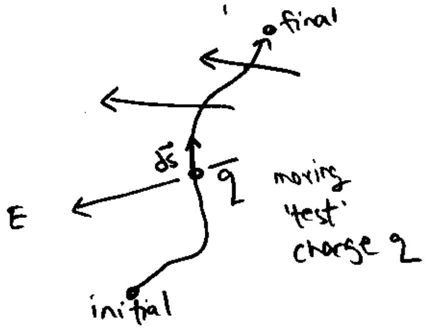
Ch 25: Electric Potential

Recall that $\vec{E}$-fields represent lines of force via $\vec{F} = q\vec{E}$.

Thus moving q across the E-field means that the E-field does work on charge q:

$$W = \int_i^f \vec{F} \cdot d\vec{s} = q \int_i^f \vec{E} \cdot d\vec{s} = \text{work done on } q \text{ by } \vec{E}.$$



Recall Eqn 8-1 of Chap 8:

work done on particle = $-1 \times$ change in p's Potential Energy (PE)

so $W = -\Delta U = -(U_{final} - U_{init})$

q's final and init PE.

where U = q's electric potential energy, or PE

Suppose we delivered charge q from far away (ie, from infinity).

We usually set the PE @ infinity to zero, ie

$$U_{init} = 0$$

The work done on q by the E-field is W_∞, so

$$\Rightarrow U = -W_\infty = \text{PE of } q \text{ at the 'final' position}$$

note I dropped the 'final' subsript

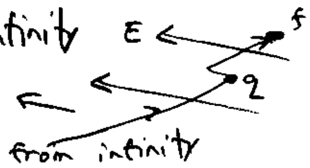
$\Rightarrow$ a particle's PE = $-1 \times$ work done by $\vec{E}$-field as q is delivered in from infinity

The electric potential V

Let U = potential energy of charge q.

$= -W_\infty$

$= -1\times$ work done on q by $\vec{E}$ as it is delivered from infinity



What if you instead delivered charge $2q$ across this $\vec{E}$-field.

How much work did $\vec{E}$ do on q?

$2W_\infty$

What is the PE of charge $2q$ when @ position f?

$U = -2W_\infty$

$\Rightarrow$ The PE U scales with charge q.

It will be convenient to introduce the

<u>electric potential</u> $V = \frac{U}{q} = -\frac{W_\infty}{q}$

= potential energy per unit charge due to $\vec{E}$-field.

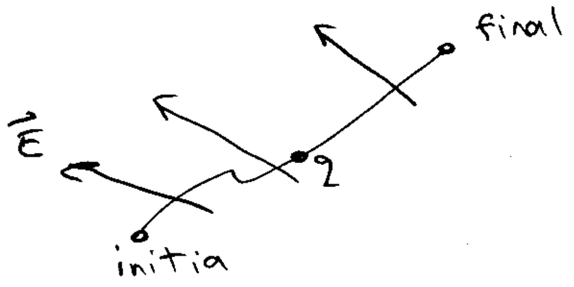
so $V = -1 \times$ work done on unit charge by $\vec{E}$

What units does V have?

energy/charge, or J/C in SI units

1 <u>volt</u> = 1 J/C

Suppose instead you want to deliver q from some site $i \to f$



Then $\Delta V = V_f - V_i =$ potential difference between sites i & f

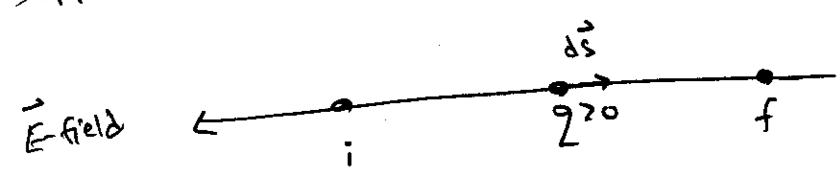
$$= \frac{1}{q}(U_f - U_i)$$

$$= \frac{\Delta U}{q} = -\frac{W}{q}$$

so $W = -\Delta U = -q\,\Delta V$ = work done by $\vec{E}$ as q is delivered across the potential difference (PD) ΔV.

Work done by an applied force:

Suppose $q > 0$:



Suppose $W = \underline{\text{work done by } \vec{E}}$ as q is pushed by an applied force from site $i \to f$.

$$= q\int_i^f \vec{E}\cdot d\vec{s}$$

Is $W > 0$ or $W < 0$?

How much work must an <u>applied force</u> do to push q 'up' this electric potetial ?

$W_{app} = -W$ = work that the 'applied' force, aka, <u>YOU</u>, must do to push q from $i \to f$.

$\Rightarrow$ $\boxed{W_{app} = \Delta U = U_f - U_i = q\,\Delta V}$

the relationship between work done by you (or some 'applied force') and

ΔU = change in PE and ΔV = potential difference.

