Lecture Notes for PHY 405
Classical Mechanics

From Thorton & Marion's *Classical Mechanics*

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Chapter 6: Calculus of Variations

A mathematical method that will be used in Chapter 7 to obtain
the Lagrange equations and Hamilton's principle,
which are very useful reformulations of Newtonian mechanics.

Functions and functionals

The problem: determine the path $y(x)$ such that

$$J = \int_{x_1}^{x_2} f\left[y(x), y'(x); x\right] dx \quad \text{is an extremum (ie, a min or max)}$$

where $x =$ independent variable (might be time, distance, angle, etc)
$y(x) =$ dependant function
$y' = dy/dx$
$J =$ integral of the function of $y(x)$, ie, a *functional*
x_1 and x_2 are fixed integration endpoints (could be times, distances, etc)

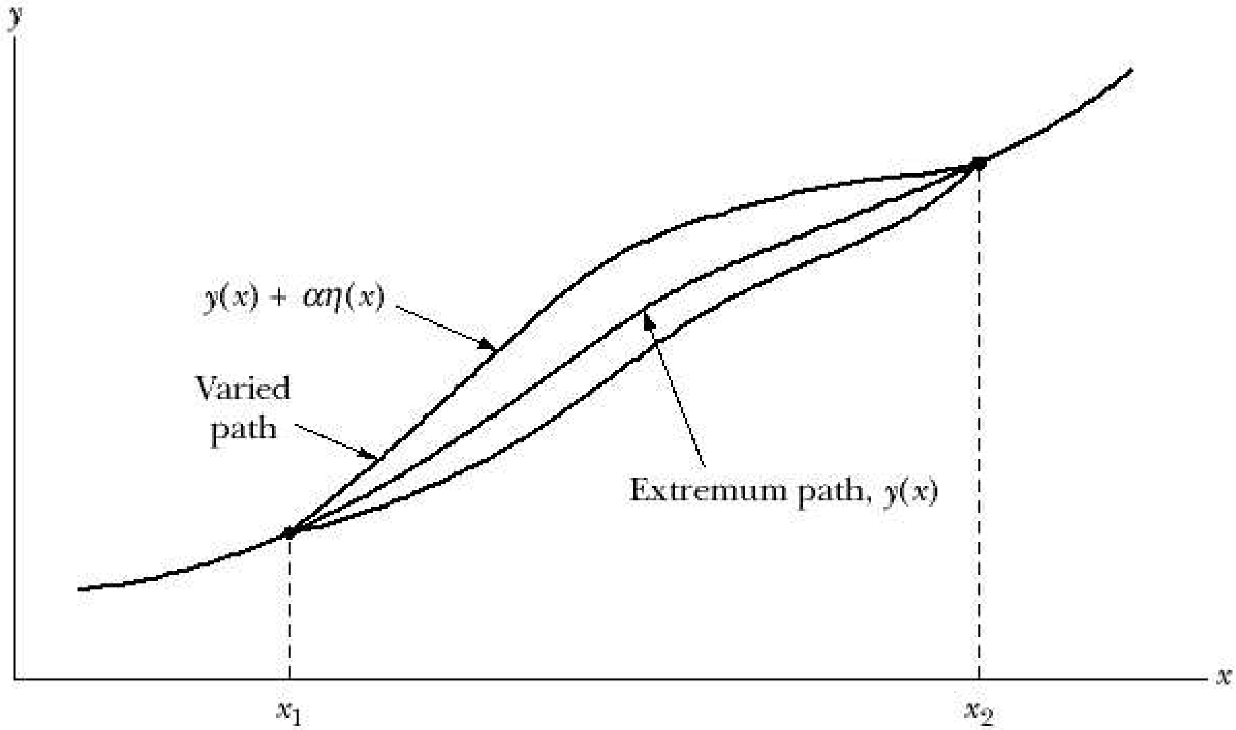

Fig. 6–1.

