Lecture Notes for PHY 405
Classical Mechanics

From Thorton & Marion's *Classical Mechanics*

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September 23, 2005

Chapters 5 (Gravity) and 8 (Central–Force Motion)

Gravity

Newton's law of gravity (Principia, 1687):

$$\text{force on point mass } m \quad \mathbf{F} = -\frac{GM'm}{r^2}\hat{\mathbf{e}}_r \quad \text{due to mass } M'$$

where $\hat{\mathbf{e}}_r$ = unit vector pointing from source mass M' to target mass m

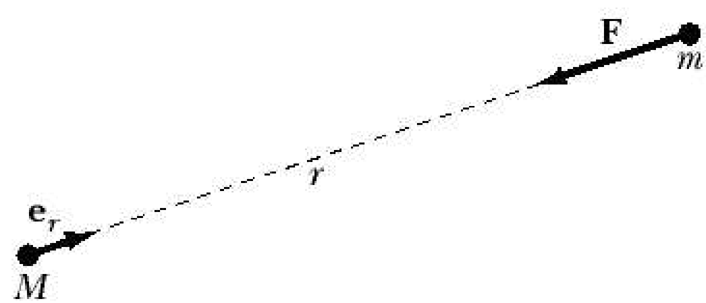


Fig. 5-1

$$\text{but note that}\quad \hat{\mathbf{e}}_{\mathbf{r}} = \hat{\mathbf{r}} = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

$$\text{so}\quad \mathbf{F}(\mathbf{r}) = -\frac{GM'm(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

which is the notation I prefer—its less prone to sign errors.

What if M' was not a point mass,
but an extended body having volume V' and density $\rho(\mathbf{r}')$?

$$\begin{aligned}
\text{then}\quad M' &\Rightarrow \rho(\mathbf{r}')dV' \\
d\mathbf{F} &= -\frac{Gm(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')dV'}{|\mathbf{r}-\mathbf{r}'|^3} = \text{differential force due to } \rho(\mathbf{r}')dV' \\
\text{so}\quad \mathbf{F}(\mathbf{r}) &= -Gm\int_{V'}\frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dV' = \text{ total force on } m \text{ due to } M'
\end{aligned}$$

Gravitational *field* $\mathbf{g} = \mathbf{F}/m =$ acceleration experienced by m due to M':

$$\mathbf{g}(\mathbf{r}) = -G\int_{V'}\frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dV'$$

Note that mass m can be zero.
In this case, the EOM for this massless test particle is $\ddot{\mathbf{r}} = \mathbf{g}$.

Keep in mind that $\mathbf{r} =$ field point (the point where we are calculating $\mathbf{g}$)
and $\mathbf{r}' =$ source point (spot of mass $\rho dV'$ that contributes grav' force $d\mathbf{F}$).

Gravitational potential $\Phi(\mathbf{r})$

The gravitational *field* created by source mass of density $\rho(\mathbf{r}')$ is:

$$\mathbf{g}(\mathbf{r}) \;=\; -G\int_{V'}\frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dV'$$

$$\text{note that}\quad \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \;=\; -\nabla_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)$$

where $\nabla_{\mathbf{r}}$ = gradient operator that acts on vector $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, and *not* on vector $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}}$, which is treated as a constant:

$$\begin{aligned}
\nabla_{\mathbf{r}} \;&=\; \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}} \text{ in Cartesian coord's} \\
\text{and}\quad |\mathbf{r}-\mathbf{r}'| \;&=\; [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2} \\
\text{so RHS} \;&=\; -\nabla_{\mathbf{r}}[\cdots]^{-1/2} \\
&=\; \frac{1}{2}[\cdots]^{-3/2}[2(x-x')\hat{\mathbf{x}} + 2(y-y')\hat{\mathbf{y}} + 2(z-z')\hat{\mathbf{z}}] \\
&=\; \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \quad \surd
\end{aligned}$$

Thus the gravitational field can be written

$$\begin{aligned}
\mathbf{g} &= G\int_{V'}\rho(\mathbf{r}')\nabla_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dV' \\
&= \nabla_{\mathbf{r}}G\int_{V'}\frac{\rho(\mathbf{r}')dV'}{|\mathbf{r}-\mathbf{r}'|} \\
&\equiv -\nabla\Phi(\mathbf{r}) \\
\text{where}\quad \Phi(\mathbf{r}) &= -G\int_{V'}\frac{\rho(\mathbf{r}')dV'}{|\mathbf{r}-\mathbf{r}'|} = \text{ grav' potential of object } V'
\end{aligned}$$

Note that if the source is a point mass M' at the origin ($\mathbf{r}'=0$), then $\mathbf{r}$ is exterior to V' and

$$\Phi(\mathbf{r}) = -\frac{G}{r}\int_{V'}\rho(\mathbf{r}')dV' = -\frac{GM'}{r},$$

which is the familiar potential for a point mass.

Work and energy

Now calculate the work W that must be done to move a mass m thru a gravitational field $\mathbf{g}$:

The differential work dW that an *outside agent* must do to move m a small distance $d\mathbf{r}$ in a grav' field $\mathbf{g}$ is

$$dW = -\mathbf{F}\cdot d\mathbf{r} \;=\; -m\mathbf{g}\cdot d\mathbf{r} = +m(\nabla\Phi)\cdot d\mathbf{r}$$

where $\Phi=\Phi(\mathbf{r})=\Phi(x,y,z)$, and the $-$sign accounts for the fact that the *outside agent* is doing the work.

$$\begin{aligned}
\text{so}\quad dW &= m\left(\frac{\partial\Phi}{\partial x}\hat{\mathbf{x}}+\frac{\partial\Phi}{\partial y}\hat{\mathbf{y}}+\frac{\partial\Phi}{\partial z}\hat{\mathbf{z}}+\right)\cdot(dx\hat{\mathbf{x}}+dy\hat{\mathbf{y}}+dz\hat{\mathbf{z}}) \\
&= m\sum_i\frac{\partial\Phi}{\partial x_i}dx_i = md\Phi
\end{aligned}$$

so the total work done to deliver m from position $\mathbf{r_1}$ to $\mathbf{r_s}$ is

$$W = m\int_{\mathbf{r_1}}^{\mathbf{r_2}} d\Phi = m(\Phi_2 - \Phi_1)$$

Start the particle far away, ie $|\mathbf{r_1}| = \infty$,
which is also a convenient reference point at which to set potential to zero:
$\Phi_1 = \Phi(\mathbf{r_1}) = 0 \Rightarrow W = m\Phi(\mathbf{r})$,

which is the work W that the outside agent must do to deliver m
thru a gravity field from ∞ to $\mathbf{r}$

$$\begin{aligned} \Rightarrow W &= m\text{'s potential energy } U \\ U(\mathbf{r}) &= m\Phi(\mathbf{r}) \end{aligned}$$

From which we recover

$$\begin{aligned} \mathbf{F} &= -\nabla U = -m\nabla\Phi = \text{ force on } m \text{ due to gravity field} \\ \text{or} \quad \mathbf{g} &= \mathbf{F}/m = -\nabla\Phi \end{aligned}$$

Example 5.2: Potential of a shell

Calculate the gravitational potential $\Phi(R)$ and field $\mathbf{g}(R)$ of a shell having inner, outer radii b, a and uniform density ρ.