An electron-volt (eV):

1 eV = work required to move one |electron| across a PD of 1 volt:

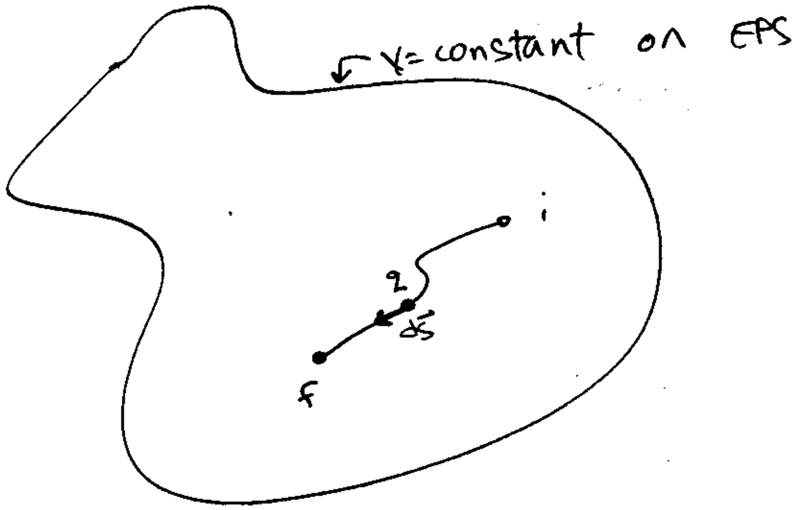
$$W = 1eV = (1e)\cdot(1\ \text{volt})$$

$$= (1.6\times10^{-19}\ C)(1\ V = 1\ J/C)$$

$$= 1.6\times10^{-19}\ J$$

Equipotential Surface

an equipotential surface (EPS) = adjacent sites where potential V = constant, or $\Delta V = 0$



Suppose you want to move charge q around an EPS. How much work will this cost you?

$$W = q\,\Delta V = 0$$

Recall that
$$W = -\int_i^f \vec{F} \cdot d\vec{s} = -q \int_i^f \vec{E} \cdot d\vec{s}$$

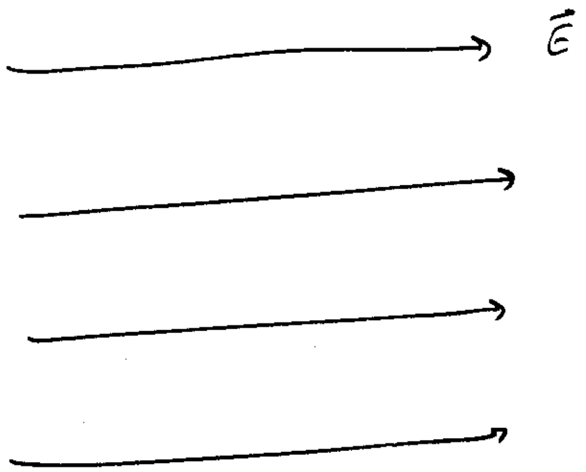
sign indicates this is the work you do to move q about $\vec{E}$-field.

Note that this line-segment $d\vec{s}$ lies in the EPS

since $W = 0 \Rightarrow$ all path segments $d\vec{s}$ are perpendicular to $\vec{E}$

$\Rightarrow$ The $\vec{E}$-field lines are perpendicular to the EPS

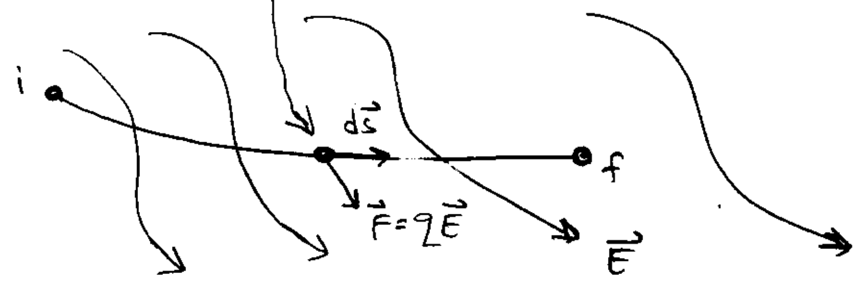
Draw the EPS for a uniform field $\vec{E}$:



Ditto for an isolated charge $q>0$:

Calculate the potential V from field $\vec{E}$:

suppose test-charge q moves across $\vec{E}$ from $i \to f$:



$$W = \int_i^f \vec{F} \cdot d\vec{s} = q \int_i^f \vec{E} \cdot d\vec{s}$$

= work done on q by $\vec{E}$

$= -W_{app}$ = $-1 \times$ work done by you/applied force

$= -q\,\Delta V$

so

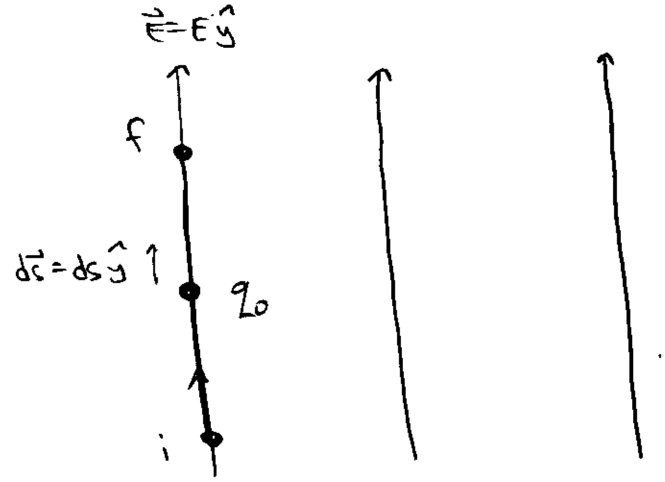
$$\Delta V = V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}$$ = PD between sites i & f

Note that we can choose $V_i = 0$ without altering the physics.

often we set $V_i = 0$ for when q is at "infinity", ie, far away.

