Chapter 24: Gauss' Law

See sections 1-9 of the text book

Gauss' Law (~1800 by Carl Friedrich Gauss)
provides an alternate formulation of
the electrostatic force law, $F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$

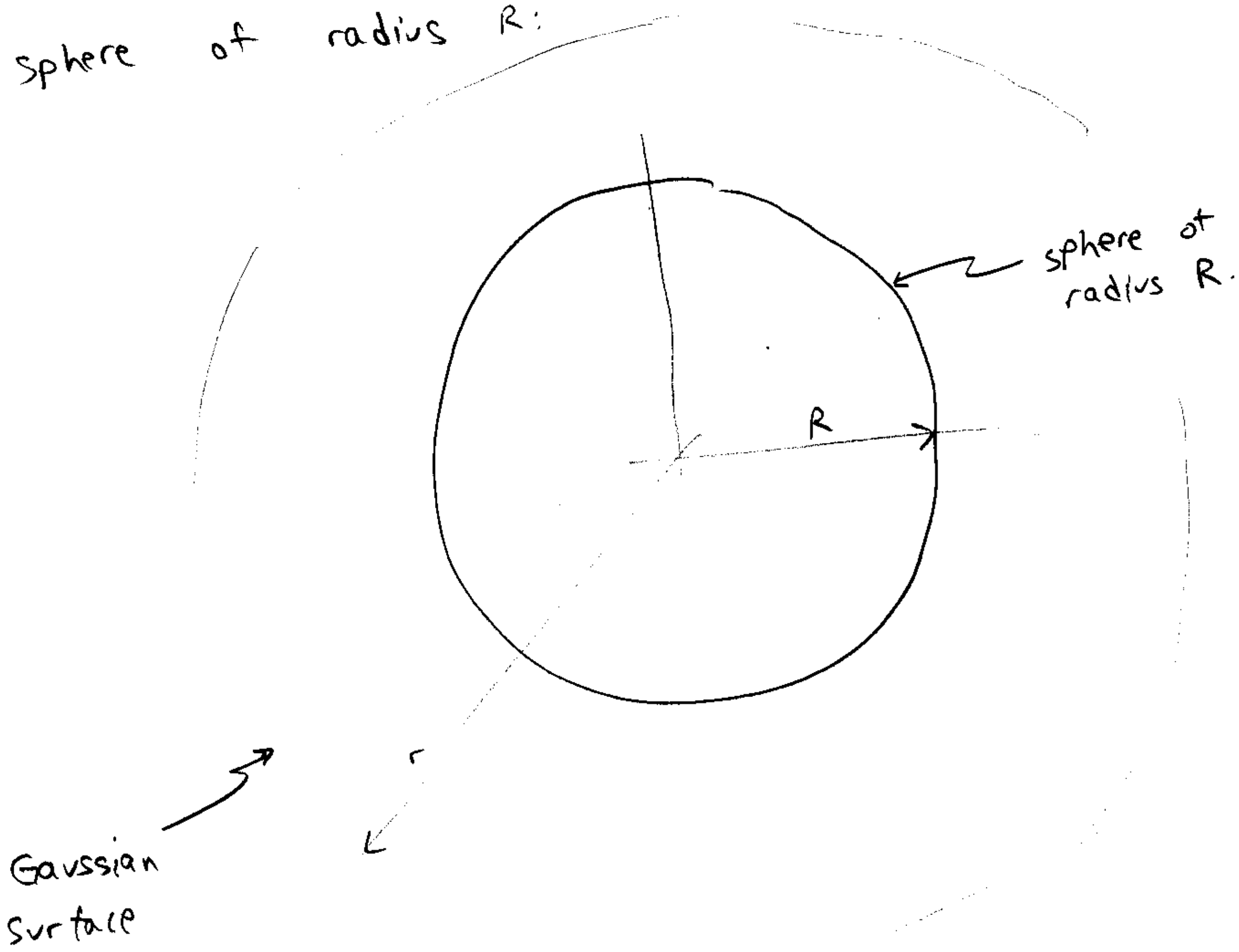
However it is very handy when dealing
with symmetric charge distributions, ie,
a charged sheet, sphere, cylinder, etc

To proceed, we need to introduce two concepts:

i) Gaussian surface: an imaginary surface,
designed to help you solve the problem
at hand.

It: must be closed,
can extend to infinity,
might contain charge.

Suppose we were dealing with a charged sphere of radius R:



We would likely adopt a spherical Gaussian surface of radius r,

where r might be $r > R$ (ie, the charged sphere is enclosed by the Gaussian surface)

<u>or</u> $r < R$ (ie, the charged sphere is only partially enclosed)

ii) <u>The electrostatic flux through a Gaussian surface</u>

The electric flux through some Gaussian surface S is

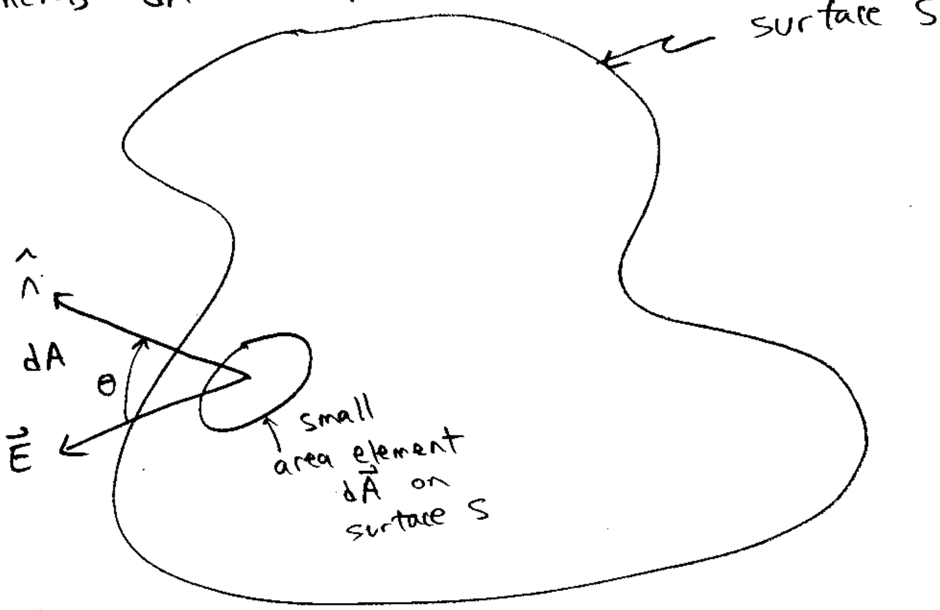
$$\Phi = \int_S \vec{E} \cdot d\vec{A} \quad = \text{electric flux (or just 'flux')}$$

what does this mean?