Solution: Euler's equation

$$
\begin{aligned}
\text{consider } y(\alpha, x) &= y(0, x) + \alpha\eta(x) \\
\text{where } \quad y(x) &= y(0, x) = \text{ desired path that minimizes } J(\alpha) \\
\text{so } \quad y(\alpha, x) &= \text{ alternate paths that result in larger } J \text{ when } \alpha > 0 \\
\text{and } \quad \eta(x) &= \text{ an (almost) arbitrary function obeying } \eta(x_1) = \eta(x_2) = 0 \\
\Rightarrow J(\alpha) &= \int_{x_1}^{x_2} f\left[y(\alpha, x), y'(\alpha, x); x\right] dx
\end{aligned}
$$

We want α to be such that J is an extremum—
what does this tell us about $J(\alpha)$?

$$
\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0 \quad \text{for } J \text{ to have an extremum}
$$

$$
\begin{aligned}
\text{by Chain Rule,} \quad \frac{\partial J}{\partial \alpha} &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha} \right) dx \quad (\text{since } x \text{ is indep' of } \alpha) \\
\text{where} \quad \frac{\partial y}{\partial \alpha} &= \eta \\
\text{and} \quad \frac{\partial y'}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\frac{dy}{dx} = \frac{\partial}{\partial \alpha}\left(\left.\frac{dy}{dx}\right|_{\alpha=0} + \alpha\frac{d\eta}{dx} \right) = \frac{d\eta}{dx} \\
\text{so} \quad \frac{\partial J}{\partial \alpha} &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\frac{d\eta}{dx} \right) dx = 0
\end{aligned}
$$

Do the 2^{nd} integral by parts:

$$
\begin{aligned}
\int_{x_1}^{x_2} u dv &= uv|_{x_1}^{x_2} - \int_{x_1}^{x_2} v du \\
u &= \frac{\partial f}{\partial y'} \qquad\qquad dv = \frac{d\eta}{dx}dx = d\eta \\
du &= \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx \qquad v = \eta \\
\text{so } 2^{nd} \text{ integral} &= \left.\frac{\partial f}{\partial y'}\eta\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)dx \\
\text{and recall that } \eta(x_1) &= \eta(x_2) = 0 \quad \text{by definition}
\end{aligned}
$$

$$
\text{so} \quad \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right] \eta(x)dx = 0
$$

for any $\eta(x)$ that is (almost) arbitrary.

What does this tell us about the integrand?

$$
\Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0
$$

This is Euler's equation,
whose solution yields the path $y(x)$ that minimizes/maximizes J.

Note that many physics problems (optics, mechanics, etc) also seek the path $y(x)$ that minimize J.

Ex. 6.1, the brachistochrone problem

Brachistochrone=Greek for path of shortest delay.
This classic physics problem was solved by Bernoulli in 1696.

A bead slides along a frictionless wire due to gravity, from rest at point $(x_1, y_1) = (0, 0)$ to point (x_2, y_2).

What is the shape of the wire $y(x)$ that minimizes the travel time?

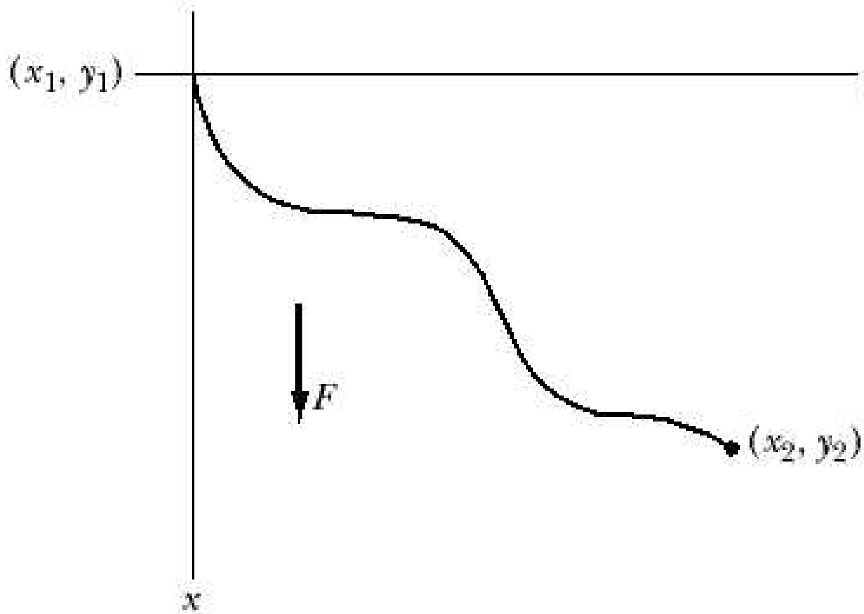

Fig. 6–3.

where independent coordinate x = bead's vertical position and $y(x)$ = its horizontal position.