Sample Problem 25-2

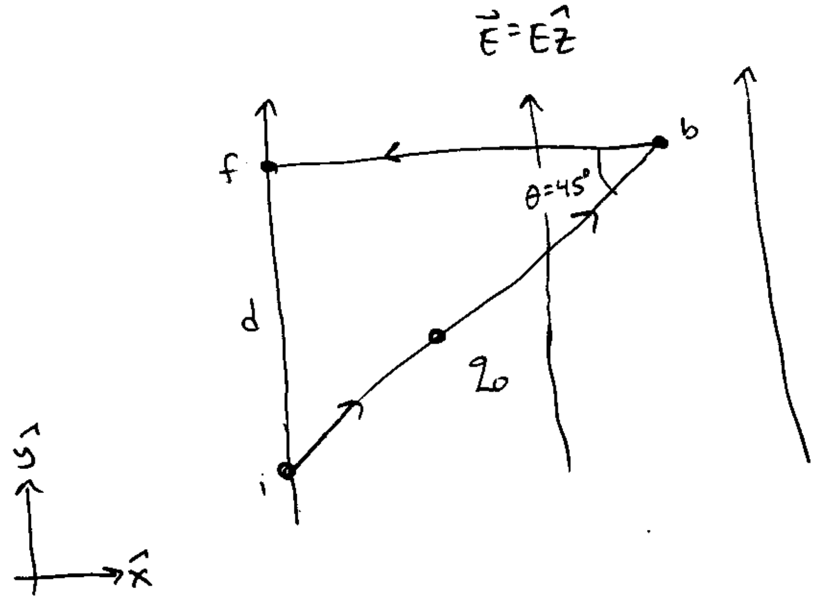
a.) Consider a uniform $\vec{E}$-field $E = E\hat{y}$. Let charge q_0 travel a distance d along a field line. What is the PD $\Delta V = V_f - V_i$?



$$\Delta V = V_f - V_i = -\int_i^f \underbrace{\vec{E}\cdot d\vec{s}}_{\vec{E}\cdot d\vec{s} = E\,ds} = -E\int_0^d ds = -Ed$$

b.) suppose instead q_0 moves along this path

What is ΔV?



Confirm that $\Delta V = -Ed$:

what is $\vec{E}\cdot d\vec{s}$ along this path segment

$$\Delta V = -\int_i^b \vec{E}\cdot d\vec{s} \quad - \underbrace{\int_b^f \vec{E}\cdot d\vec{s}}$$

$$d\vec{s} = \frac{ds}{\sqrt{2}}(\hat{x}+\hat{y})$$

$$\text{so } \vec{E}\cdot d\vec{s} = \frac{1}{\sqrt{2}} E\, ds$$

$$\text{so } \quad \Delta V = -\frac{E}{\sqrt{2}} \underbrace{\int_i^b ds}_{\text{what is this?} \quad \sqrt{2}d}$$

so $\Delta V = -Ed$, as expected

$\Rightarrow$ the calculation of the potentials V and ΔV are independant of the selected path

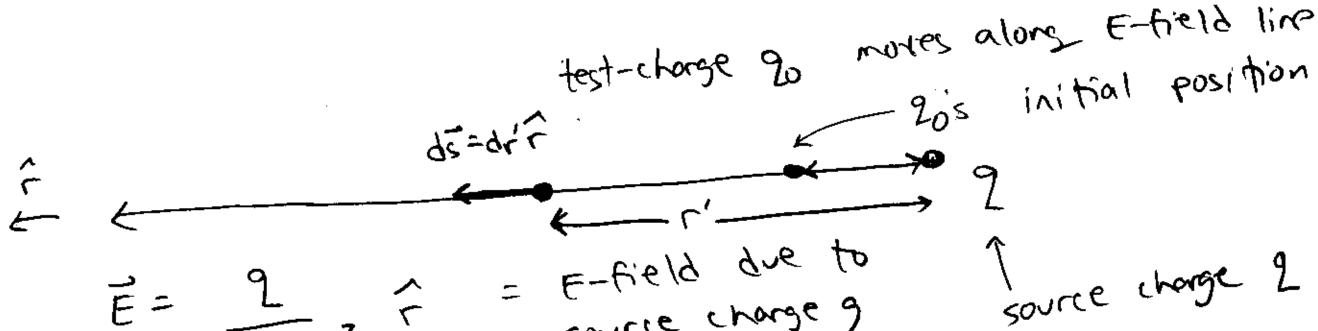
This is because $\Delta V = \frac{W_{app}}{q}$

where W_{app} = applied work (ie, work you must do on q) is also path independant.

Potential Due to a Point Charge

Calculate the potential V due to charge q while at a distance r away

Solve this by placing a test-charge q_0 a distance r' away, and letting it move radially away and off to infinity along a single field line:



$\vec{E} = \dfrac{q}{4\pi\varepsilon_0 r'^2}\,\hat{r}$ = E-field due to source charge q

$$\Delta V = V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{s}$$

$$\vec{E}\cdot d\vec{s} = \frac{q\,dr'}{4\pi\varepsilon_0 r'^2}$$

so
$$V_f - V_i = -\frac{q}{4\pi\varepsilon_0}\int_r^\infty \frac{dr'}{r'^2}$$

$V_f = V$ at final position, @ $r' = \infty$, where $V_f = 0$ by convention so $\quad V_i = \frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{r'}\right)\Big|_r^{\infty} = \frac{-q}{4\pi\varepsilon_0}\left(0-\frac{1}{r}\right)$

$\Rightarrow V(r) = \frac{q}{4\pi\varepsilon_0 r}$ = electric potential at a distance r away from charge q.

I dropped the i-subscript

what is V when $r \to \infty$?