1. invent a Gaussian surface S that encloses some $\vec{E}$-field
2. break it up into lots of little area elements $d\vec{A} = dA\,\hat{n}$,

where $|d\vec{A}|$ = area

$\hat{n}$ = unit vector that points normal (ie, perpendicular) to area element dA



the <u>integrand</u> is

$\vec{E} \cdot d\vec{A} = \underbrace{E\cos\theta}_{\text{component}}\,dA$ of $\vec{E}$ that is <u>perpendicular</u> to surface S

= "flux" of E-field lines passing thru area dA.

3. so the total flux is

$$\Phi = \int_S \vec{E} \cdot d\vec{A} = \text{sum of electric flux that passes thru surface } S$$

integrate over surface S

also written as $\Phi = \oint \vec{E} \cdot d\vec{A}$

means: integrate over a <u>closed</u> surface

Note that Φ is proportional to the number of field lines that pass through surface S.

<u>Gauss' Law</u>:

$$\varepsilon_0 \Phi = q_{enc}$$

total charge that is enclosed by the Gaussian surface S.

ie,

$$\varepsilon_0 \int_S \vec{E} \cdot d\vec{A} = q_{enc}$$

(you text uses $\oint$ instead of $\int_S$)

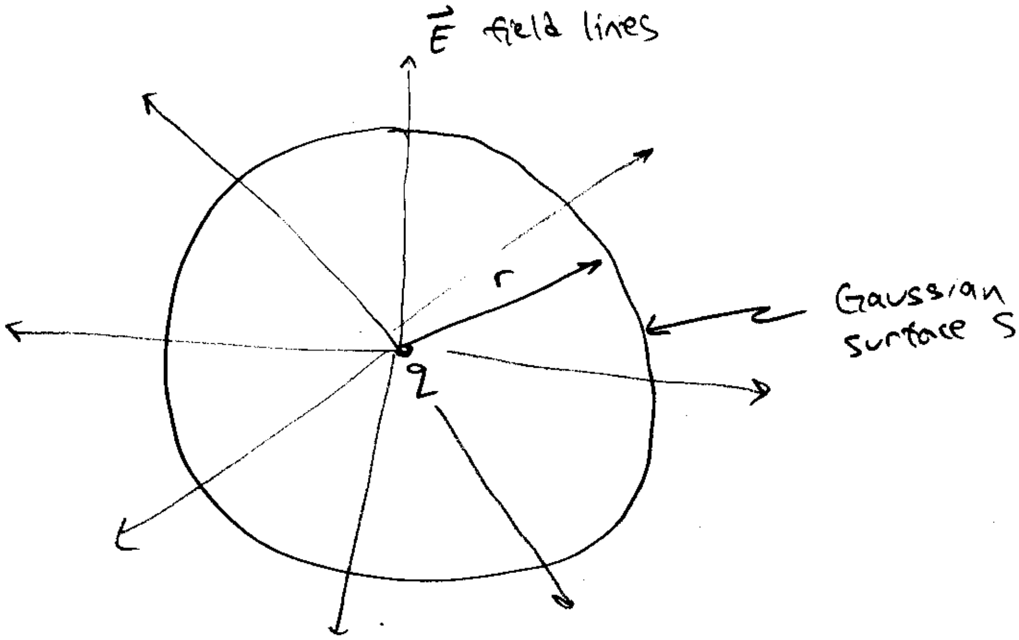
Gauss' Law is an <u>alternate</u> way of writing the Electrostatic / Coulomb force law.

Consider a point charge q.

We want to solve for the $\vec{E}$-field.

To solve for $\vec{E}$,

Use a Gaussian sphere, centered on the charge:



Gauss' Law: $\varepsilon_0 \underbrace{\oint \vec{E} \cdot d\vec{A}}_{\text{can we simplify this any?}} = q$

Note that $|\vec{E}|$ = constant on sphere,

and that $d\vec{A} = dA \cdot \hat{n}$ is parallel to $\vec{E}$

everywhere on sphere, so

$$\vec{E} \cdot d\vec{A} = E\,dA$$

and Gauss' law becomes

$$\varepsilon_0 E \underbrace{\oint dA}_{\text{what is this ?}} = q$$

$\oint dA$ = total area of sphere $= 4\pi r^2$

$\Rightarrow \quad E = \dfrac{q}{4\pi\varepsilon_0 r^2}$ = electric field due to point charge q.

Which is the electric field you would have obtained had you started with electrostatic force law...

Note: If you want to use Gauss' Law to solve for $\vec{E}$:

- Find the Gaussian surface over which $\vec{E}$ is a constant, so that you can easily the E-flux through the surface:

$$\varepsilon_0 \Phi = q_{enc}$$ Gauss' Law

$$\Phi = \int_S \vec{E} \cdot d\vec{A} = \text{flux}$$

If successful, you will find that

$$\int_S \vec{E} \cdot dA = E \cdot \text{Area}$$

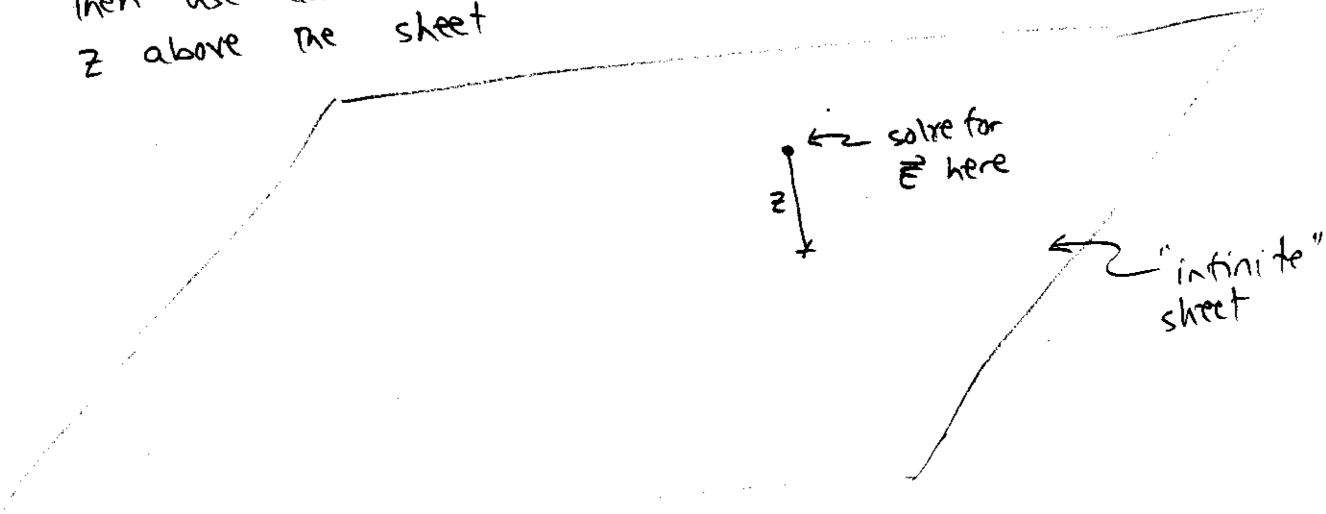
E-field on surfaces

so $$E = \frac{q_{enc}}{\varepsilon_0 \cdot \text{Area}}$$

Gauss' Law & an Infinite Sheet of Charge

solve for the $\vec{E}$-field due to an infinite sheet having constant surface charge density $\sigma =$ charge/area

Then use Gauss' Law to solve for $\vec{E}$ at some spot z above the sheet



which way are the $\vec{E}$-field lines pointing?

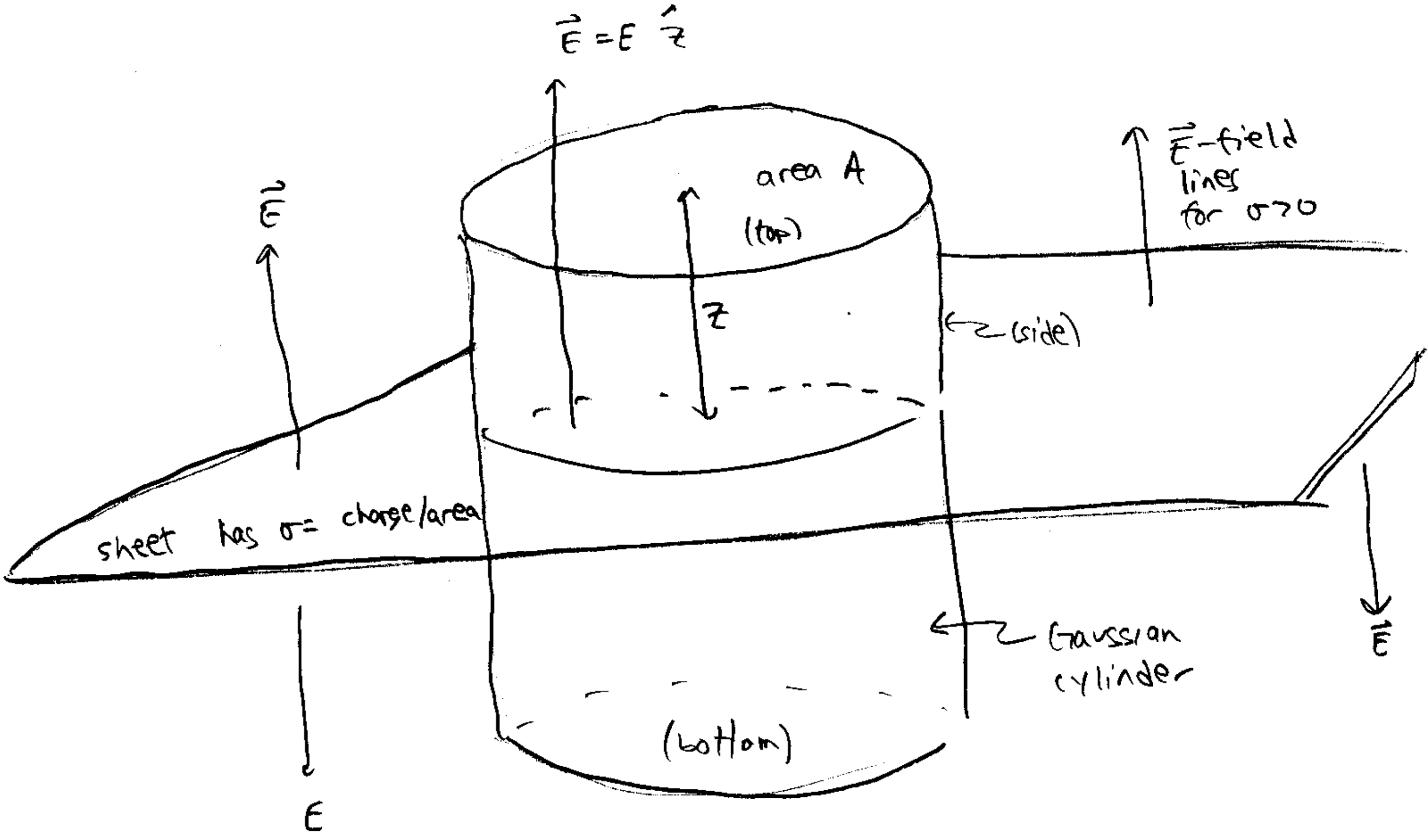
we anticipate $\vec{E} = s E \hat{z}$ where $s = \text{sign}(z) = \pm 1$. why?

