Lecture Notes for PHY 405
Classical Mechanics

From Thorton & Marion's *Classical Mechanics*

Prepared by
Dr. Joseph M. Hahn
Saint Mary's University
Department of Astronomy & Physics
September 14, 2005

Chapter 3: Oscillations

SHO—simple harmonic oscillator

Start with mass m attached to a spring.
If m is displaced a distance x from equilibrium,
the spring pulls back with a force $F = -kx$, known as Hooke's Law
(Robert Hooke, 1676), where $k > 0$ is the spring constant.

Solve this problem with Newton II, $\mathbf{F} = m\ddot{\mathbf{r}}$. The EOM for mass m is

$$\begin{aligned}
\text{or} \quad F = -kx &= m\ddot{x} \quad \text{for this 1D problem} \\
\text{so} \quad \ddot{x} + \omega_0^2 x &= 0 \\
\text{where} \quad \omega_0^2 &\equiv k/m \\
\text{this EOM has solution} \quad x(t) &= A\sin(\omega_0 t - \delta) \\
\text{where} \quad A &= \text{amplitude of the motion} \\
\omega_0 &= \text{spring's angular frequency (radians/time)} \\
\delta &= \text{phase constant}
\end{aligned}$$

Energy of a SHO

$$
\begin{aligned}
\text{the KE is} \quad T &= \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\omega_0^2\cos^2(\omega_0 t - \delta) \\
\text{use} \quad \mathbf{F} &= -\nabla U, \quad \text{ie} \quad F = -\frac{\partial U}{\partial x}, \quad \text{to get the system's PE:} \\
U(x) &= -\int_0^x F\,dx = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\sin^2(\omega_0 t - \delta) \\
\text{so} \quad E &= T + U = \frac{1}{2}kA^2 = \frac{1}{2}mA^2\omega_0^2
\end{aligned}
$$

or E =constant, as expected for a conservative force.

Period of a SHO

$$
\begin{aligned}
\text{period } \tau &= \text{time for the motion to repeat} \\
\tau &= \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}} \\
\text{frequency } \nu_0 &= \frac{1}{\tau} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{\omega_0}{2\pi}
\end{aligned}
$$

Don't confuse the angular frequency ω_0 with the frequency $\nu_0 = \omega_0/2\pi$.

The plane pendulum

Mass m hangs at the end of a massless rod of length ℓ.
Use Newton II, $\mathbf{F} = m\ddot{\mathbf{r}}$, to solve for the pendulum's motion.

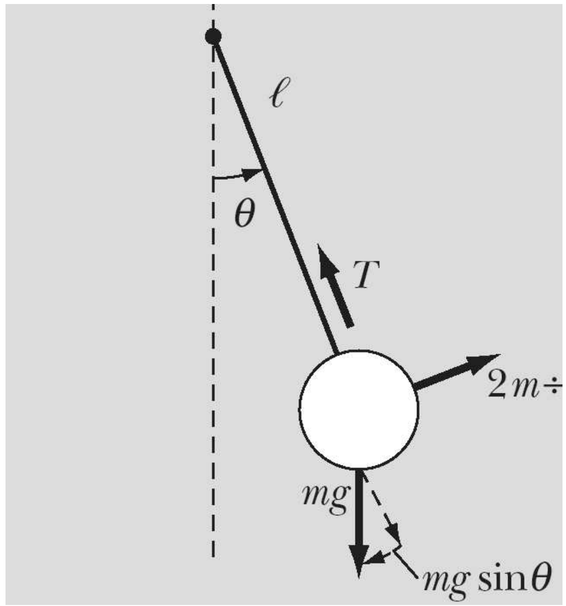


Fig. 3-13.

The pendulum is confined to a plane, so this is seemingly a 2D problem.

What is this problem's natural coordinate system?

Choose a coordinate system that confines the motion to 1D,
ie, use polar coordinates $\hat{\mathbf{r}}$ and $\hat{\theta}$

What is the mass's radial acceleration? its tangential acceleration?

$$\begin{aligned}
\text{force perpendicular to rod} \quad F_\theta &= -mg\sin\theta \\
m \text{ is an arclength} \quad & \quad \ell\theta \text{ from equilibrium position,} \\
\text{so its velocity} &= \ell\dot{\theta} \quad \text{and acceleration} = \ell\ddot{\theta} \\
\text{so NII is} \quad -mg\sin\theta &= m\ell\ddot{\theta} \\
\text{or} \quad \ddot{\theta} + \omega_0^2\sin\theta &= 0 \quad \text{where } \omega_0^2 \equiv \frac{g}{\ell}
\end{aligned}$$

This nonlinear EOM is discussed in greater detail in Chap. 4.