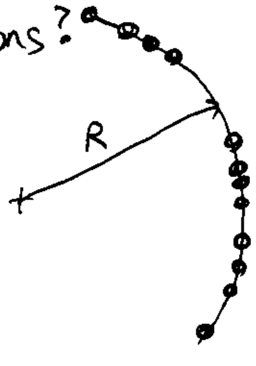
Multiple Point Charges

what if I had N multiple charges q_i, all at distances r_i away?

$$V = \sum_{i=1}^{N} V_i = \frac{1}{4\pi\varepsilon_0}\sum_{i=1} \frac{q_i}{r_i}$$

Sample Problem 25-4

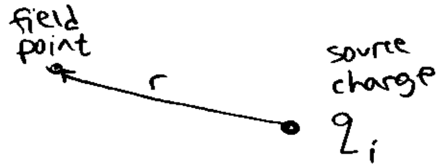
Suppose I had 12 electrons that were placed along an arc. What is V at the spot that is equidistant from the electrons?



$$V = \frac{-12e}{4\pi\varepsilon_0 R}$$

where $-e$ = charge on an electron.

Potential due to an Electric Dipole:

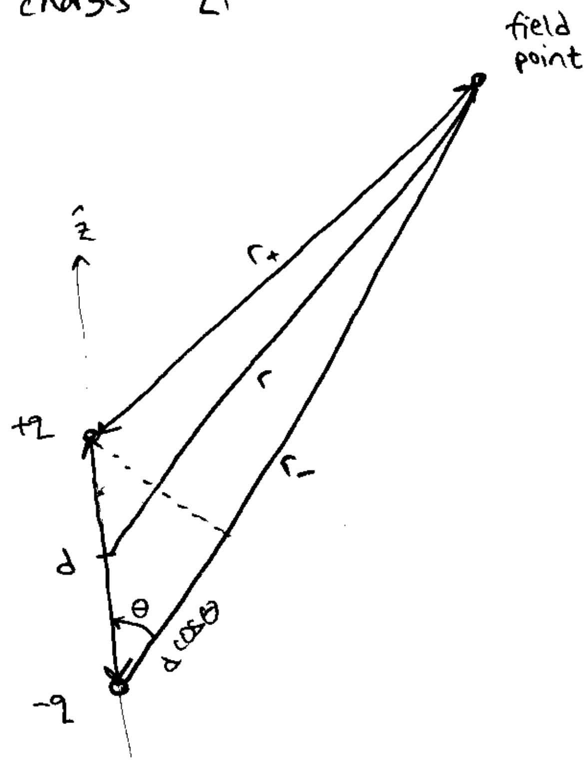


recall that $V_i = \dfrac{q_i}{4\pi\varepsilon_0 r}$

= potential due to charge q_i

and that $V = \sum_{i=1}^{n} V_i$ = total potential due to charges q_i

Consider a dipole:



$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{r_- - r_+}{r_- r_+}\right)$$

but note that $r_- - r_+ \simeq d\cos\theta$

and $r_- r_+ \simeq r^2$

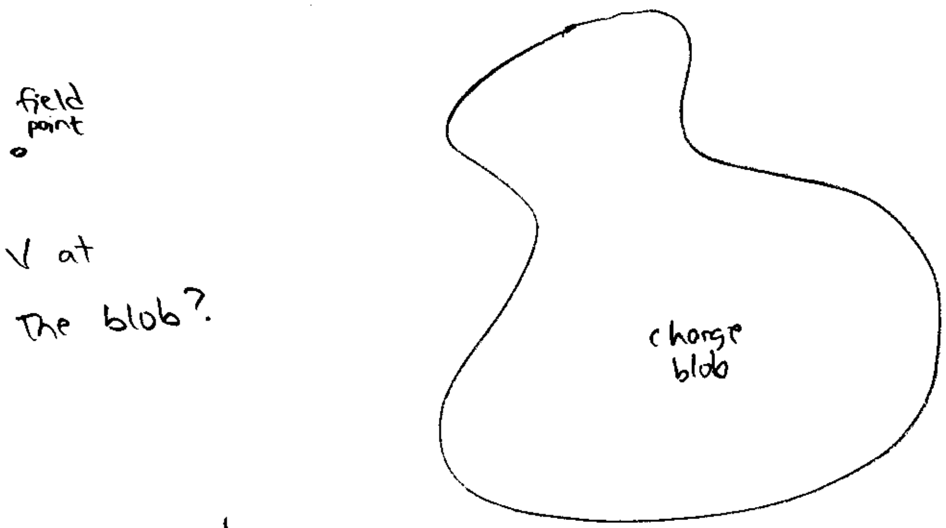
$\Big\}$ valid when $r \gg d$ (ie, far from dipole)

So $V = \dfrac{qd\cos\theta}{4\pi\varepsilon_0 r^2} = \dfrac{p\cos\theta}{4\pi\varepsilon_0 r^2}$ = potential far from dipole.

where $p = qd$ = dipole moment

Potential Due to Continous Charge Distribution

Suppose we have a blob of charge:



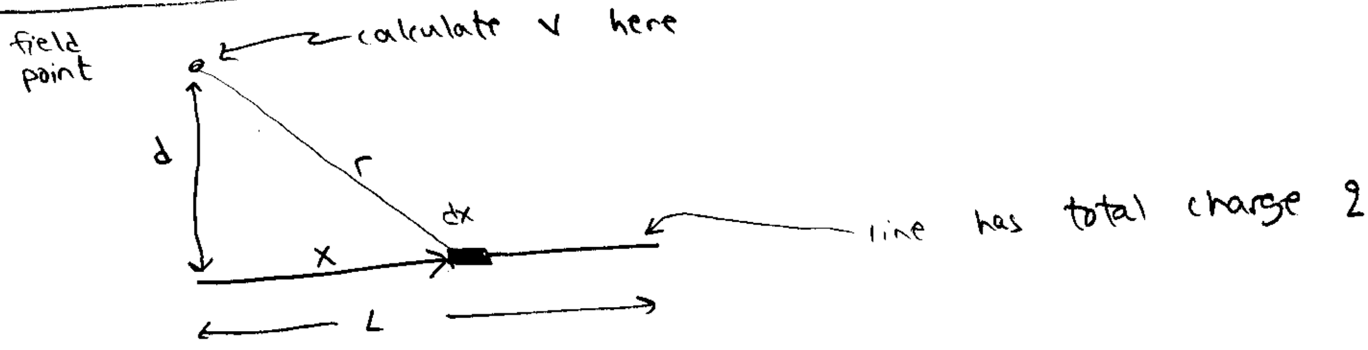
How do you calculate V at the field point due to the blob?

Break the blob up into lots of little charges dq having potentials $dV = \dfrac{dq}{4\pi\varepsilon_0 r}$

How do you calculate the total potential V?

$$V = \int dV = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}.$$

Example: V due to a line of charge:



What do I do next?

Break line up into lots of little charges

$dq = \lambda\, dx$ where $\lambda = \frac{q}{L}$ = linear charge density

so $dV = \frac{dq}{4\pi\varepsilon_0 r} = \frac{\lambda\, dx}{4\pi\varepsilon_0 \sqrt{d^2+x^2}}$ = potential due to charge dq.

The total potential is

what are my integration limits?

$$V = \int dV = \frac{\lambda}{4\pi\varepsilon_0} \underbrace{\int_0^L \frac{dx}{\sqrt{d^2+x^2}}}_{}$$

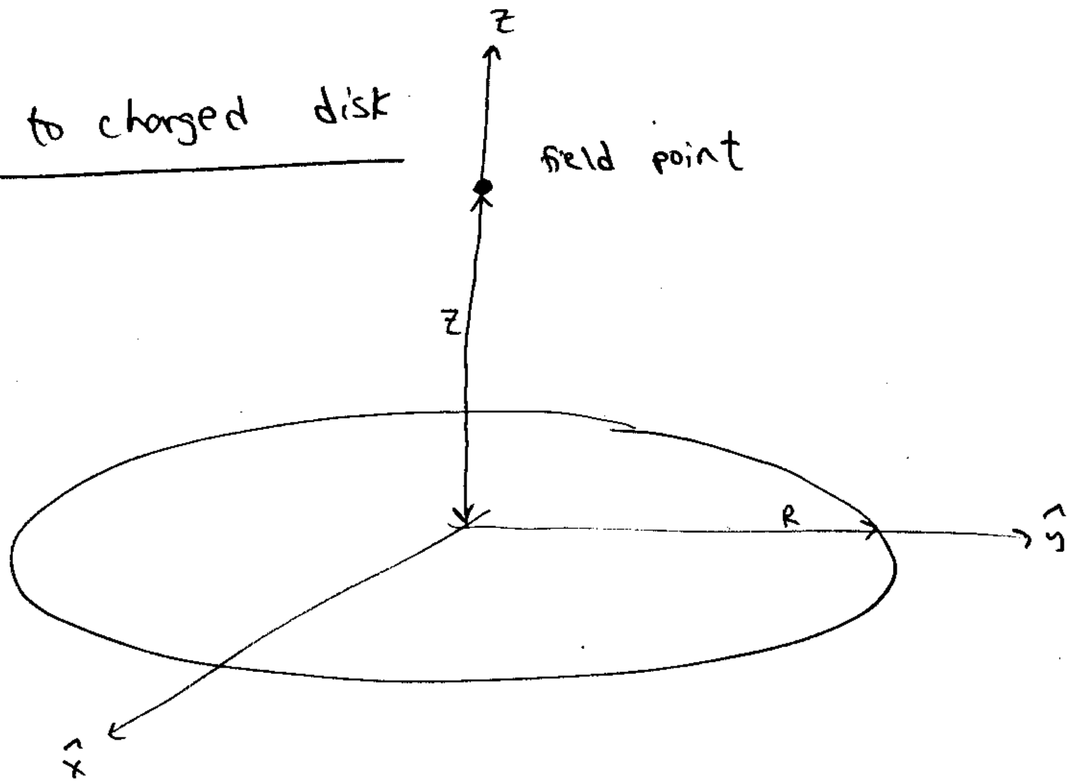
see Appendix E:

$$\ln\left(x+\sqrt{x^2+d^2}\right)\Big|_0^L = \ln\left(\frac{L+\sqrt{L^2+d^2}}{d}\right)$$

so $V = \dfrac{\lambda}{4\pi\varepsilon_0} \ln\left(\dfrac{L + \sqrt{L^2+d^2}}{d} \right)$