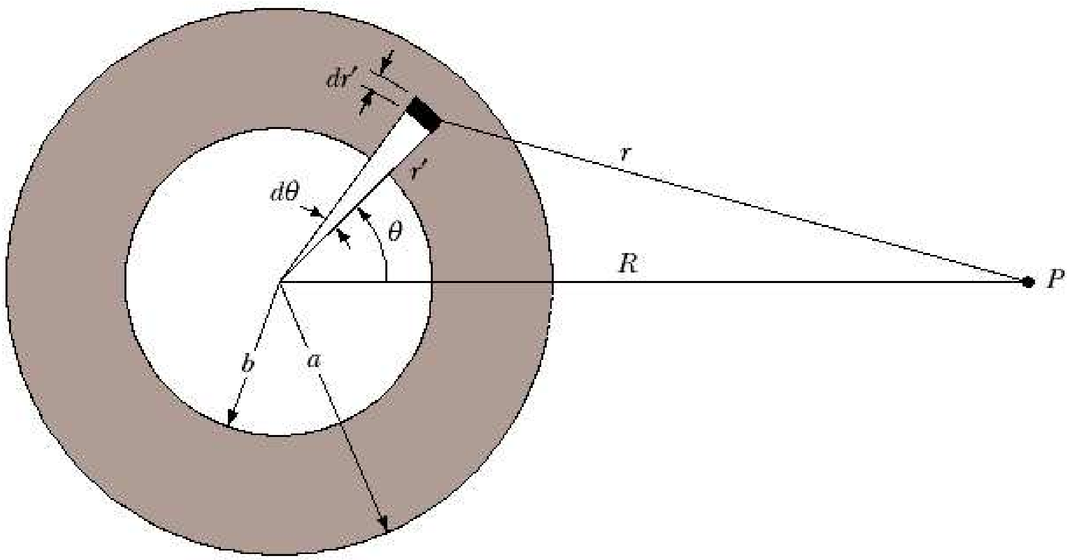


Fig. 5–3

Get Φ by direct integration (this is the laborious method...)

$$\Phi(R) = -G\int_{V'} \frac{\rho dV'}{r}$$

where $r^2 = r'^2 + R^2 - 2Rr'\cos\theta'$

note $R =$ distance of field point P, $r' =$ distance of source mass $\rho dV'$

and $dV' = r'^2 \sin\theta' d\theta' d\phi' dr' =$ small differential volume

(see Fig. F–4 and Eqn. F-16)

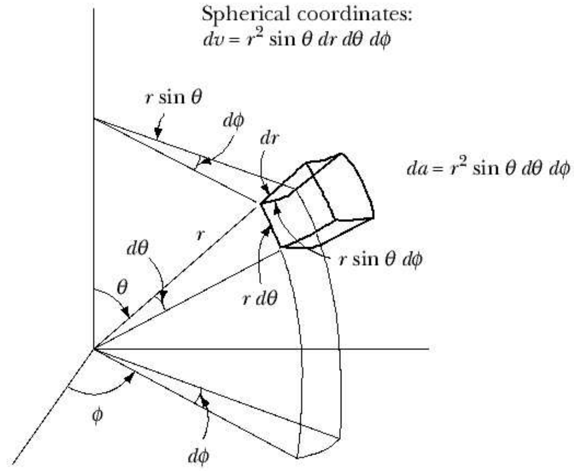


Fig. F–4

$$\begin{aligned}
\text{so} \quad \Phi(R) &= -G \int_b^a \rho r'^2 dr' \int_0^{2\pi} d\phi' \int_0^{\pi} \frac{\sin\theta' d\theta'}{\sqrt{r'^2 + R^2 - 2Rr'\cos\theta'}} \\
&= -\frac{2\pi G}{R} \int_b^a \rho r'^2 dr' \int_0^{\pi} \frac{\sin\theta' d\theta'}{\sqrt{1 + \alpha^2 - 2\alpha\cos\theta'}} \quad \text{where } \alpha \equiv \frac{r'}{R} \\
&= -\frac{2\pi G}{R} \int_b^a \rho r'^2 dr' \int_{(1-\alpha)^2}^{(1+\alpha)^2} \frac{u^{-1/2} du}{2\alpha} \\
\text{where } u &= 1 + \alpha^2 - 2\alpha\cos\theta' \quad \text{and} \quad du = 2\alpha\sin\theta
\end{aligned}$$

$$\begin{aligned}
\text{note that the right integral} \quad &= \left. \frac{u^{1/2}}{\alpha} \right|_{(1-\alpha)^2}^{(1+\alpha)^2} \\
&= \frac{1}{\alpha}(|1+\alpha| - |1-\alpha|) \\
&= \begin{cases} 2 & \alpha < 1 \text{ or } r' < R \\ \frac{2}{\alpha} = \frac{2R}{r'} & \alpha > 1 \text{ or } r' > R \end{cases}
\end{aligned}$$

So the potential exterior to the shell is

$$
\begin{aligned}
\Phi(R>a) &= -\frac{G}{R}\int_b^a 4\pi\rho r'^2 dr' = -\frac{GM}{R} \text{ as expected} \\
\text{since} \quad M &= \int_b^a 4\pi\rho r'^2 dr' = \text{ shell's total mass}
\end{aligned}
$$

And the potential in the inner cavity at $r<b$ is

$$
\begin{aligned}
\Phi(R<b) &= -4\pi G\int_b^a \rho r' dr' = -2\pi G\rho(a^2-b^2) \\
&= \text{ a constant}
\end{aligned}
$$

While the potential inside the shell is

$$
\begin{aligned}
\Phi(b<R<a) &= -\frac{4\pi G\rho}{R}\left[\int_b^R r'^2 dr' + \int_R^a R r' dr'\right] \\
&= -\frac{4\pi G\rho}{R}\left[\frac{1}{3}(R^3-b^3)+\frac{1}{2}R(a^2-R^2)\right] \\
&= -4\pi G\rho\left(\frac{a^2}{2}-\frac{b^3}{3R}-\frac{R^2}{6}\right)
\end{aligned}
$$

The gravitational acceleration is

$$
\begin{aligned}
\mathbf{g} &= -\nabla\Phi = -\frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} \\
\text{where} \quad g &= -\frac{\partial\Phi}{\partial R} = \begin{cases} 0 & R<b \\ -\frac{4\pi}{3}G\rho\left(R-\frac{b^3}{R^2}\right) & b<R<a \\ -\frac{GM}{R^2} & R>a \end{cases}
\end{aligned}
$$

Note that potential $\Phi(R)$ is continuous, while the acceleration $g(R)$ is not.