Gauss' Law: $\varepsilon_0 \Phi = \varepsilon_0 \int \vec{E} \cdot d\vec{A} = q_{enc}$

Next: select your Gaussian surface S.

Choose S so that the integral is easy,
preferably so that the integrand
$\vec{E} \cdot d\vec{A}$ = a constant (if possible)

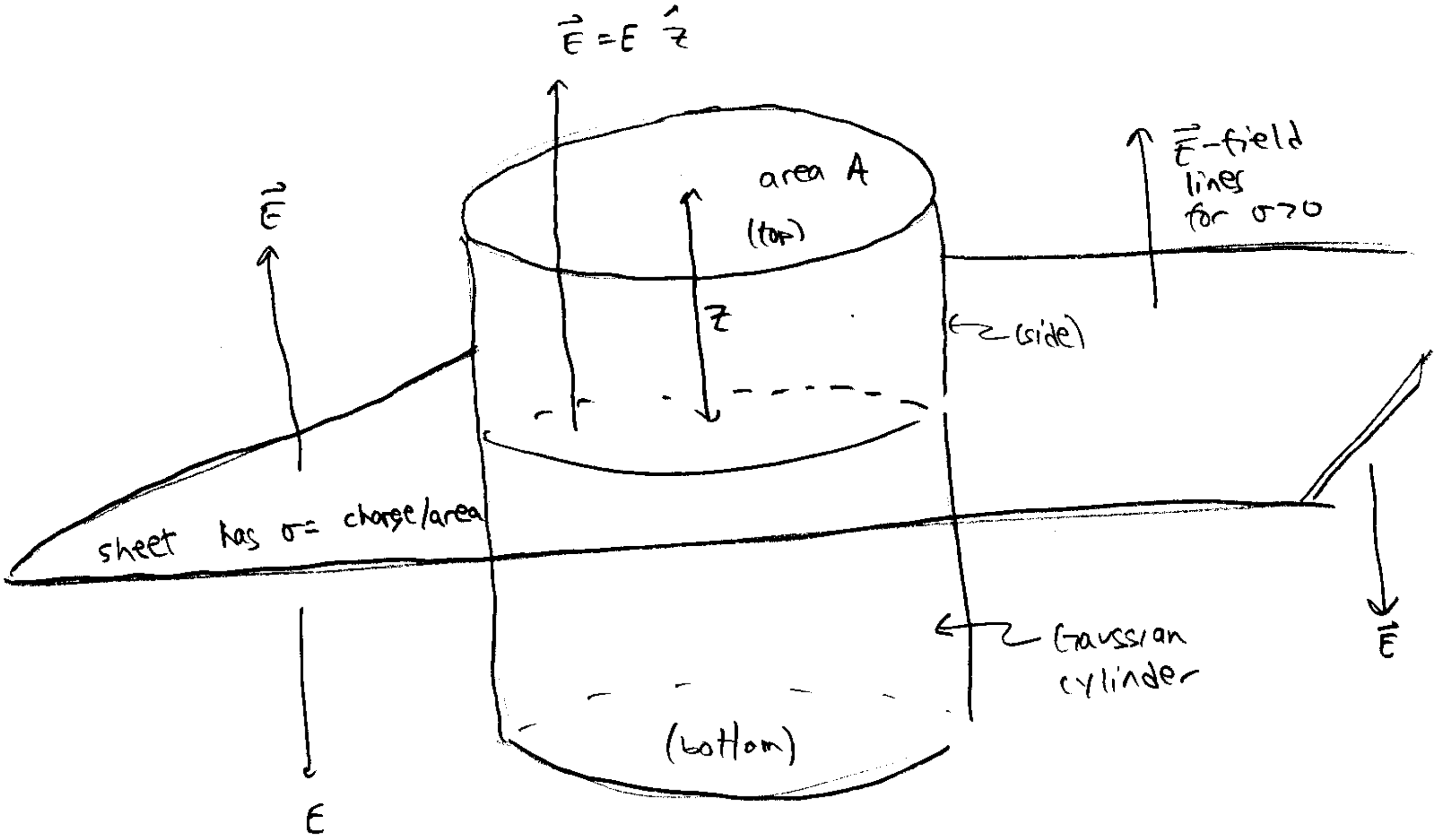
Try a Gaussian cylinder that is perpendicular to the sheet, and extends $\pm z$ above/below the sheet, and has area A on its ends



What is q_{enc} = charge enclosed by surface?

$$= \sigma A$$

Try a Gaussian cylinder that is perpendicular to the sheet, and extends $\pm z$ above/below the sheet, and has area A on its ends



What is q_{enc} = charge enclosed by surface?

$= \sigma A$

Now calculate the $\vec{E}$-flux thru the Gaussian cylinder:

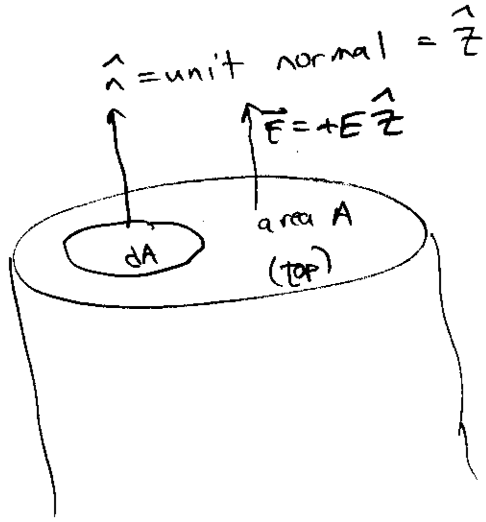
$$= \Phi_{top} + \Phi_{bottom} + \Phi_{side}$$

$$\Phi = \int \vec{E}\cdot d\vec{A} = \underbrace{\int_{top} \vec{E}\cdot d\vec{A} + \int_{bottom} \vec{E}\cdot d\vec{A} + \int_{side} \vec{E}\cdot d\vec{A}}_{\text{contribution to } \Phi \text{ from the cylinders' 3 parts.}}$$