Since $dt = ds/v$ = time for bead to traverse small distance ds,

$$
\begin{aligned}
\text{the total travel time is} \quad t &= \int_{(0,0)}^{(x_2,y_2)} \frac{ds}{v} \\
\text{where} \quad ds = \sqrt{dx^2 + dy^2} &= (1 + y'^2)^{1/2} dx = \text{ small path segment} \\
\text{particle's energy} \quad E &= \frac{1}{2} m v^2 - m g x = 0 \quad \text{(recall zeropoint is arbitrary)} \\
\text{so velocity} \quad v(x) &= \sqrt{2gx} \\
\text{and} \quad t &= \int_0^{x_2} \sqrt{\frac{1 + y'^2}{2gx}} dx
\end{aligned}
$$

What is our functional J? What is f?

The total travel time t is thus minimized when the path $y(x)$ satisfies Euler's Eqn.:

$$
\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0
$$

$$
\begin{aligned}
\text{where} \quad f(y, y'; x) &= \sqrt{\frac{1 + y'^2}{2gx}} \\
\text{since} \quad \frac{\partial f}{\partial y} &= 0, \\
\Rightarrow \frac{\partial f}{\partial y'} &= \text{constant C} = \frac{(1 + y'^2)^{-1/2} 2y'}{2\sqrt{2gx}} = \frac{y'}{\sqrt{2gx(1 + y'^2)}} \\
\text{so} \quad y'^2 &= 2gC^2 x(1 + y'^2) \\
y'^2(1 - 2gC^2 x) &= 2gC^2 x \\
\text{so} \quad y' &= \frac{dy}{dx} = \sqrt{\frac{2gC^2 x}{1 - 2gC^2 x}}
\end{aligned}
$$

Now what?

$$\int_0^{x_2} dy \ = \ y(x) = \int_0^x \sqrt{\frac{2gC^2x}{1-2gC^2x}}dx$$

Recall that $C =$ some unknown constant,
so set $2gC^2 = 1/2a$ where a is some other constant:

$$y(x_2) \ = \ \int_0^{x_2} \sqrt{\frac{x/2a}{1-x/2a}} = \int_0^{x_2} \frac{xdx}{\sqrt{2ax-x^2}}$$

after multiplying by $\sqrt{2ax}$ upstairs & downstairs.

To solve this integral, change variables:

$$\begin{aligned}
x \ &= \ a(1-\cos\theta) \\
\text{so} \quad dx \ &= \ a\sin\theta d\theta \\
\text{and} \quad y \ &= \ \int_0^\theta \frac{a^2(1-\cos\theta)\sin\theta d\theta}{\sqrt{2a^2(1-\cos\theta)-a^2(1-2\cos\theta+\cos^2\theta)}} \\
\text{note denominator} \ &= \ \sqrt{a^2(1-\cos^2\theta)} = a\sin\theta \\
\text{so} \quad y \ &= \ \int_0^\theta a(1-\cos\theta)d\theta \\
\text{or} \quad y(\theta) \ &= \ a(\theta-\sin\theta) \\
\text{and} \quad x(\theta) \ &= \ a(1-\cos\theta)
\end{aligned}$$

This is the equation for a *cycloid*= a curve traced by a point on a circle of radius a rolling along the $x = 0$ plane:

$$
\begin{aligned}
x(\theta) &= x_c + \Delta x(\theta) \\
y(\theta) &= y_c + \Delta y(\theta) \\
\text{where} \quad x + c &= a \quad \& \quad y_c = a\theta \quad \text{is the motion of the cycloid's center} \\
\text{and} \quad \Delta x(\theta) &= -a\cos\theta \\
\Delta y(\theta) &= -a\sin\theta \text{ are the bead's displacements from the center}
\end{aligned}
$$

The radius a is chosen so that the bead passes thru endpoint (x_2, y_2).