However we can solve this EOM when the motion is small, ie, $|\theta| \ll 1$:

$$\begin{aligned} \text{use small } \theta \text{ approx:} \quad \sin\theta &\simeq \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 \cdots \qquad \text{(D.28)} \\ \Rightarrow \quad \sin\theta &\simeq \theta \quad \text{to lowest order in small } \theta \\ \ddot{\theta} + \omega_0^2\theta &\simeq 0 \Leftarrow \text{ EOM for SHO} \end{aligned}$$

What is the solution to this EOM?

$$\theta(t) \simeq A\sin(\omega_0 t - \delta)$$

Initial conditions

The constants A & δ are determined by your initial conditions.

First note that $\dot{\theta} = A\omega_0\cos(\omega_0 t - \delta)$.

Suppose your initial conditions are

$$\begin{aligned} \theta(t=0) &= \theta_0 \quad \text{pendulum's initial displacement} \\ \dot{\theta}(t=0) &= \dot{\theta}_0 \quad \text{initial angular velocity} \end{aligned}$$

We need to write the express the unknown A, δ in terms of the known $\theta_0, \dot{\theta}_0$:

$$\begin{aligned} \text{then} \quad -A\sin\delta &= \theta_0 \\ \text{and} \quad A\cos\delta &= \dot{\theta}_0/\omega_0 \\ \text{ratio the above:} \quad \tan\delta &= -\theta_0\frac{\omega_0}{\dot{\theta}_0} \quad \text{is the phase} \\ \text{sum the squares:} \quad A^2 &= \theta_0^2 + (\dot{\theta}_0/\omega_0)^2 \quad \text{to get the amplitude} \end{aligned}$$

Phase diagrams

Note that a particle's trajectory, $x(t)$, is completely specified as a function of time once its initial conditions, $x(0)$ and $\dot{x}(0)$ are specified.

The point $P(x(t), \dot{x}(t))$ at some time t can be regarded as a spot in *phase space* at time t.

A system with n degrees of freedom has a $2n$–dimensional phase space.

As t varies over time,
$P(x, \dot{x})$ describes the particle's trajectory through this phase space.

To graphically illustrates the particle's time–history, plot $\dot{x}(t)$ versus $x(t)$. Plotting $\dot{x}$ vs x for various initial conditions thus reveals the particle's range of possible motions—this is a *phase diagram*.