where $d\vec{A} = dA \cdot \hat{n}$

size of area element

unit normal to small area, <u>always</u> points exterior to Gaussian surface



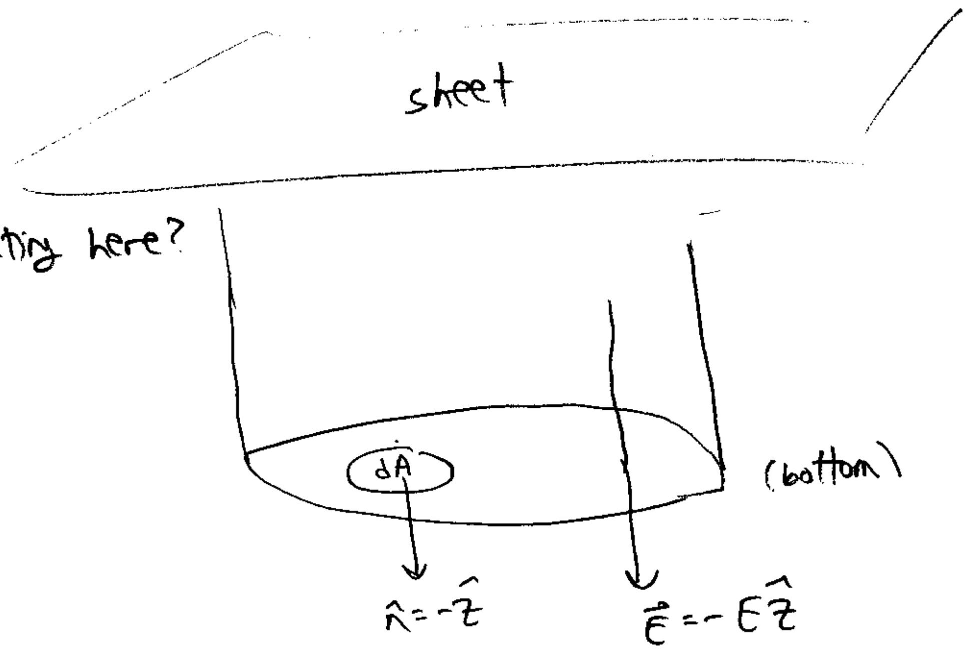
so $\vec{E}\cdot d\vec{A} = (E\,\hat{z})\cdot(dA\,\hat{z})$

$= E\,dA$ at cylinder's top end

and $\int_{top} \vec{E}\cdot d\vec{A} = \int_{top} E\,dA = E\int dA = EA$

constant

Now consider the bottom of cylinder:



which way is $\vec{E}$ pointing here?

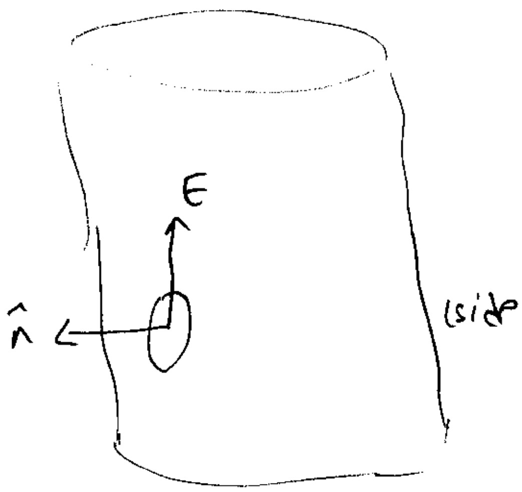
$$\vec{E} = -E\hat{z}$$

And $d\vec{A}$?

$$d\vec{A} = -dA\hat{z}$$

so $\vec{E} \cdot dA = (-E\hat{z}) \cdot (-dA\hat{z}) = E\,dA$

and $\int_{bottom} \vec{E} \cdot dA = EA$



What about the side?

$$\int_{side} \vec{E} \cdot d\vec{A} = ?$$

= zero since $\vec{E} \cdot d\vec{A} = 0$

since $\vec{E}$ and $\hat{n}$ are perpendicular

The total flux through this cylinder is

$$\Phi = \Phi_{top} + \Phi_{bottom} + \Phi_{side}$$

$$= EA + EA + 0$$

$$= 2EA$$

According to Gauss Law:

$$\varepsilon_0 \Phi = \varepsilon_0 2EA = q_{enc} = \sigma A$$

$$\Rightarrow E = \frac{\sigma}{2\varepsilon_0} =$$ magnitude E-field due to infinite sheet of charge, as expected.

Usually, Gauss' Law merely tells you the magnitude of $\vec{E}$, not its direction.

You still have to use your head to get the direction.

For example: Suppose $\sigma < 0$. Which way is $\vec{E}$ pointing in the $z < 0$ region below the sheet?

the $+\hat{z}$ direction

$\vec{E}$-field due to an infinite line of charge :

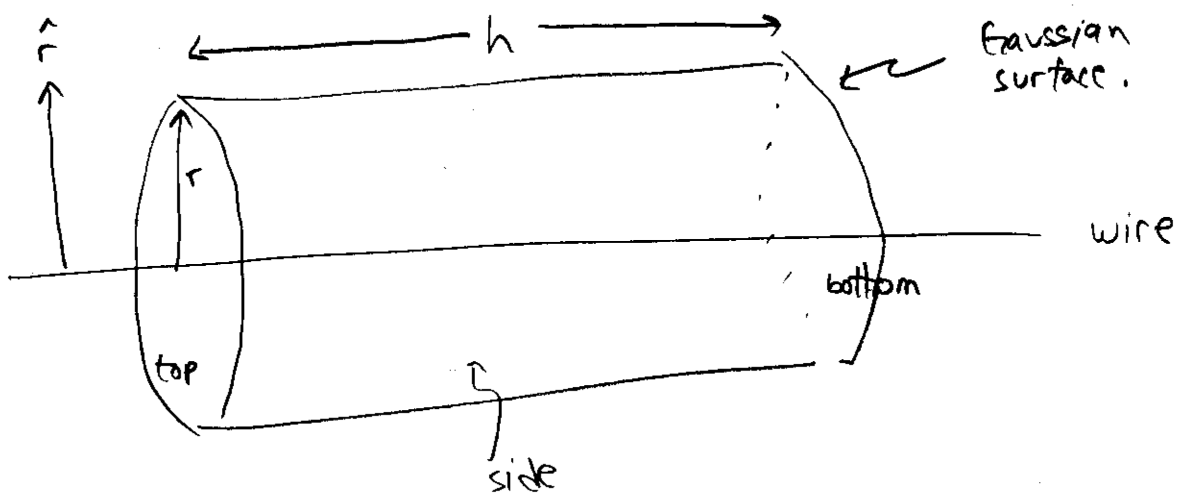
wire has λ = charge/length = linear charge density