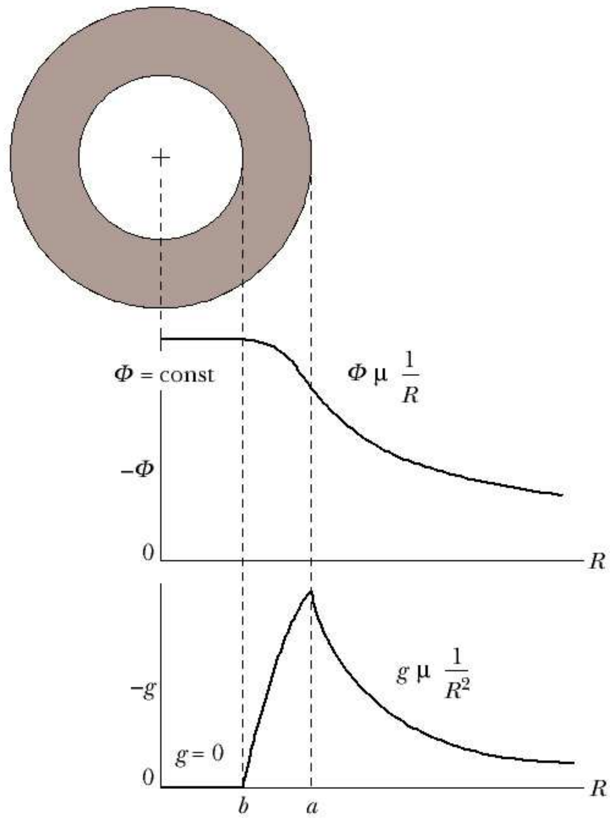


Fig. 5-4

Note that this also recovers a familiar result:
the interior of shell of uniform density ρ has a constant potential Φ and zero gravitational acceleration, $\mathbf{g} = \mathbf{0}$.

You've already obtained this same result in your previous studies of the Coulomb force law $\mathbf{F} = qQ\hat{\mathbf{r}}/4\pi\epsilon_0 r^2$:
the electrostatic field inside a shell is $\mathbf{E} = \mathbf{0}$,
while its electrostatic potential = constant.

Gauss' Law

Calculate the gravitational flux ψ thru an arbitrary surface S due to mass m:

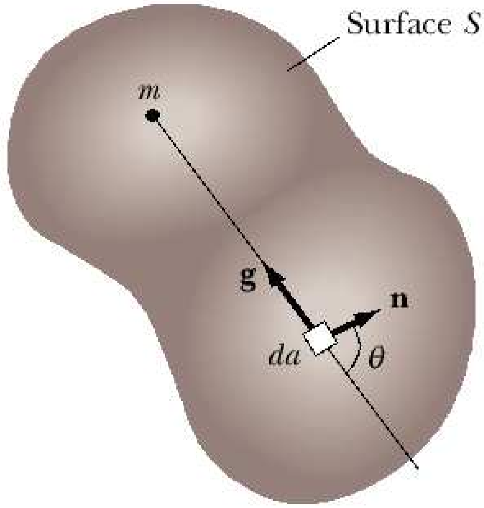


Fig. 5-7

Place origin at m. The gravitational flux thru S is

$$\begin{aligned} \psi &\equiv \int_S \mathbf{g} \cdot \hat{\mathbf{n}} da \\ \hat{\mathbf{n}} &= \text{ unit vector normal to area } da \\ \mathbf{g} &= -\frac{Gm\mathbf{r}}{r^3} \\ \text{and} \quad \mathbf{g} \cdot \hat{\mathbf{n}} &= -\frac{Gm\cos\theta}{r^2} \end{aligned}$$

$$\begin{aligned} \text{but} \quad d\Omega &= da\cos\theta/r^2 = \text{ solid angle that area } da \text{ subtends} \\ \text{so} \quad \psi &= -Gm\int_S d\Omega \quad \Leftarrow \quad \text{what does } \int_S d\Omega = ? \\ \text{so} \quad \psi &= -4\pi Gm \text{ for an enclosed mass } m \end{aligned}$$

If S encloses multiple masses $\sum_i m_i$, then

$$\int_S \mathbf{g} \cdot \hat{\mathbf{n}} da \;=\; -4\pi G \sum_i m_i = -4\pi G M_{enclosed}$$

where $M_{enclosed}$ is the total mass enclosed by area S.

This is known as Gauss' Law.

As we shall see, the application of Gauss' Law to problems that have a high degree of symmetry can make problem-solving a *lot* easier.

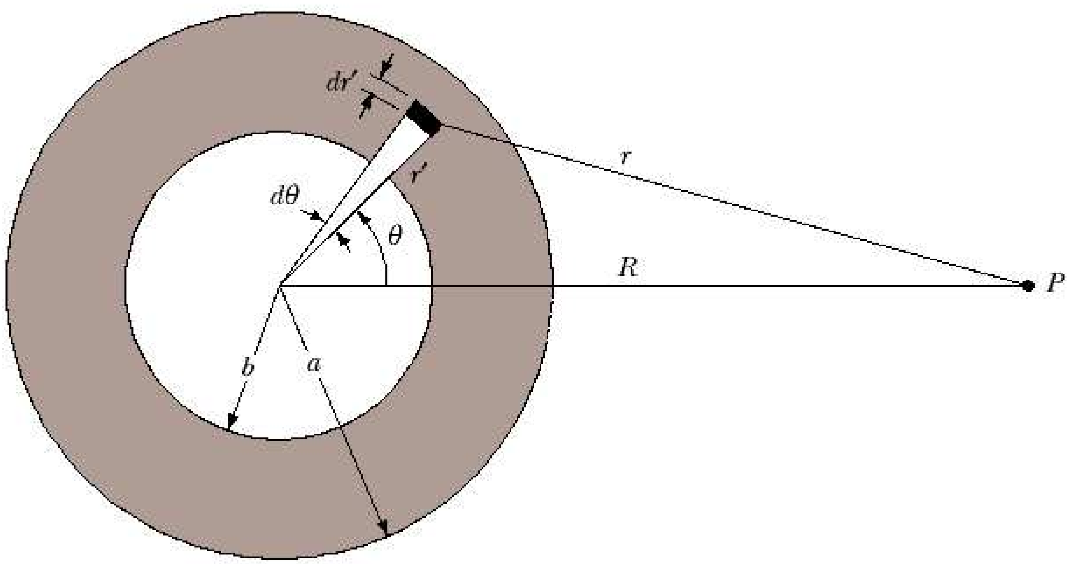


Fig. 5–3

Potential of a shell, again

Redo Example 5.1—use Gauss' law to calculate the sphere's $g(R)$ and $\Phi(R)$;this is the easy solution.

$$\int_S \mathbf{g} \cdot \hat{\mathbf{n}} da \;=\; -4\pi G M_{enclosed}$$

What kind of surface S should we use? What will be most convenient?