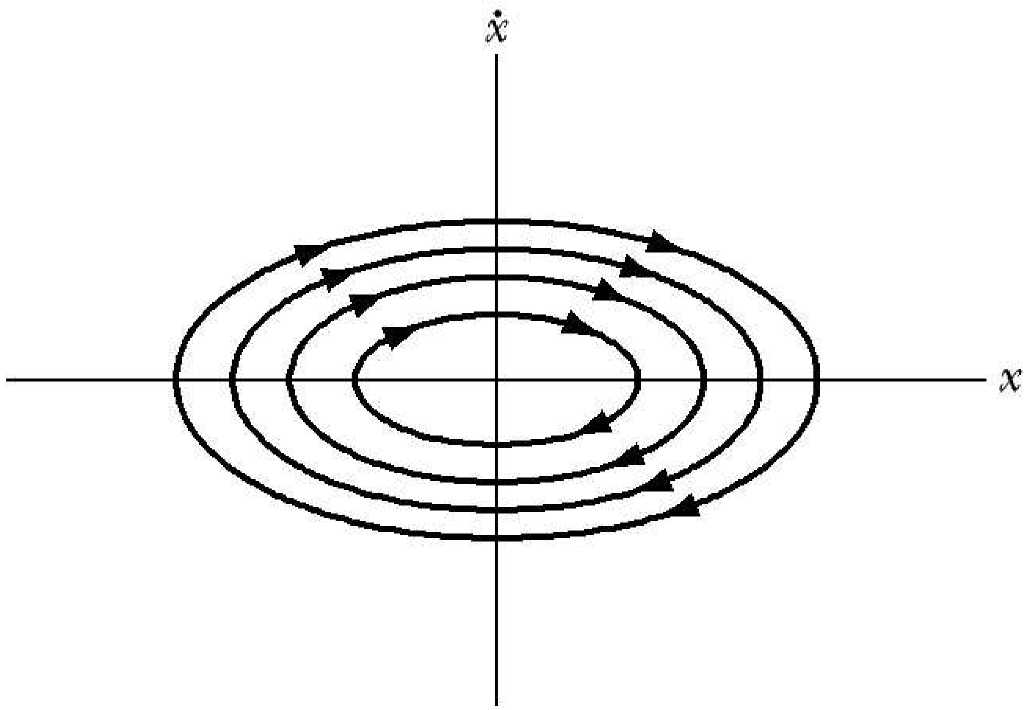


Fig. 3-5.

The SHO has a rather simple phase diagram:

$$
\begin{aligned}
x(t) &= A\sin(\omega_0 t - \delta) \\
\dot{x}(t) &= A\omega_0\cos(\omega_0 t - \delta)
\end{aligned}
$$

$$
\text{so} \quad \left(\frac{x}{A}\right)^2 + \left(\frac{\dot{x}}{A\omega_0}\right)^2 = 1,
$$

which is the equation for an ellipse where
$A =$ ellipse's semimajor axis,
$A\omega_0 =$ ellipse's semiminor axis.

The phase diagram for more complicated systems, like the NL pendulum, can be much more interesting...

Can trajectories in phase space cross?
Keep in mind that solutions to NII are *unique*.

Phase Diagram for a Nonlinear (NL) Pendulum

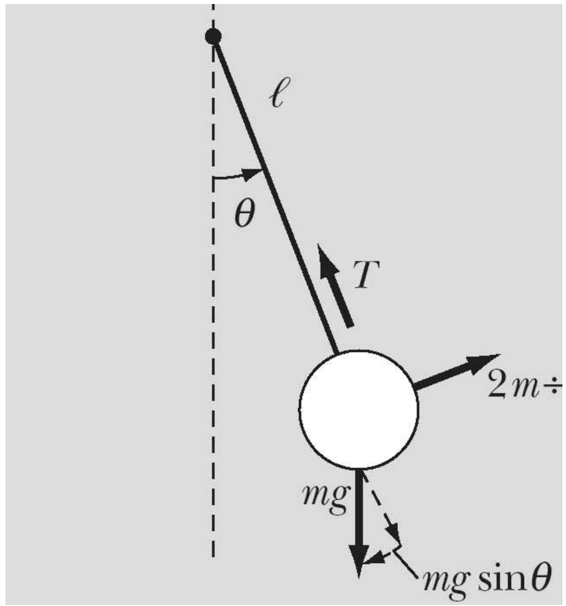


Fig. 3-13.

This discussion comes from Section 4.4.

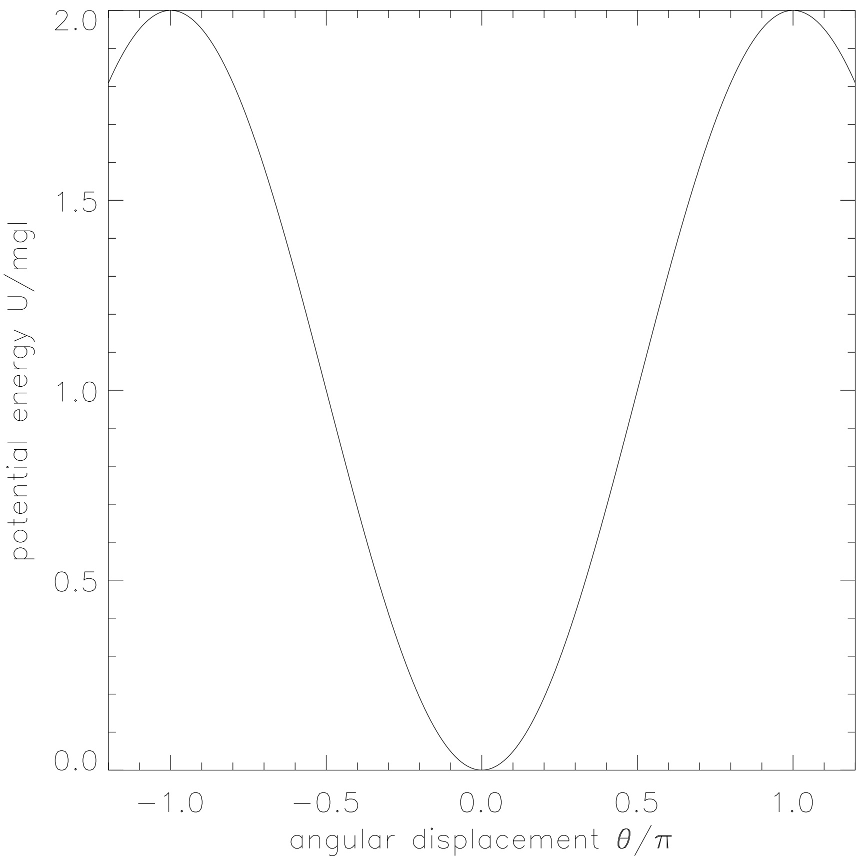
Recall that the exact EOM for mass m is $\ddot{\theta} + \omega_0^2 \sin\theta = 0$ where $\omega_0^2 = g/\ell$.