what does the $\vec{E}$-field look like ? $\vec{E} = E(r)\hat{r}$

solve for $\vec{E}$ using Gauss' Law.

What Gaussian surface should I use?

A cylinder of radius r, length h :



Gauss Law: $\varepsilon_0 \Phi = q_{enc}$.

what is q_{enc}?

$q_{enc} = \lambda h$

Also, $\Phi = \int \vec{E} \cdot d\vec{A} = \Phi_{top} + \Phi_{bottom} + \Phi_{side}$

what is Φ_{top}?

Φ_{bottom}?

$\Phi_{side} = \int_{side} \vec{E} \cdot d\vec{A}$

integrate over cylinder's 'side'

what is $\vec{E} \cdot d\vec{A}$? $\quad d\vec{A} = dA\,\hat{r}$

so $\vec{E} \cdot d\vec{A} = (E(r)\hat{r}) \cdot (dA\,\hat{r})$

$= E(r)\,dA$

and $\Phi_{side} = \int_{side} E\,dA = E \int dA = EA$

where A = area of cylinder's side

$= ?$

$A = 2\pi r \cdot h$

$\Rightarrow \Phi = E \cdot 2\pi r h$

And Gauss' Law is

$$\varepsilon_0 \Phi = \varepsilon_0 E \cdot 2\pi r h = q_{enc} = \lambda h$$

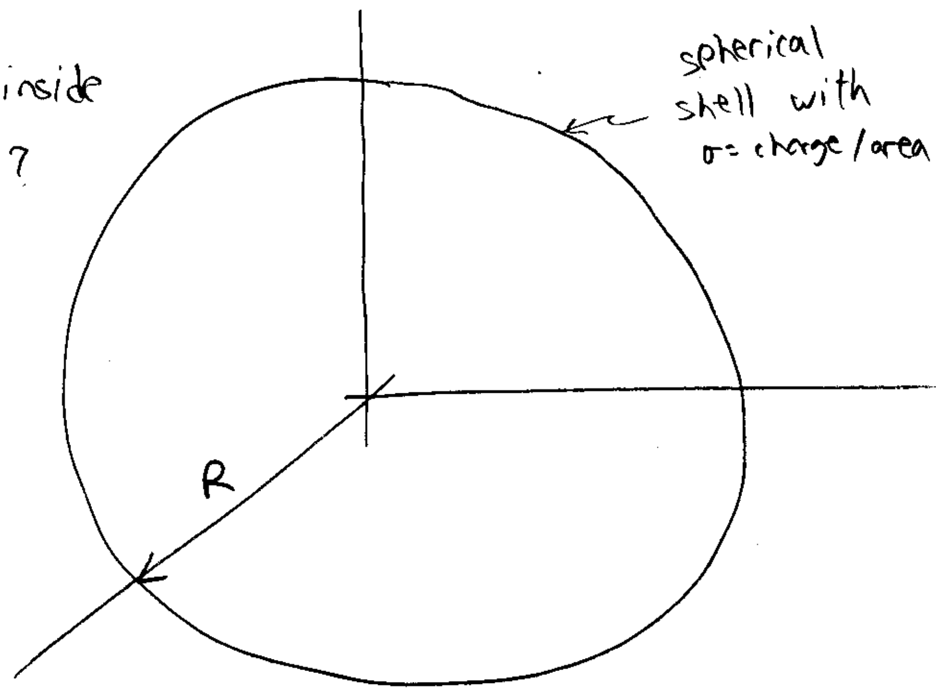
$$\text{so} \quad E = \frac{\lambda}{2\pi \varepsilon_0 r}$$

Which way are the field lines pointing if $\lambda < 0$?

Uniform Spherical Shell of Charge

Use Gauss' Law to calculate $\vec{E}$ due to a spherical shell of radius R having a uniform surface charge density σ:

What is $\vec{E}$ inside & outside shell?

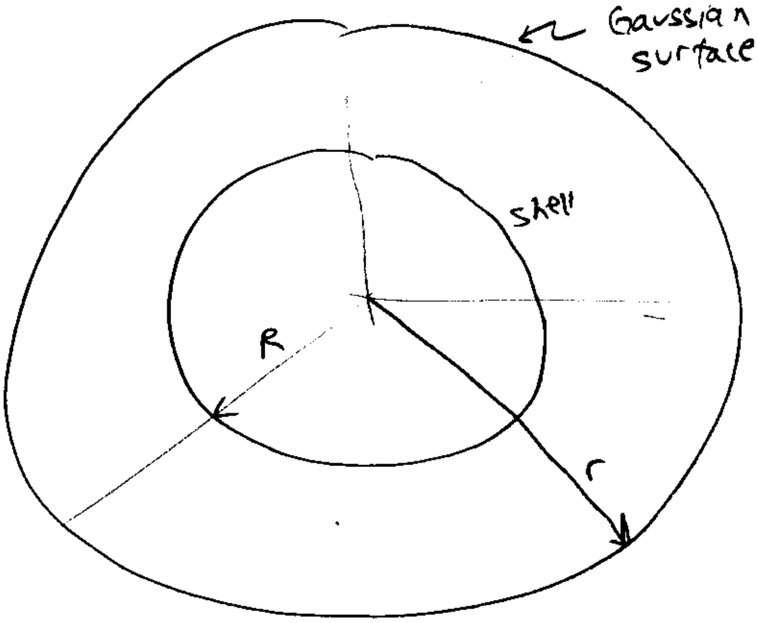


What Gaussian surface should I adopt?