Chose S to be sphere of area $A = 4\pi R^2$.

Note that the field point P sits on the sphere, and that $\mathbf{g} \cdot \hat{\mathbf{n}} = g(R)$, so

$$\begin{aligned}
\int_S \mathbf{g} \cdot \hat{\mathbf{n}} da &= g(R)A = -4\pi G M_{enclosed} \\
\text{so} \quad g(R) &= -\frac{G M_{enclosed}(R)}{R^2} \\
&= \begin{cases} 0 & R < b \\ -\frac{4\pi G\rho}{3R^2}(R^3 - b^3) & b < R < a \\ -\frac{GM}{R^2} & R > a \end{cases}
\end{aligned}$$

which recovers our earlier answer.

The potential $\Phi(R)$ is then easily computed from $\mathbf{g} = -\nabla\mathbf{\Phi}$,
so $\Phi(R) = -\int^R g(R')dR'$.

$\Rightarrow$Problems with a high degree of symmetry (eg, spherical, cylindrical, planar symmetry, etc) can at times obey $\mathbf{g} \cdot \hat{\mathbf{n}} = \pm g$. In this instance, Gauss' law can then be used to calculate g without doing any brute–force integrations.

Another example—an infinite wire

What is the gravitational field $\mathbf{g}$ for an infinitely long thin wire having a mass–per–length λ? What is the wire's potential Φ? If a particle is released from a distance r_0 away from the wire, how long until it hits the wire?

Use Gauss' Law to obtain $\mathbf{g}$:

$$\psi = \int_S \mathbf{g} \cdot \hat{\mathbf{n}} da \ = \ -4\pi G M_{enclosed}$$

What Gaussian surface S should we use?

Use a cylinder of radius r and length ℓ.
Since $\mathbf{g}$ points radial to the wire, $\mathbf{g} \cdot \hat{\mathbf{n}} = g(r)$.

Do the ends of the cylinder contribute to ψ?

$$\begin{aligned} \psi = \int_S \mathbf{g} \cdot \hat{\mathbf{n}} da \ &= \ gA = -4\pi G M_{enclosed} = -4\pi G \lambda \ell \\ \text{where} \quad A \ &= \ 2\pi r \ell \\ \text{so} \quad g(r) \ &= \ -\frac{2G\lambda}{r} \end{aligned}$$

The wire's gravitational potential is obtained from $g = -\partial\Phi/\partial r$ so

$$\Phi(r) = -\int_{r_{ref}}^{r} g(r')dr' = \int_{r_{ref}}^{r} \frac{2G\lambda dr'}{r'} = 2G\lambda \ln(r/r_{ref})$$

What is an appropriate value for the reference distance r_{ref}?

We will use the particle's initial distance r_0
(but we could have used any distance).

Now find the time required for a particle starting a distance r_0 away to fall onto the wire.

The EOF according to NII is $g = \ddot{r} = -2G\lambda/r$.
How do you solve this?

An alternate approach is to use the system's total energy:

$$\begin{aligned} E &= T + U \\ \text{where} \quad T &= \frac{1}{2}mv^2 \\ \text{and} \quad U &= m\Phi = 2Gm\lambda \ln(r/r_0) \end{aligned}$$

Is E conserved? What is the particle's initial E?

$$\begin{aligned} E = 0 \Rightarrow v^2 &= (dr/dt)^2 = -4G\lambda \ln(r/r_0) \\ \text{so} \quad \frac{dr}{dt} &= \pm\sqrt{-4G\lambda \ln(r/r_0)} \end{aligned}$$

Which sign should we use?

$$
\begin{aligned}
\frac{dr}{\sqrt{-\ln(r/r_0)}} &= -\sqrt{4G\lambda}dt \\
\text{so} \quad \int_{r_0}^{0} \frac{dr'}{\sqrt{-\ln(r'/r_0)}} &= r_0 \int_1^0 \frac{du}{\sqrt{-\ln(u)}} = -\int_0^t \sqrt{4G\lambda}dt' \\
\text{the LHS is (E.19b)} &= -r_0\Gamma(1/2) = -\sqrt{\pi}r_0, \\
\text{while RHS is} &= -\sqrt{4G\lambda}t \\
\Rightarrow t &= \sqrt{\frac{\pi r_0^2}{4G\lambda}}
\end{aligned}
$$

is the time for the particle to fall onto the wire.

Problem Set #3
due Tuesday October 18
at start of class

Problem 1 (below) + text problems 5–5, 5–9, 8–3, 8–6, & 8–12

Problem 1: A gaseous cylinder has an infinite length, radius R, and a constant density ρ. Calculate the cylinder's gravitational potential $\Phi(r)$ and its gravitational acceleration $\mathbf{g}(\mathbf{r})$ at all distances r from the cylinder's long axis (ie, for $r \geq R$ and $r < R$). A massless particle is released from rest a distance $r > R$ from the cylinder's center. How much time is required for the particle to pass through the cylinder's center? Assume the particle can travel through the gaseous cylinder with no resistance.

Poisson's Eqn.

Recall divergence theorem from PHY 335, also known as Gauss' Theorem:

$$\int_S \mathbf{A} \cdot \mathbf{da} = \int_V \nabla \cdot \mathbf{A} dV \quad \text{Eq. (1.130)}$$
$$\text{where} \quad \mathbf{da} = \hat{\mathbf{n}} da$$

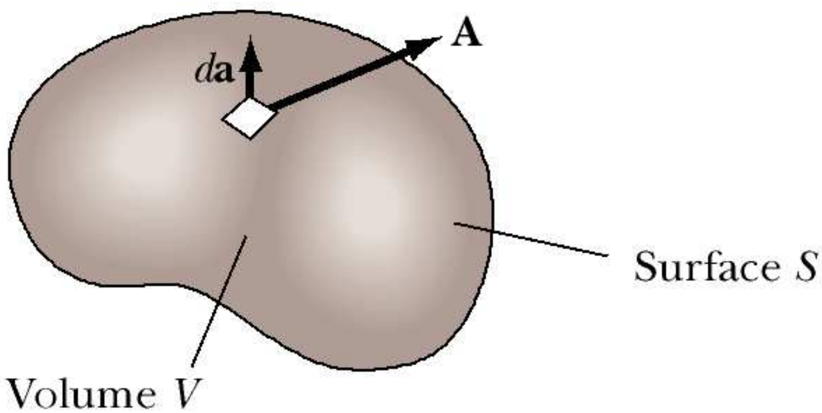


Fig. 1–23

According to the divergence theorem, Gauss' Law is

$$\begin{aligned}
\psi &= \int_S \mathbf{g} \cdot \hat{\mathbf{n}} da = -4\pi G M_{enclosed} \\
\text{so} \quad \psi &= \int_V \nabla \cdot \mathbf{g} dV = -4\pi G M_{enclosed} \\
\text{but} \quad \mathbf{g} &= -\nabla\Phi \quad \text{and } M_{enclosed} = \int_V \rho dV \\
\text{so} \quad \psi &= -\int_V \nabla^2 \Phi dV = -4\pi G \int_V \rho dV \\
\text{so} \quad & \int_V (\nabla^2 \Phi - 4\pi G \rho) dV = 0
\end{aligned}$$

Note that the volume V is arbitrary.
What does that tell us about the integrand?

$$\Rightarrow \nabla^2 \Phi = 4\pi G \rho \quad \text{which is Poisson's equation}$$

This equation is very useful in astrophysical problems.

