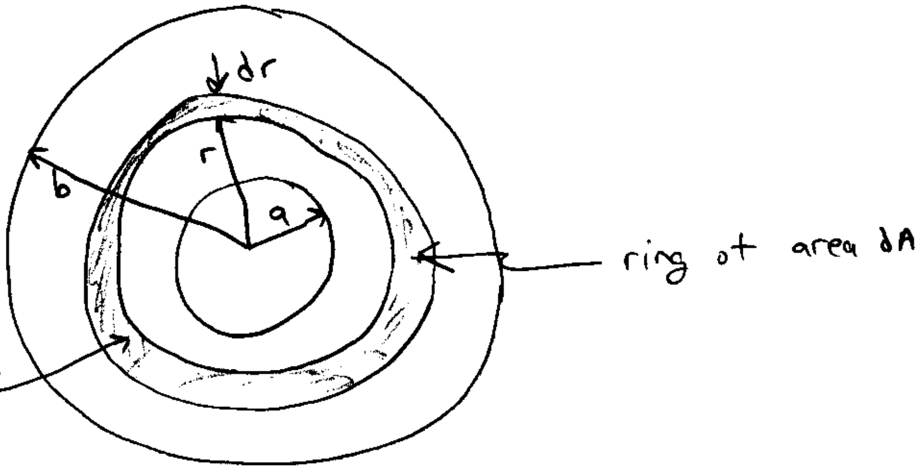
Use Ampere's law $\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc}$ to calculate $\vec{B}$ everywhere.

What kind of Amperian loop should I use?

What is i_{enc}? Since $J = \frac{di}{dA}$,

$$|i_{enc}| = \int_a^r J(r)\, dA$$

To do this integral, break up the wire's cross-section into rings of radius r, width dr.
Ring has small area
$dA = ? = 2\pi r\, dr$



so $|i_{enc}| = \int_a^r cr^2 \, 2\pi r \, dr = 2\pi c \int_a^r r^3 \, dr$

or $|i_{enc}| = \frac{1}{2}\pi c \, (r^4 - a^4)$ provided $a < r < b$

what is $|i_{enc}|$ when $r < a$?
$r > b$? set $|i_{enc}(b)| = i_{total}$

We also need $\vec{B} \cdot d\vec{s}$. Which way does $\vec{B}$ point?
Does $|\vec{B}|$ vary along the loop?

Note that $\vec{B} \cdot d\vec{s} = \pm B \, ds$
which sign?

Calculate $\vec{B}$ in $a < r < b$ zone:

$$\oint \vec{B} \cdot d\vec{s} = -B \int ds = -B \cdot 2\pi r = \mu_0 i_{enc} = \pm \mu_0 \frac{\pi}{2} c \, (r^4 - a^4)$$

which sign?

$$= -\frac{\mu_0}{2} \pi c \, (r^4 - a^4)$$

$$\Rightarrow B = \frac{\mu_0 c}{4} \left(r^3 - a^4/r \right)$$ in $a < r < b$ zone.

What is B in $r < a$ zone? $B = 0$

in $r > b$ zone? $-B \cdot 2\pi r = -\mu_0 i_{total}$

so $B = \dfrac{\mu_0 i_{total}}{2\pi r}$ as expected for any wire