How are the $\vec{E}$-field lines oriented outside the shell?

we anticipate $\vec{E} = E(r)\hat{r}$ due to the spherical symmetry.

calculate $\vec{E}$ outside the shell:



Gauss' Law: $\varepsilon_0 \Phi = q_{enc}$

what is q_{enc}?

$q_{enc} = 4\pi R^2 \sigma$

what is the flux?

$\Phi = \int \vec{E} \cdot d\vec{A} = E(r)\, 4\pi r^2$

Gauss' Law: $\varepsilon_0 \Phi = \varepsilon_0 E\, 4\pi r^2 = q_{enc}$

so $E = \dfrac{q_{enc}}{4\pi \varepsilon_0 r^2} = \dfrac{\sigma}{\varepsilon_0}\left(\dfrac{R}{r}\right)^2$

is this consistent with the 'outside shell theorem'?

What does the 'inside shell theorem' say about the $\vec{E}$-field inside the shell?

Check with Gauss' Law:

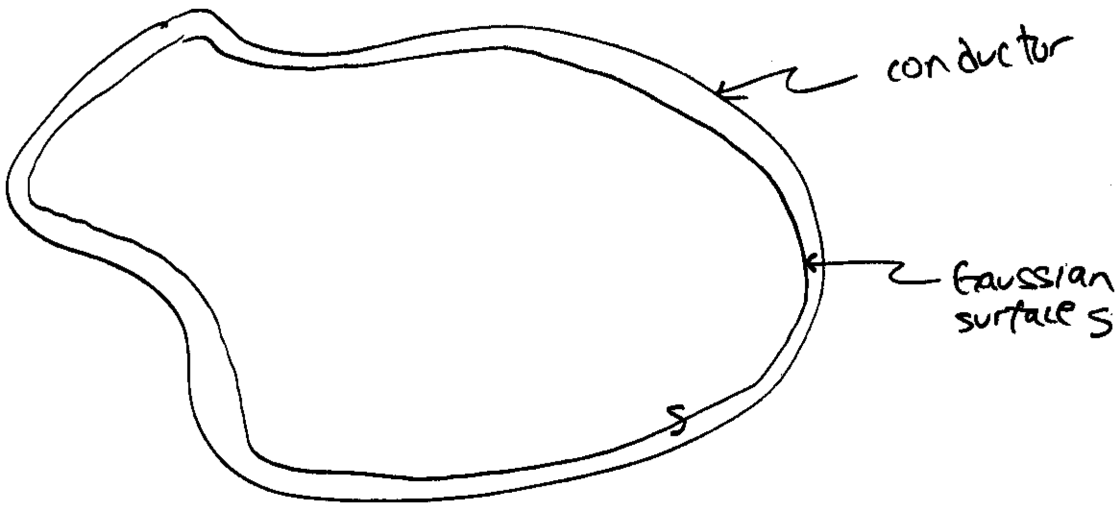
$$\varepsilon_0 \int \vec{E} \cdot dA = q_{enc} = 0$$

$\Rightarrow E = 0$ inside the shell ✓

Isolated, Charged conductors

So far, we have considered bodies having 'fixed', or immobile charges - as if we embedded charge within a plastic insulator (eg, charged sheet, wire, sphere, etc).

Now lets add some mobile charges (ie, electrons) that can roam about an arbitrarily shaped conductor that is <u>isolated</u> (ie, there are no <u>external</u> $\vec{E}$-fields present)



Next, place a Gaussian surface just inside the conductor's surface

Can there be any charge inside this surface, ie, inside the conductor?

If the charge q inside the conductor were nonzero, then Gauss' Law, $\varepsilon_0 \oint_S \vec{E} \cdot dA = q \neq 0$,

tells us that $\vec{E} \neq 0$ inside,

so the force on the charge, $\vec{F} = q\vec{E} \neq 0$, which pushes that charge until it moves out of the conductor's interior, and onto the conductor's <u>surface</u>

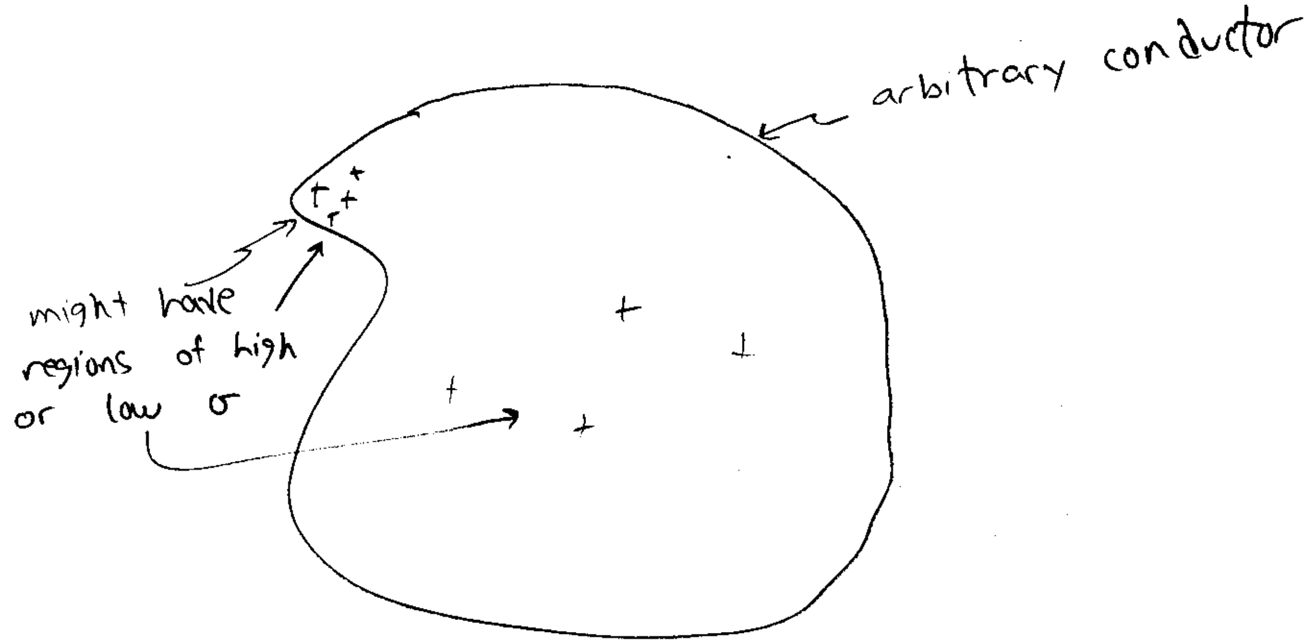
Equilibrium occurs when <u>all</u> charge inside a conductor moves to the surface.