Suppose you know how the matter in your system is distributed, ie, $\rho(\mathbf{r})$.
Then you can try to solve Poisson's eqn' for the system's potential $\Phi(\mathbf{r})$, which then yields the gravitational field $\mathbf{g}(\mathbf{r}) = -\nabla\Phi$.
Then you can try to solve NII, $\ddot{\mathbf{r}} = \mathbf{g}(\mathbf{r})$
to see how your system evolves over time.
This approach is often difficult to do by hand,
but can be implemented on a computer via a hydrodynamic code.

If, however, you are interested in the motion of massless test particles that are perturbed by some external gravitational forces described by $\Phi(\mathbf{r})$, then $\rho(\mathbf{r}) = 0$ and Poisson's eqn' becomes the Laplace eqn: $\nabla^2\Phi = 0$.

We won't actually use Poisson & Laplace eqn's in this class,
but you will see it again in your electrodynamics class, as well as in grad school.

Of course, these preceding equations also occur in electrodynamics since Newton's Law of gravity has the same form as the Coulomb force law:

$$F_G = -\frac{Gm_1m_2}{r^2} \rightarrow F_C = +\frac{q_1q_2}{4\pi\epsilon_0 r^2}$$

so replace $G \rightarrow -1/4\pi\epsilon_0$ if you want to apply our results to electrostatics.

In this case, interpret mass density ρ as the charge density,
and replace the gravity field $\mathbf{g} \rightarrow \mathbf{E}$ with the electric field.
then you recover Gauss' eqn. for the electric field.

Poisson's eqn' is

$$\begin{aligned} \nabla^2\Phi = -\nabla \cdot \mathbf{g} &= 4\pi G\rho, \\ \rightarrow \nabla \cdot \mathbf{E} &= \rho/\epsilon_0 \end{aligned}$$

which is Gauss' eqn' for the electric field $\mathbf{E}$.

Evidently, your earlier studies of electrodynamics can also be regarded as a a first course in celestial mechanics, since the solutions obtained in E&M class often apply to gravitating systems.

Exam #1
Tuesday October 25
on text chapters 2, 3, 5, & 8, and Problem Sets 1-3

The 2–Body Problem & Central Force motion

The following discussion is drawn from Chap 8. These notes will use Newton's Laws to solve the 2–body problem, while the text starts with the Lagrange's eqn's. Although the problem is attacked using different equations, the resulting solutions will be the same.

Consider 2 gravitating point masses m_1 and m_2 at positions $\mathbf{r_1}$ and $\mathbf{r_1}$:

The EOM are

$$\begin{aligned} m_1\ddot{\mathbf{r}}_{\mathbf{1}} &= -\frac{Gm_1m_2(\mathbf{r_1}-\mathbf{r_2})}{|\mathbf{r_1}-\mathbf{r_2}|^3} \quad \text{and} \quad m_2\ddot{\mathbf{r}}_{\mathbf{2}} = -\frac{Gm_2m_1(\mathbf{r_2}-\mathbf{r_1})}{|\mathbf{r_1}-\mathbf{r_2}|^3} \\ \text{so} \quad \ddot{\mathbf{r}}_{\mathbf{1}} &= +\frac{Gm_2(\mathbf{r_2}-\mathbf{r_1})}{|\mathbf{r_2}-\mathbf{r_1}|^3} \quad \text{and} \quad \ddot{\mathbf{r}}_{\mathbf{2}} = -\frac{Gm_1(\mathbf{r_2}-\mathbf{r_1})}{|\mathbf{r_2}-\mathbf{r_1}|^3} \end{aligned}$$

I will adopt the convention that $m_1 > m_2$,
and will often call m_1 the primary object and m_2 the secondary.

If we were considering the motion of a simple star–planet system,
then the primary m_1 would be the star and m_2 the planet.

However our results will also apply to any 2–body system, like a binary star system, or a galaxy that is orbited by a satellite galaxy, etc.

Usually we are interested in the motion of the secondary m_2 (which might be a planet), whose coordinates related to the primary are

$$\begin{aligned} \mathbf{r} &= \mathbf{r_2} - \mathbf{r_1} \\ \text{so} \quad \ddot{\mathbf{r}} &= \ddot{\mathbf{r}}_\mathbf{2} - \ddot{\mathbf{r}}_\mathbf{1} = -\frac{G(m_1 + m_2)\mathbf{r}}{|\mathbf{r}|^3} \end{aligned}$$

This is the EOM for the relative coordinate $\mathbf{r}$.

Where is the origin of the relative coordinate system?

Is this coordinate system inertial?

Note also that the above EOM can be written as

$$\ddot{\mathbf{r}} = -\nabla\Phi'(r) \quad \text{where} \quad \Phi' = -\frac{G(m_1 + m_2)}{r}$$

resembles a gravitational potential.

The reduced mass

The above EOM for 2 bodies can be recast
so that it resembles a 1–body problem:

Recall that the primary's gravitational potential is $\Phi_1(r) = -Gm_1/r$.

Next, add the secondary m_2 to the problem.
The system's total PE is then $U(r) = m_2\Phi_1 = -Gm_1m_2/r$.

The reduced mass for this system is

$$\mu \;=\; \frac{m_1 m_2}{m_1 + m_2}$$

note that $\mu < m_1$ and m_2.

Multiply the EOM by μ:

$$\mu\ddot{\mathbf{r}} \;=\; -\nabla\left(-\frac{Gm_1m_2}{r}\right) = -\nabla U(r)$$

$\Rightarrow$ the EOM for the relative motion of 2 bodies can be written like the
EOM for a single particle of mass μ as it moves about the
central–force–field $\mathbf{F} = -\nabla U(r) = -Gm_1m_2\hat{\mathbf{r}}/r^2$.

Angular momentum conservation

Recall the EOM

$$\ddot{\mathbf{r}} = -\frac{G(m_1 + m_2)\mathbf{r}}{|\mathbf{r}|^3}$$

and note that

$$\mathbf{r} \times \ddot{\mathbf{r}} = \frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = 0$$

This tells us that the motion of m_1 and m_2 is always confined to the same plane. Why?