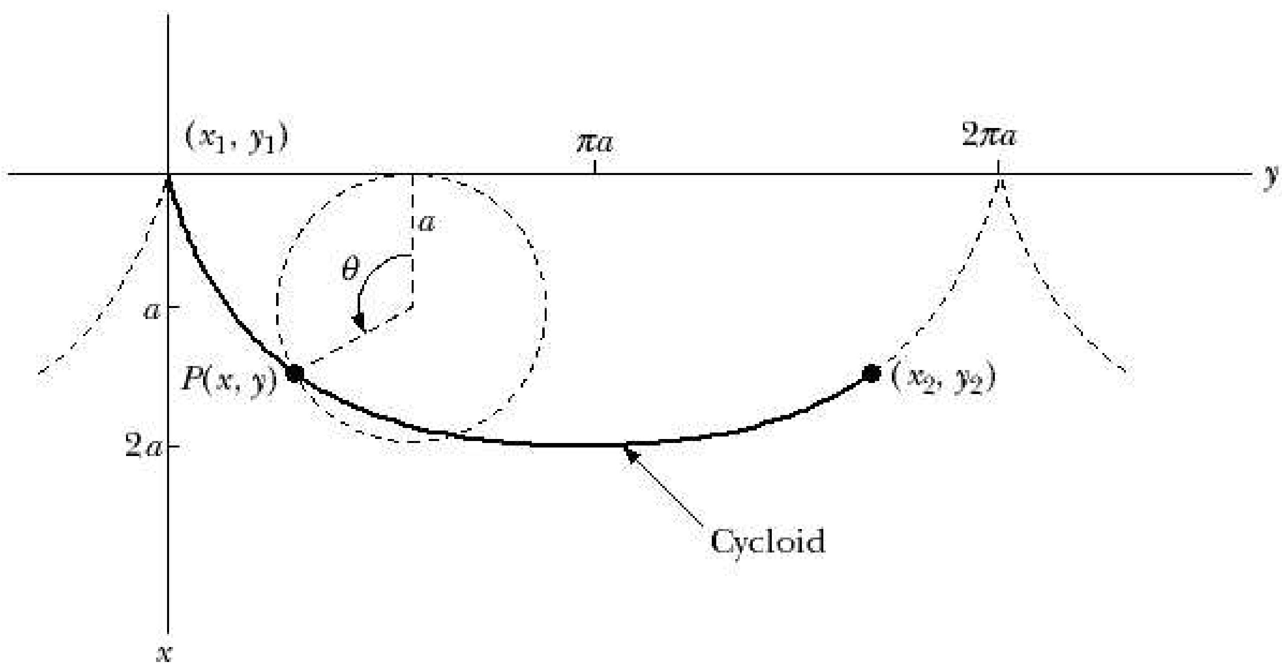

Fig. 6–4.

Note also that a straight wire does *not* minimize travel time.

Euler's Eqn. for N dimensional problem

$$\begin{aligned}
\text{suppose} \quad f &= f(y_1, y_2, y_3, \ldots, y_1', y_2', y_3', \ldots; x) \\
&\equiv f(y_i, y_i'; x) \quad \text{where } i = 1, 2, 3, \ldots, N \\
\text{write} \quad y_i(\alpha, x) &= y_i(0, x) + \alpha \eta_i(x) \\
\text{where} \quad y_i(0, x) &= y_i(x) = \text{ trajectory along } i\text{–axis that minimizes } J \\
\text{and} \quad \eta_i(x_1) &= \eta_i(x_2) = 0 \text{ at trajectory endpoints} \\
\text{recall} \quad J &= \int_{x_1}^{x_2} f(y_i, y_i'; x)dx \\
\text{so} \quad \frac{dJ}{d\alpha} &= \int_{x_1}^{x_2} \sum_{i=1}^{N} \left(\frac{\partial f}{\partial y_i}\frac{\partial y_i}{\partial \alpha} + \frac{\partial f}{\partial y_i'}\frac{\partial y_i'}{\partial \alpha} \right) dx \\
\text{and note that} \quad \frac{\partial y_i}{\partial \alpha} &= \eta_i
\end{aligned}$$

Integrate 2^{nd} term by parts:

$$\begin{aligned}
u &= \frac{\partial f}{\partial y_i'} \qquad\qquad dv = \frac{\partial y_i'}{\partial \alpha}dx = \frac{\partial \eta_i}{\partial x}dx = d\eta_i \\
du &= \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) dx \qquad v = \eta_i \\
\text{so the } 2^{nd} \text{ term} &= \sum_{i=1}^{N} \left[\frac{\partial f}{\partial y_i'}\eta_i \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) \eta_i dx \right] \\
\text{and} \quad \frac{dJ}{d\alpha} &= \sum_{i=1}^{N} \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) \right] \eta_i(x)dx \\
&= 0 \quad \text{when } J \text{ is an extremum}
\end{aligned}$$

Since the $\eta_i(x)$ are (almost) arbitrary functions of x, each individual i^{th} integrand in the [] must also be zero:

$$\Rightarrow \frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0$$

which is Euler's eqn. in N–dimensions