= potential at field point due to line of charge

V due to charged disk



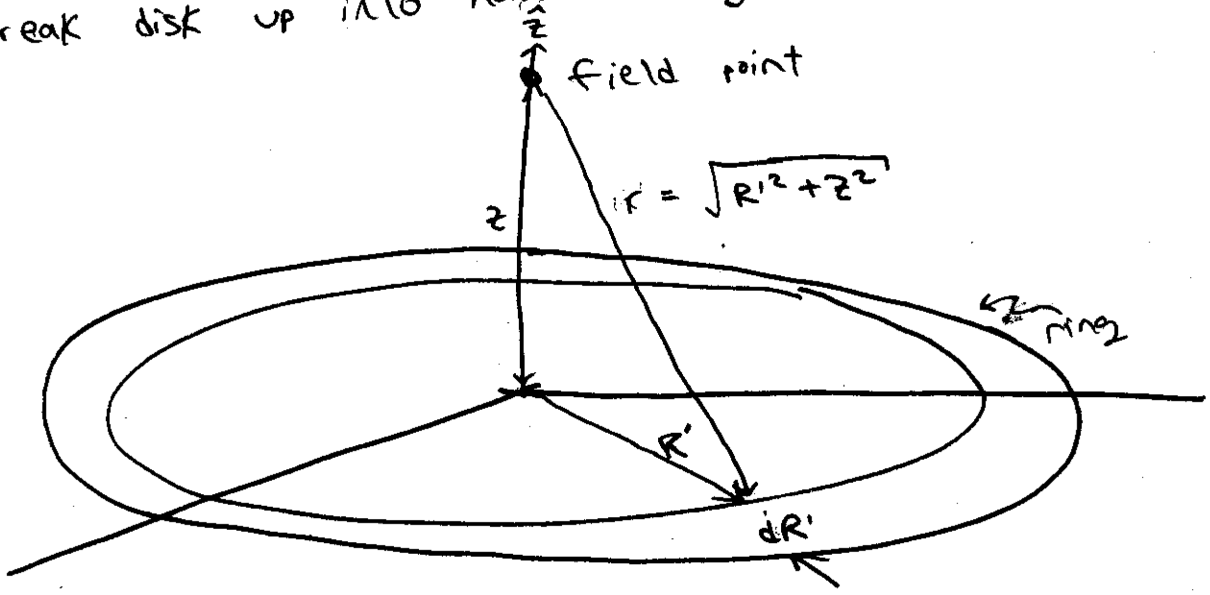
Disk has radius R and charge q.

What is σ = surface charge density?

$= \dfrac{q}{\pi R^2}$

What do I do next? What did we do when we calculated $\vec{E}$ due to disk?

Break disk up into narrow rings of radius R' with dR'



$$dV = \frac{dq}{4\pi\varepsilon_0 r} = \text{potential due to ring}$$

note that the ring is equidistant from the field point - very useful!

What is dq = total charge on ring?

$= \sigma \times (\text{ring area})$

$= \sigma \cdot \underbrace{2\pi R'}_{\text{ring circumference}} dR'$ ← ring width

So $dV = \dfrac{2\pi\sigma R'\,dR'}{4\pi\varepsilon_0\sqrt{R'^2+z^2}}$ = potential due to ring along z-axis

and $V = \int dV = \dfrac{\sigma}{2\varepsilon_0}\int \dfrac{R'\,dR'}{\sqrt{R'^2+z^2}}$

what are my integration limits

$$V = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2} - z\right)$$ valid for $z \geq 0$

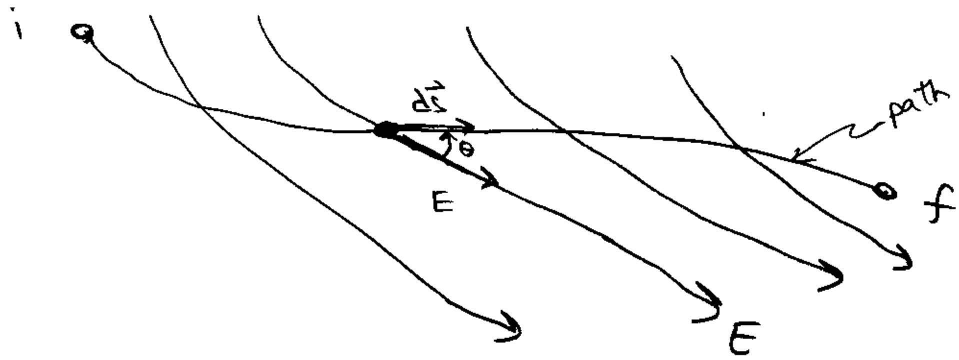
along $\hat{z}$ axis

when the integral is evaluated — you should check this.

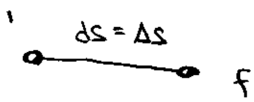
Calculating $\vec{E}$-field from V

Recall that $\Delta V = V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{s}$

is the relationship between $\vec{E}$ and $\Delta V = V_f - V_i$:



Note that when i and f are very close,



$\vec{E}$ is constant over small step $ds = \Delta S$

so $\vec{E}\cdot d\vec{s} = \underbrace{E\cos\theta}\,\Delta S$

where E_s = component of $\vec{E}$ parallel to $d\vec{s}$.

so $\Delta V = -E_s\,\Delta S$ for small steps

or $E_s = -\dfrac{\Delta V}{\Delta S} = -\dfrac{\partial V}{\partial s}$

the component of E in the direction of step $d\vec{s}$

In Cartesian coordinates, we would write this as

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

partial derivative,

ie $\frac{\partial V}{\partial x}$ = derivative of $V(x,y,z)$ w.r.t. x, with y & z held constant

Sample Problem 25-5

Recall $V = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2} - z\right)$ = potential along disk's $\hat{z}$-axis

Calculate $\vec{E}$ along z-axis.