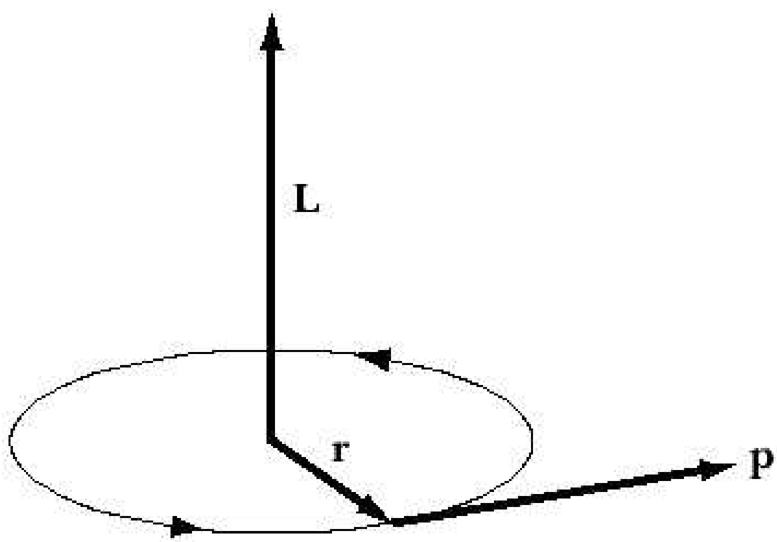


Fig. 8–2.

For a particle having the reduced mass μ,
the system's linear momentum is $\mathbf{p} = \mu\dot{\mathbf{r}}$,
and its angular momentum is $\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mu\mathbf{r} \times \dot{\mathbf{r}}$.

Note that $\mathbf{L}$ is conserved.

The EOM

Since m_1 and m_2 are always coplanar, what coordinate system should I use?

The EOM is

$$\begin{aligned}
\ddot{\mathbf{r}} &= -\frac{G(m_1+m_2)}{r^2}\hat{\mathbf{r}} \\
\text{where} \quad \mathbf{r} &= r\hat{\mathbf{r}} \\
\text{and} \quad \dot{\mathbf{r}} &= \frac{d}{dt}(r\hat{\mathbf{r}}) = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} \quad \text{(see Section 1.14)} \\
\text{and} \quad \ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -\frac{G(m_1+m_2)}{r^2}\hat{\mathbf{r}}
\end{aligned}$$

Separate into radial and tangential components:

$$\begin{aligned}
\hat{\mathbf{r}}: \quad \ddot{r} - r\dot{\theta}^2 &= -\frac{G(m_1+m_2)}{r^2} \\
\hat{\theta}: \quad \frac{d}{dt}(r^2\dot{\theta}) &= 0
\end{aligned}$$

The latter is simply angular momentum conservation:

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mu\mathbf{r}\times\dot{\mathbf{r}}) = \frac{d}{dt}\mu[r\hat{\mathbf{r}}\times(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta})] = \frac{d}{dt}(\mu r^2\dot{\theta}\hat{\mathbf{r}}\times\hat{\theta}) = \frac{d}{dt}(\mu r^2\dot{\theta}\hat{\mathbf{z}}) = 0$$

We usually define $\ell = |\mathbf{L}| = \mu r^2\dot{\theta}$ = system's constant angular momentum.

Kepler's Law's of Planetary Motion

Tycho Brahe & Johannes Kepler made extensive observations of the motions of the planet during late 1500's, and in 1609 Kepler showed that the motions of the planets obeyed 3 empirical laws:

1. planets travel on ellipses with the Sun at one focus.

2. a planet's position vector **r** sweeps out equal areas in equal times.

3. $(P/1\text{ yr})^2 = (a/1\text{ AU})^3$.

Newton showed that Kepler's laws are a consequence of his laws of motion + law of gravity (Principia, 1687).

For instance, the 2nd law is a consequence of **L** conservation:

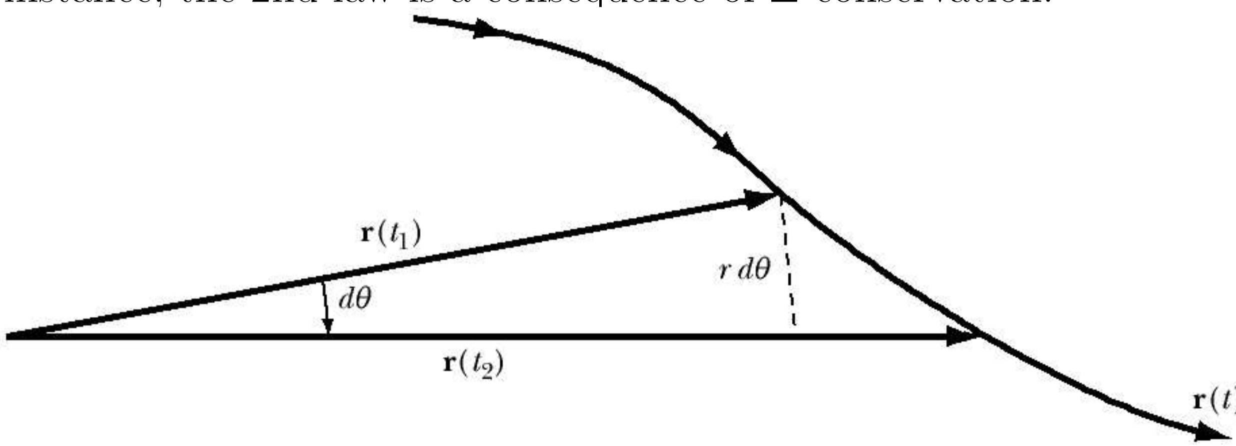


Fig. 8–3.

Radius vector **r** sweeps out angle $d\theta$ in short time dt, so

$$
\begin{aligned}
\text{the area swept is} \quad dA &= \frac{1}{2}r^2 d\theta \\
\text{so} \quad \frac{dA}{dt} &= \frac{1}{2}r^2\dot{\theta} \\
\text{but} \quad \dot{\theta} &= \frac{\ell}{\mu r^2} \quad \text{so} \quad \frac{dA}{dt} = \frac{\ell}{2\mu} = \text{constant}
\end{aligned}
$$

which is Kepler's 2^{nd} law (equal areas in equal times) and is a consequence of ℓ conservation.