We say this EOM is NL due to the term $\sin\theta = \theta - \theta^3/6 + \cdots$ (D.28)

First, examine the system's PE, $U(\theta)$:
Recall that

$$\begin{aligned}
U(\theta) &= -\int_{\theta_{ref}=?}^{\theta} \mathbf{F}\cdot\mathbf{dr} \\
\text{where position vector} \quad \mathbf{r} &= \ell\theta\hat{\mathbf{r}} \\
\text{so differential path length} \quad \mathbf{dr} &= \ell d\theta\hat{\theta} \\
\text{and} \quad \mathbf{F} &= F_r\hat{\mathbf{r}} + F_\theta\hat{\theta} = mg\cos\theta\hat{\mathbf{r}} - mg\sin\theta\hat{\theta} \\
\text{so} \quad \mathbf{F}\cdot\mathbf{dr} &= -mg\ell\sin\theta d\theta \\
\text{and} \quad U(\theta) &= mg\ell\int_{\theta_{ref}=0}^{\theta} \sin\theta' d\theta' = -\left. mg\ell\cos\theta'\right|_0^\theta \\
&= mg\ell(1-\cos\theta) = m\ell^2\omega_0^2(1-\cos\theta)
\end{aligned}$$

Note that U is periodic in θ.



Find the equilibrium sites θ_{eq}, which is easily done by inspecting $U(\theta)$.

More formally, θ_{eq} satisfies

$$\begin{aligned} \left.\frac{\partial U}{\partial \theta}\right|_{\theta_{eq}} &= m\ell^2\omega_0^2 \sin\theta_{eq} = 0 \\ \Rightarrow \theta_{eq} &= 0, \pm\pi \end{aligned}$$

Which equilibria are stable? unstable? Again, inspect $U(\theta)$.

Alternatively, check sign of k:

$$\begin{aligned} k &= \left.\frac{\partial^2 U}{\partial \theta^2}\right|_{\theta_{eq}} = m\ell^2\omega_0^2 \cos\theta_{eq} \\ \text{for } \theta_{eq} = 0, \quad k &= m\ell^2\omega_0^2 > 0 \Rightarrow \text{ stable} \\ \text{for } \theta_{eq} = \pm\pi, \quad k &= -m\ell^2\omega_0^2 < 0 \Rightarrow \text{ unstable} \end{aligned}$$

Now calculate the system's KE, $T = \frac{1}{2}mv^2$.

If the end of the pendulum is moving with an angular velocity $\dot{\theta}$, what is the pendulum's linear velocity?

Thus $T = \frac{1}{2}m\ell^2\dot{\theta}^2$.

The system's total energy is

$$\begin{aligned} E &= T + U = \frac{1}{2}m\ell^2\dot{\theta}^2 + m\ell^2\omega_0^2(1 - \cos\theta) \\ &= \frac{1}{2}m\ell^2\omega_0^2\left[\left(\frac{\dot{\theta}}{\omega_0}\right)^2 + 2(1 - \cos\theta)\right] \end{aligned}$$

Is E conserved? Why?
Can we use this expression to draw a phase diagram, $\dot{\theta}(\theta)$? How?