What is $\vec{E}$ inside a conductor?

Since $q_{enc} = 0$ inside a conductor, $\vec{E} = 0$ inside, by Gauss' law.

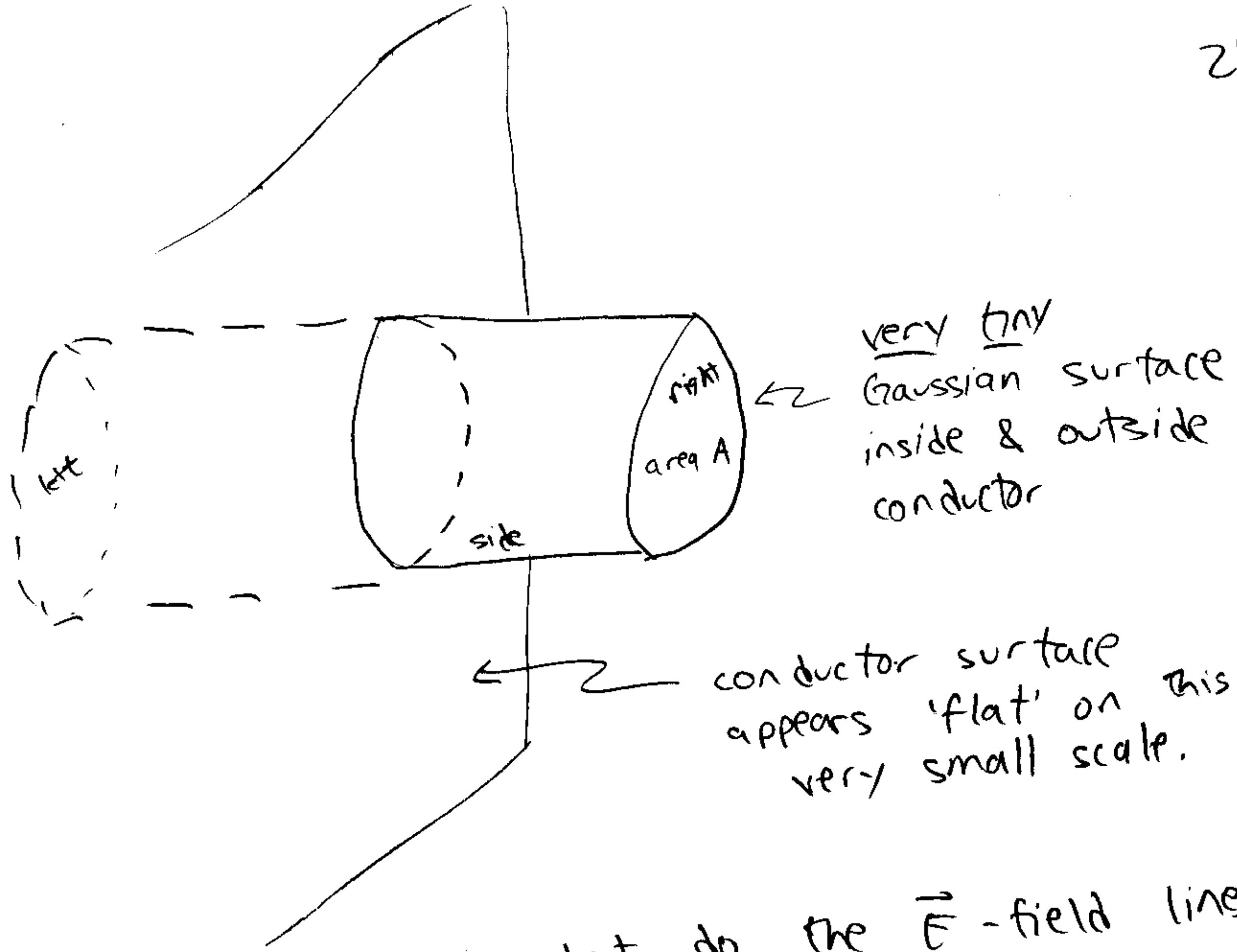
$\vec{E}$ just outside a conductor

Use Gauss' Law to calculate $\vec{E}$ just exterior to the conductor:



The charged conductor has an arbitrary shape, and so σ = charge surface density can vary across its surface, too.

Place a very small cylindrical Gaussian surface S at the conductor's surface. Choose S to be so small that: the conductor's surface is flat there, and σ = constant there.



what do the $\vec{E}$-field lines look like outside the conductor?

And inside?

Use Gauss Law: $\varepsilon_0 \Phi = q_{enc} = \sigma A$

$$\Phi = \int \vec{E} \cdot dA$$

$$\Phi = \Phi_{\substack{left\\end}} + \Phi_{\substack{right\\end}} + \Phi_{side}$$

what is Φ_{left}? Φ_{side}?

$$\Phi_{right} = \int_{\substack{right\\end}} \vec{E} \cdot dA = EA$$

so $\Phi = EA$

and $\varepsilon_0 EA = \sigma A$

$\Rightarrow E = \frac{\sigma}{\varepsilon_0}$ = electric field just exterior to a conductor.

Note that the E-field at a conductor's surface is twice that of an infinite sheet.

Why?

The flux $\Phi \propto EA$ differs by 2.