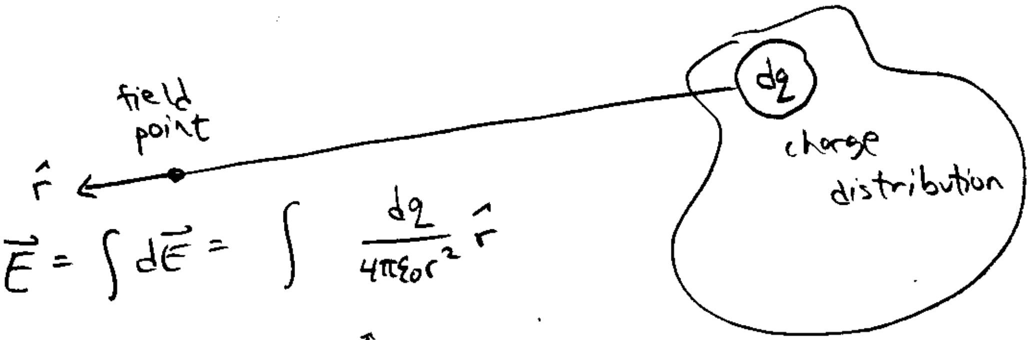
What is E_x? E_y?

so $E_z = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\varepsilon_0}\left[\frac{1}{2}(z^2+R^2)^{-1/2} \cdot 2z - 1\right]$

or $\vec{E} = +\frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)\hat{z}$

which recovers Eqn 23-26.

Thus we have 2 distinct ways to calculate $\vec{E}$ due to some charge distribution:



1: $\vec{E} = \int d\vec{E} = \int \frac{dq}{4\pi\varepsilon_0 r^2}\hat{r}$

which can be a laborious integral of a vector quantity

OR

2. Calculate potential $V = \int \frac{dq}{4\pi\varepsilon_0 r}$ first

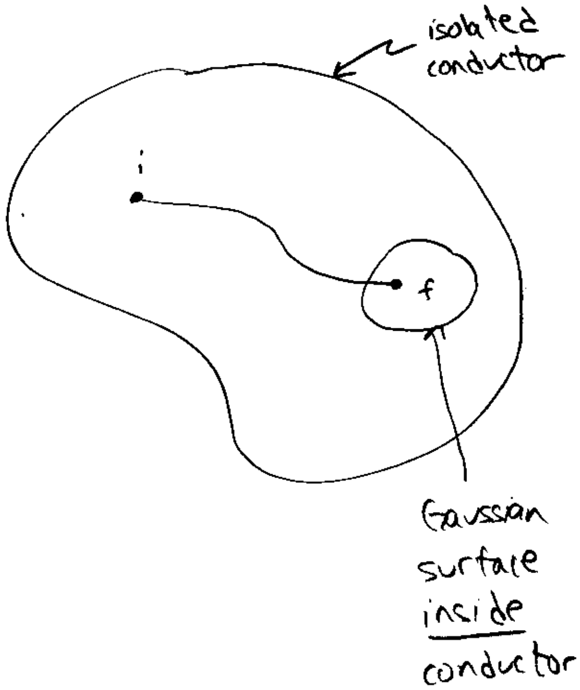
less laborious intergral over a scalar quantity.

Then use $E_x = -\frac{\partial V}{\partial x}$, $E_y = -\frac{\partial V}{\partial y}$, etc

to form $\vec{E}$. This can be a lot easier at times...

Potential of a Charged Conductor

what is the potential V inside an isolated conductor?



Use $V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}$

Now use Gauss' Law to calculate $\vec{E}$ inside conductor:

$$\varepsilon_0 \int \vec{E} \cdot d\vec{A} = q_{enc}$$

what is q_{enc}?

what does that say about $\vec{E}$ inside the conductor?

what does that tell us about

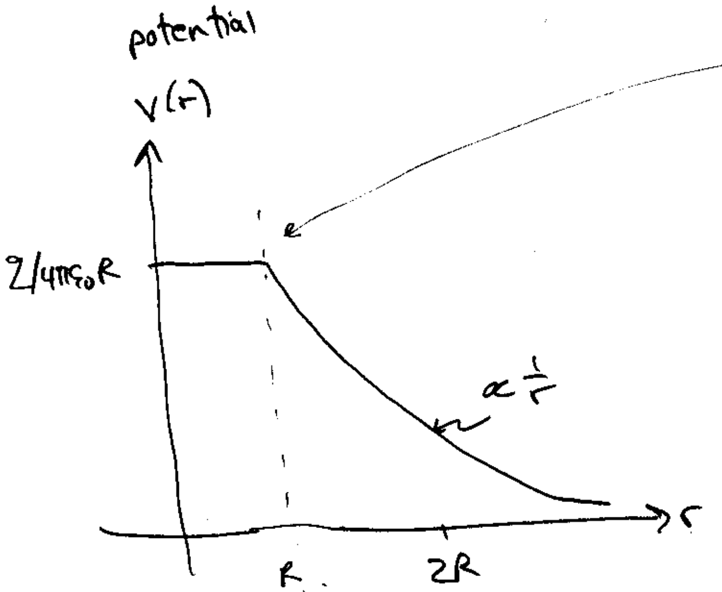
$\Delta V = V_f - V_i =$ potential inside conductor?