Note that

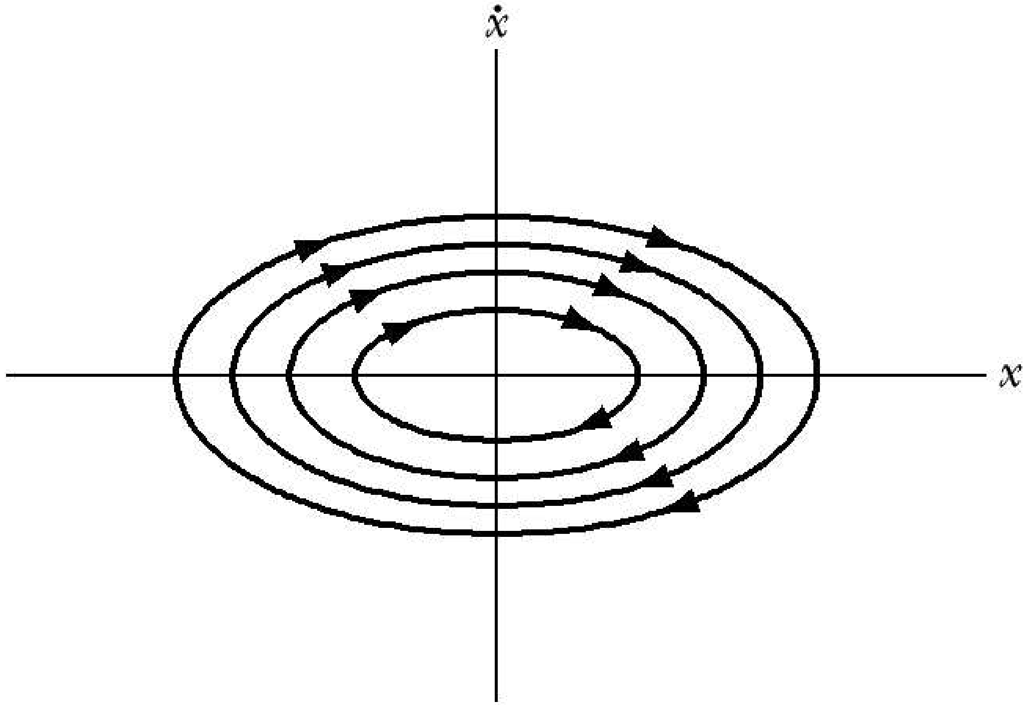
$$\begin{aligned} \left(\frac{\dot{\theta}}{\omega_0}\right)^2 + 2(1 - \cos\theta) &= \frac{2E}{m\ell^2\omega_0^2} \equiv E'^2 \\ \text{where} \quad E'^2 &= \text{a (constant) dimensionless energy} \end{aligned}$$

First, consider small–amplitude motion, ie, when $|\theta| \ll 1$:

$$\cos\theta \simeq 1 - \frac{1}{2}\theta^2 + \cdots \qquad \text{Eqn. D.29 of text}$$

$$\text{so} \quad \left(\frac{\dot{\theta}}{E'\omega_0}\right)^2 + \left(\frac{\theta}{E'}\right)^2 = 1$$

which again is the same eqn' for an ellipse as obtained previous for the SHO:



Where is the stable equilibrium site for this system?

What would happen to these trajectories if a small amount of friction were added to the system?

Friction causes particles to settle towards the stable equilibria; such sites in phase space are known as *attractors*.

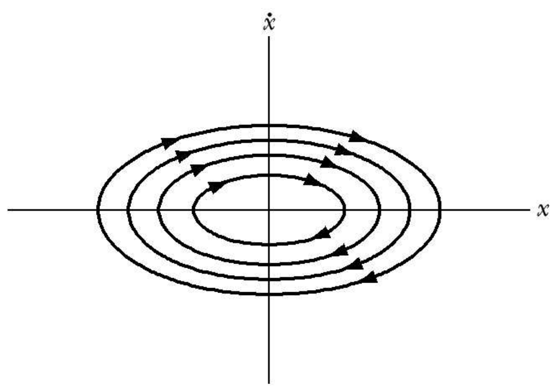
Now lets consider the more general case where the amplitude of the motion $|\theta|$ is no longer small:

$$\left(\frac{\dot{\theta}}{\omega_0}\right)^2 + 2(1 - \cos\theta) \;=\; E'^2$$

Keep in mind that the first term represents the system's KE, the second is PE.