The effective potential V

The radial part of the EOM is

$$\ddot{r} \;=\; r\dot{\theta}^2 - \frac{G(m_1+m_2)}{r^2}$$

And since $\dot{\theta} = \ell/\mu r^2$, the corresponding version of NII for the equivalent particle having the reduced mass $\mu = m_1 m_2/(m_1+m_2)$ is

$$\begin{aligned}
\mu\ddot{r} \;&=\; \frac{\ell^2}{\mu r^3} - \frac{G m_1 m_2}{r^2} = \frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r} \quad \text{where } U = -\frac{G m_1 m_2}{r} \;=\; \text{PE} \\
&=\; -\frac{\partial V}{\partial r} \\
\text{and} \quad V(r) \;&=\; \frac{\ell^2}{2\mu r^2} + U = \frac{\ell^2}{2\mu r^2} - \frac{G m_1 m_2}{r}
\end{aligned}$$

is the system's *effective potential energy.*

A consequence of any system having a nonzero angular momentum ℓ is that the particle appears to suffer an additional repulsive 'force'

$$\Delta F_{cent} \;=\; -\frac{\partial}{\partial r}\left(\frac{\ell^2}{2\mu r^2}\right) = \frac{\ell^2}{\mu r^3}$$

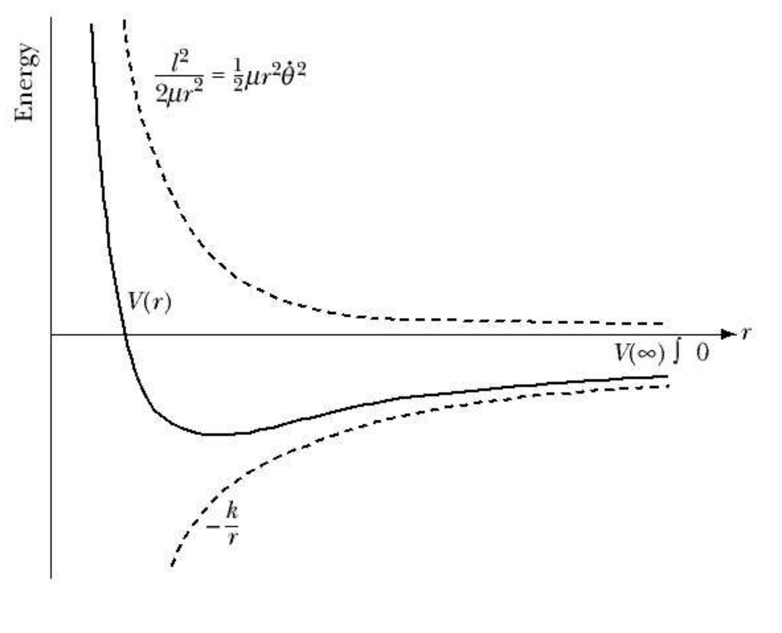


Fig. 8–5.

What value of energy $E = T + V$ corresponds to a bound 2–body system?

What about an unbound system?

Keep in mind that this fictitious force that is just the centrifugal force.

If m_2 where to hit m_1,
what does this tell us about the system's angular momentum ℓ?

Solve the radial EOM

$$\ddot{r} = r\dot{\theta}^2 - \frac{G(m_1 + m_2)}{r^2} = \frac{\ell^2}{\mu^2 r^3} - \frac{G(m_1 + m_2)}{r^2}$$

What is the solution to this equation?

Assume that the solution $r(t)$ can be written as $r = r(\theta)$ where $\theta = \theta(t)$.

Then seek a solution of the form $u(r) = \frac{1}{r(\theta(t))}$:

$$\begin{aligned}
\text{so} \quad \dot{r} &= \frac{dr}{dt} = \frac{dr}{du}\frac{du}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt} \\
\text{recall} \quad \ell &= \mu r^2 \dot{\theta} \quad \text{and} \quad r = \frac{1}{u} \\
\text{so} \quad \dot{r} &= -\frac{1}{u^2}\frac{du}{d\theta}\frac{\ell}{\mu r^2} = -\frac{\ell}{\mu}\frac{du}{d\theta} \\
\text{and} \quad \ddot{r} &= -\frac{\ell}{\mu}\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -\frac{\ell}{\mu}\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -\frac{\ell^2}{\mu^2}u^2\frac{d^2u}{d\theta^2} \\
&= \frac{\ell^2}{\mu^2}u^3 - ku^2 \quad \text{where} \quad k = G(m_1 + m_2)
\end{aligned}$$

$$\text{so the EOM is} \quad \frac{d^2u}{d\theta^2} + u = \frac{1}{\alpha} \quad \text{where} \quad \frac{1}{\alpha} = \frac{k\mu^2}{\ell^2} = \text{constant}$$

We have seen this type of EOM before—what is its solution?

Compare this to the EOM for a SHO that is driven by a constant acceleration a: $\ddot{x} + \omega_0^2 x = a$, which has solution $x(t) = A\cos(\omega_0 t) + a/\omega_0^2$.

So what is the solution $u(\theta)$?

$$
\begin{aligned}
u(\theta) &= A\cos\theta + \frac{1}{\alpha} \\
\text{so} \quad r(\theta) &= \frac{1}{u} = \frac{1}{A\cos\theta + 1/\alpha}
\end{aligned}
$$

And if we set $A = e/\alpha$ then

$$
r(\theta) = \frac{\alpha}{1 + e\cos\theta}
$$

This is the equation for an ellipse when $0 \le e < 1$.

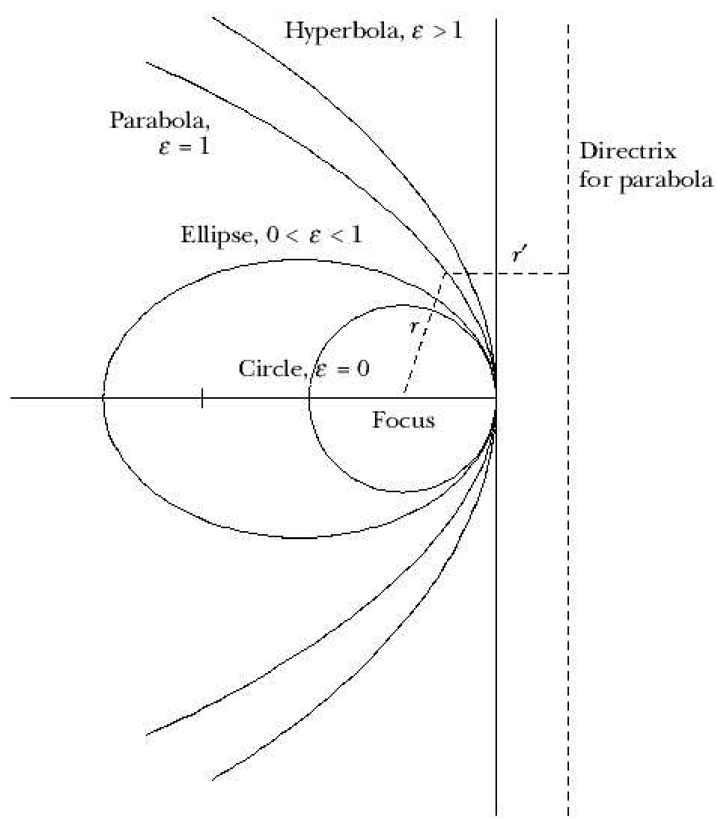


Fig. 8–8.