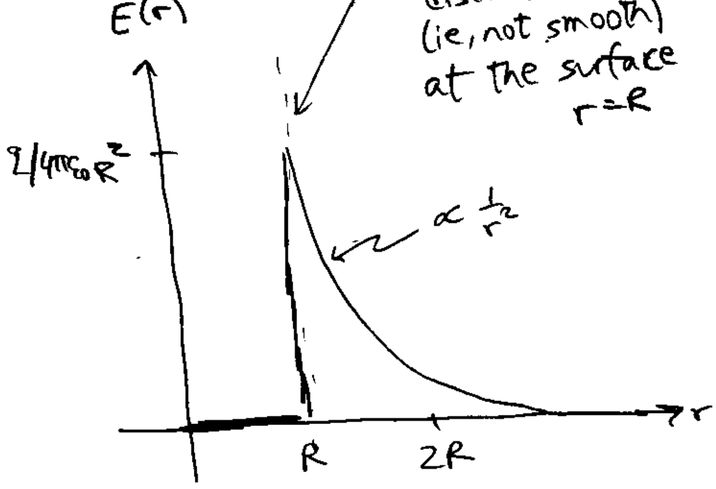
$\Delta V = 0 \Rightarrow$ potential $V =$ constant every where inside.

Example: plot $V(r)$ inside/outside a conducting shell of charge q:

$$V(r) = \begin{cases} \dfrac{q}{4\pi\varepsilon_0 r} & r \geq R \\ ? & r < R \end{cases}$$

$$|\vec{E}(r)| = \begin{cases} \dfrac{q}{4\pi\varepsilon_0 r^2} & r \geq R \\ ? & r < R \end{cases}$$

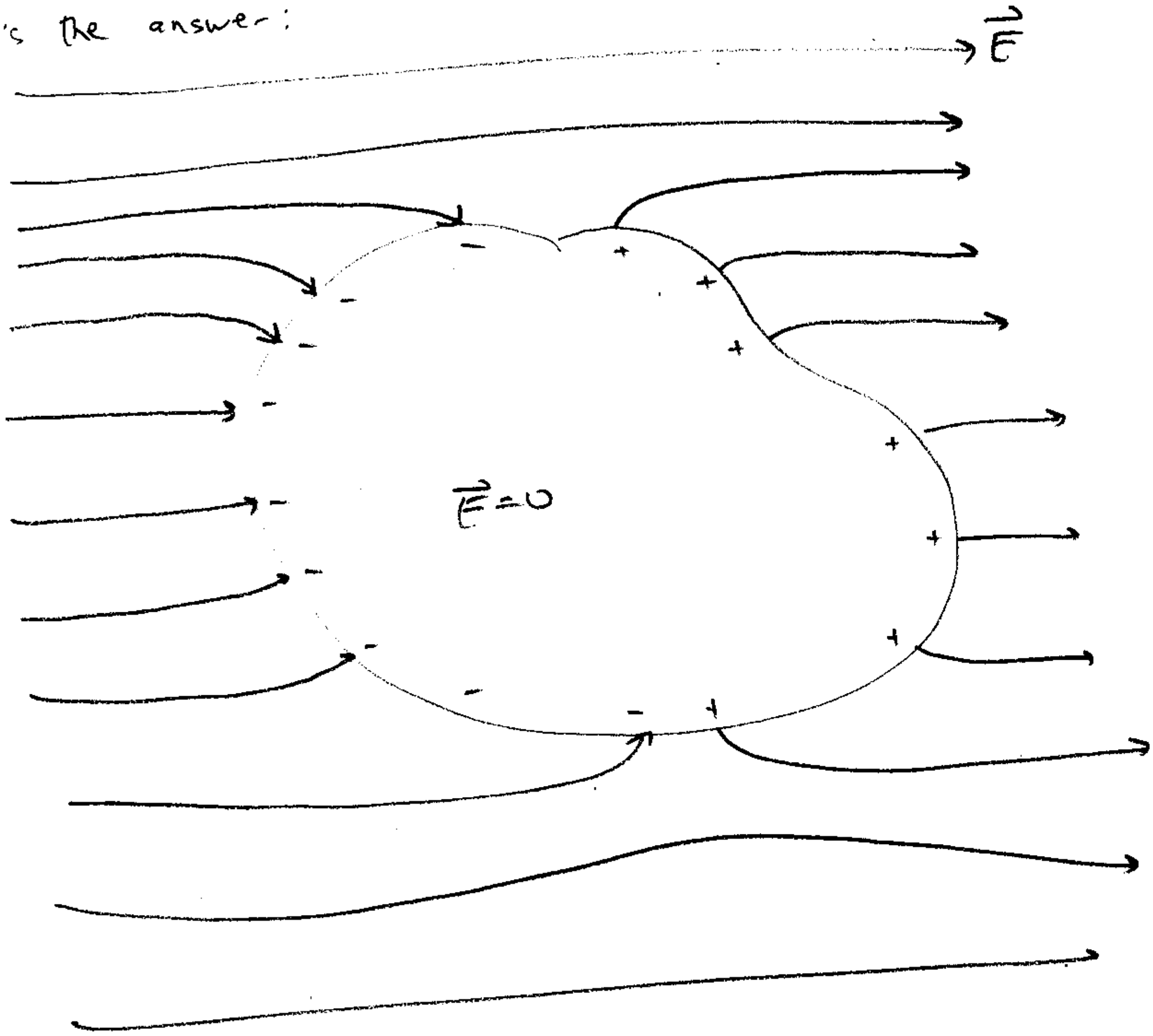




Note that $V(r)$ and $E(r)$ are discontinuous (ie, not smooth) at the surface $r=R$

Lastly, ... what if we placed an <u>uncharged</u> <u>conductor</u> inside an external $\vec{E}$-field. What is $\vec{E}$ inside the conductor ... and why?

Here's the answer:



Why is $\vec{E} = 0$ inside this conductor - despite the external field? Consider $\vec{F} = q\vec{E}$.

And why are there charges on this neutral conductor?