In particular, lets consider the case of very energetic motion, ie, $E' \gg 1$ so $\dot{\theta}/\omega_0 \gg 1$ (ie, the system's energy is nearly all KE), while the PE term is $\mathcal{O}(1)$, and is negligible.

$\Rightarrow \dot{\theta} \simeq \pm E'\omega_0 \simeq$ constant:



Which way should the arrows point?

Now consider the motion of a pendulum with just enough energy to swing up to $\theta = \pi$:

What is this system's energy, E_0?

Note that in this case, $\dot{\theta} = 0$ when $\theta = \pi$,

$$\begin{aligned} \text{so} \quad E'^2 &= 4 = \frac{2E_0}{m\ell^2\omega_0^2} \\ \Rightarrow E_0 &= 2m\ell^2\omega_0^2 = U(\theta = \pi) \quad \text{as expected.} \end{aligned}$$

Add this trajectory to the phase diagram:

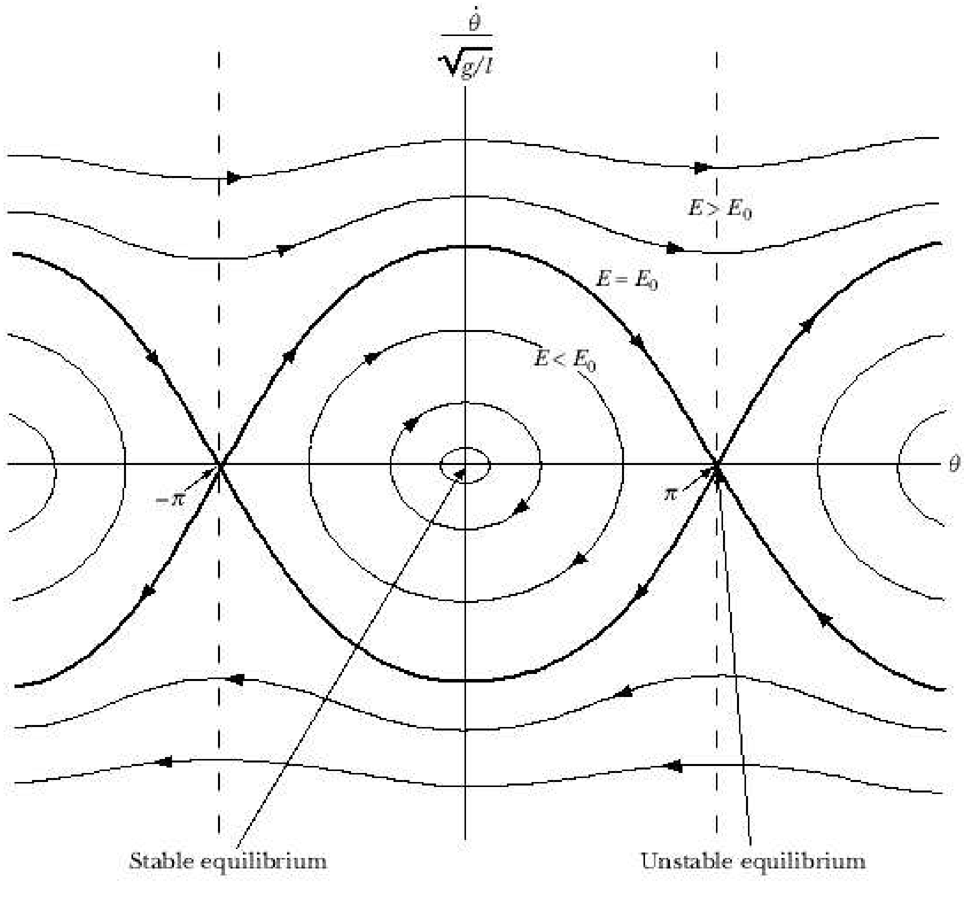


Fig. 4–11

Note that this phase diagram is periodic in θ;

The zone between $-\pi < \theta < \pi$ is referred to a *unit cell*; identical unit cells also lie left & right of the central unit cell.

What is special about the trajectory that has energy E_0?