If we define the length of the ellipse's long axis $= 2a$, then

$$\begin{aligned} 2a &= r(0) + r(\pi) = \frac{\alpha}{1+e} + \frac{\alpha}{1-e} = \frac{2\alpha}{1-e^2} \\ \text{so} \quad \alpha &= a(1-e^2) \\ \text{and} \quad r(\theta) &= \frac{a(1-e^2)}{1+e\cos\theta} \end{aligned}$$

We just recovered Kepler's 1^{st} Law:
Planets move in elliptical orbits with the Sun at one focus.

E and ℓ

The system's total angular momentum ℓ is obtained from

$$\begin{aligned} \ell^2 &= \alpha k \mu^2 \quad \text{where} \quad k = G(m_1+m_2) \\ \text{so} \quad \ell &= \mu\sqrt{G(m_1+m_2)a(1-e^2)} \end{aligned}$$

The total energy E is

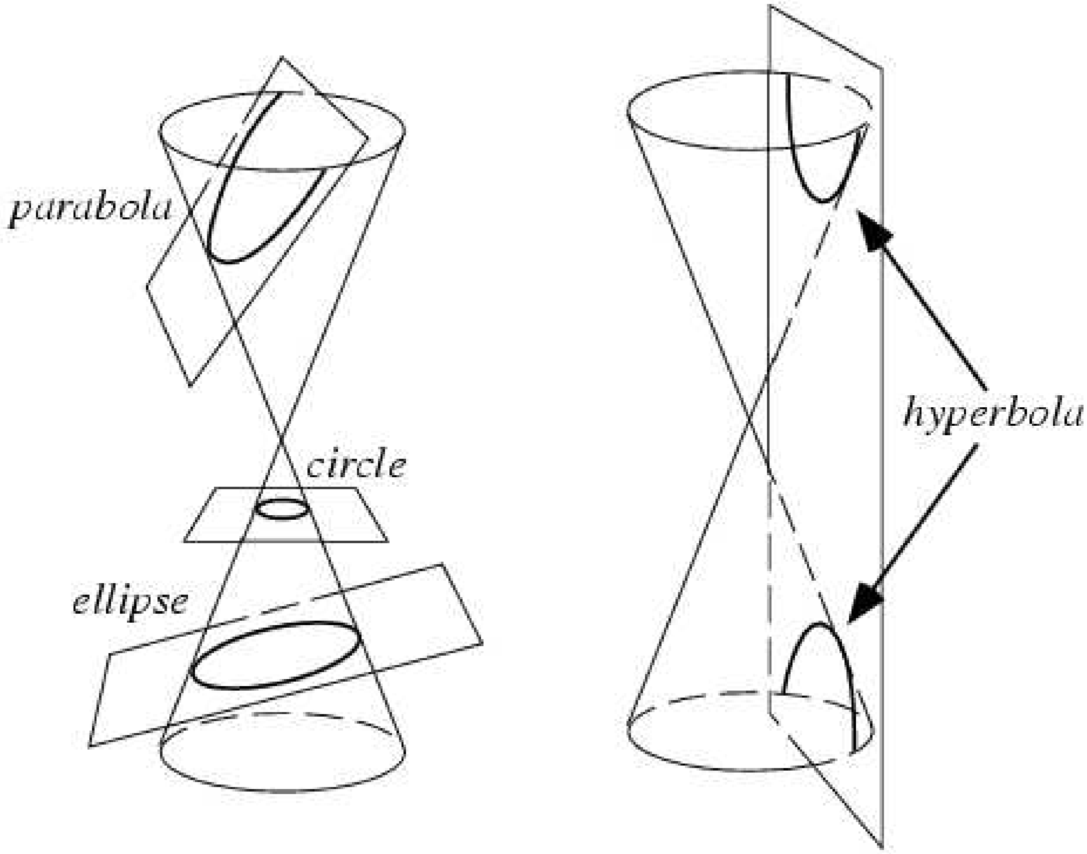
$$E \;=\; T + U = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{Gm_1m_2}{r} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{Gm_1m_2}{r} \;=\; \text{const'}$$

Since E is constant, we can evaluate it anywhere in m_2's orbit.
Choose a convenient time—what is r at that moment? $\dot{r}$?

At periapse (ie, smallest separation), $\theta = 0$, $r(\theta) = a(1-e)$, and $\dot{r} = 0$:

$$\begin{aligned} \Rightarrow E &= \frac{\mu^2 G(m_1+m_2)a(1-e^2)}{2\mu a^2(1-e)^2} - \frac{Gm_1m_2}{a(1-e)} \\ &= \frac{Gm_1m_2}{a(1-e)}\left[\frac{1-e^2}{2(1-e)} - 1\right] \\ &= \frac{Gm_1m_2}{a(1-e)}\left[\frac{1}{2}(1+e) - 1\right] \\ &= -\frac{Gm_1m_2}{2a} = \text{ total energy} \end{aligned}$$

The general solution to the two–body problem are *conic sections* ie, the intersection of a plane and a cone:



Tilting the plane increases the orbits' eccentricity e:

$$\begin{array}{rrrrl} e=0 & \text{circular orbit} & a>0 & \& \quad E<0 & \text{bound orbit} \\ 0<e<1 & \text{elliptic orbit} & a>0 & \& \quad E<0 & \text{bound orbit} \\ e=1 & \text{parabolic orbit} & r(0)=q & \& \quad E=0 & T_{orbit}\rightarrow\infty \\ e>1 & \text{hyperbolic orbit} & a<0 & \& \quad E>0 & \text{unbound}, E=\frac{1}{2}mv_{\infty}^{2} \end{array}$$

The shape of an orbit is determined by two parameters:
a = semimajor axis $\sim$ mean distance
e = eccentricity $\sim$ measure of how non–circular the orbit is.

Kepler's 3rd Law

Note that the area of an ellipse is $A = \pi ab =$, where
$a =$ ellipse's semimajor axis and $b = a\sqrt{1-e^2} =$ semiminor axis.

Now calculate the orbital period T:
Recall that the radius vector $\mathbf{r}$ sweeps across an area at the rate

$$
\begin{aligned}
\frac{dA}{dt} &= \frac{\ell}{2\mu} \\
\text{so} \quad \int_0^T \frac{dA}{dt'}dt' &= A = \pi a^2\sqrt{1-e^2} = \frac{\ell T}{2\mu} \\
\text{where} \quad \ell &= \mu\sqrt{G(m_1+m_2)a(1-e^2)} = \quad \text{ang' mom'} \\
\text{so} \quad T &= \frac{2\pi a^2\mu\sqrt{1-e^2}}{\mu\sqrt{G(m_1+m_2)a(1-e^2)}} = 2\pi\sqrt{\frac{a^3}{G(m_1+m_2)}}
\end{aligned}
$$

For a planetary system, the primary $m_1 \gg m_2$ so
$T \propto \sqrt{a^3/m_1}$, which is Kepler's third law.