This curve is known as the *separatrix*,
since it divides the phase space into two types of motion:

1. oscillatory motion: where $E < E_0$ and $|\theta| < \theta_{max}$, ie, the pendulum oscillates. Here, θ_{max} is the solution to $E = U(\theta_{max}) = m\ell^2\omega_0^2(1 - \cos\theta_{max})$

1. circulating motion: where $E > E_0$ and θ is unrestricted,
ie, the pendulum rotates.

Note that the separatrix passes through the unstable equilibrium points.

Note also that in our examination of the NL pendulum, we have *not* solved the system's EOM. Rather, all we have done is invoked energy conservation, and then analyzed $E(\theta, \dot{\theta})$ *graphically*.

This graphical analysis of a physics problem can be quite handy; even tho we have not solve the EOM analytically, we have obtained an understanding of the range of possible motions that a NL pendulum might exhibit, by simply sketching its phase diagram.

Problem Set #2
due ?

1. Draw a phase diagram for the system described in problem 2–43.
Locate stable and unstable equilibria in your sketch, and label the separatrix, if present.

Damped SHO

Now consider mass m attached to a spring, but add friction to the problem.

Many frictional forces are proportional to velocity,
so we will adopt $\mathbf{F}_{friction} = -b\dot{\mathbf{r}}$ where constant $b > 0$.

What is Newton II for this problem?

$$
\begin{aligned}
F &= -kx - b\dot{x} = m\ddot{x} \\
\text{so} \quad \ddot{x} &+ 2\beta\dot{x} + \omega_0^2 x = 0 \\
\text{where} \quad \beta &\equiv \frac{b}{2m} = \text{ damping parameter, has units of frequency} \\
\text{and} \quad \omega_0 &= \sqrt{\frac{k}{m}} = \text{ angular frequency, aka the natural oscillation frequency}
\end{aligned}
$$

Solution for when $\beta \neq \omega_0$:

$$
\begin{aligned}
x(t) \quad &= \quad e^{-\beta t}(A_1 e^{\gamma t} + A_2 e^{-\gamma t}) \\
\text{where} \quad A1, A2 \quad & \text{are constant amplitudes} \\
\text{and } \gamma \quad &\equiv \quad \sqrt{\beta^2 - \omega_0^2} \quad \text{which can be a real or imaginary frequency}
\end{aligned}
$$

Note that the coefficients A_1 and A_2 can also be complex.

Nonetheless, this expression should always result in $x(t)$ being real
since it is a physical quantity.

Additional problem for Problem Set #2:

2.A. Substitute $x(t)$ into the EOM and verify that it is a solution.

Underdamped motion

Occurs when $\beta < \omega_0$, eg, light damping.

In this case, γ is complex:

$$\begin{aligned} \gamma &= i\omega_1 \\ \text{where} \quad \omega_1^2 &\equiv \omega_0^2 - \beta^2 \\ \text{so} \quad x(t) &= e^{-\beta t}(A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}) \end{aligned}$$

Note that

$$\begin{aligned} e^{ia} &= \cos a + i \sin a, \quad \text{Eqn. (D.44)} \\ \text{so} \quad x &= e^{-\beta t}\left[A_1(\cos\omega_1 t + \sin\omega_1 t) + iA_2(\cos\omega_1 t - \sin\omega_1 t)\right] \\ &= e^{-\beta t}\left[(A_1 + A_2)\cos\omega_1 t + i(A_1 - A_2)\sin\omega_1 t\right] \end{aligned}$$

Since $x(t)$ is real, this implies that:
$A_1 + A_2$ is real and that $A_1 - A_2$ is imaginary.

However we can always replace the 2 unknowns A_1 and A_2
with two other unknowns A and δ that are a little more handy:

$$\begin{aligned} A_1 + A_2 &\rightarrow A\cos\delta \\ i(A_1 - A_2) &\rightarrow A\sin\delta \\ \text{so} \quad x(t) &= Ae^{-\beta t}(\cos\omega_1 t\cos\delta + \sin\omega_1 t\sin\delta) \end{aligned}$$

Simplify further using the trig identity $D.12$ of Appendix D:

$$\begin{aligned} \cos(a \pm b) &= \cos a\cos b \mp \sin a\sin b \\ \text{so} \quad x(t) &= Ae^{-\beta t}\cos(\omega_1 t - \delta) \\ \text{where} \quad \delta &= \text{phase coefficient} \end{aligned}$$

$\Rightarrow$underdamped motion=SHO $\times$ exponential damping of the amplitude over a e–fold damping timescale $\tau_{damp} = \beta^{-1} = 2m/b$.

Phase diagram for an underdamped SHO:

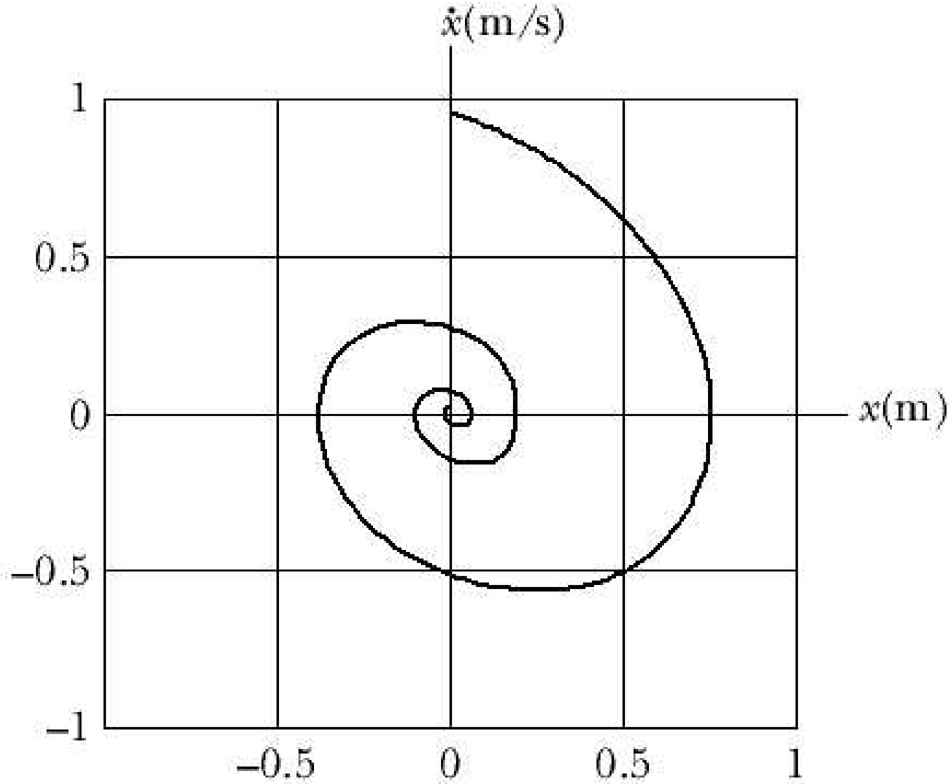


Fig. 3-10b.

The phase diagram for an underdamped SHO simply spirals towards the equilibrium point.

Note that $\omega_1 =$ is the oscillator's natural oscillation frequency,
while $T = 2\pi/\omega_1 =$ oscillation 'period'

Where in this phase diagram is this particle at time $t = T, 2T, 3T$, etc?

Also plot $x(t)$ for this underdamped oscillator.

Overdamped motion

Recall that the EOM is

$$\begin{aligned} \ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= 0 \\ \text{which has solution} \quad x(t) &= e^{-\beta t}(A_1 e^{\gamma t} + A_2 e^{\gamma t}) \quad \text{where } \gamma = \sqrt{\beta^2 - \omega_0^2} \end{aligned}$$

Overdamped motion occurs when $\beta > \omega_0$, ie, the damping is heavy:

$$\begin{aligned} \gamma &= \sqrt{\beta^2 - \omega_0^2} \quad \text{is real} \\ \text{so} \quad x(t) &= A_1 e^{-(\beta-\gamma)t} + A_2 e^{-(\beta+\gamma)t} \\ \text{where constants} \quad A_1 \ &\& \ A_2 \quad \text{are determined by initial conditions} \\ \text{overdamped systems} &\Rightarrow \text{exponential return to equilibrium} \end{aligned}$$

Overdamped systems do not oscillate.

Occurs when $\beta = \omega_0$.
Solution $x(t) = (A + Bt)e^{-\beta t}$.

Additional problem for Assignment #2:

2.B. show that this is a solution of the EOM.

Critical damping returns the particles to its equilibrium with the fewest overshoots.

Examples of critically damped systems: the pneumatic tube on a screen door, auto shock absorbers.

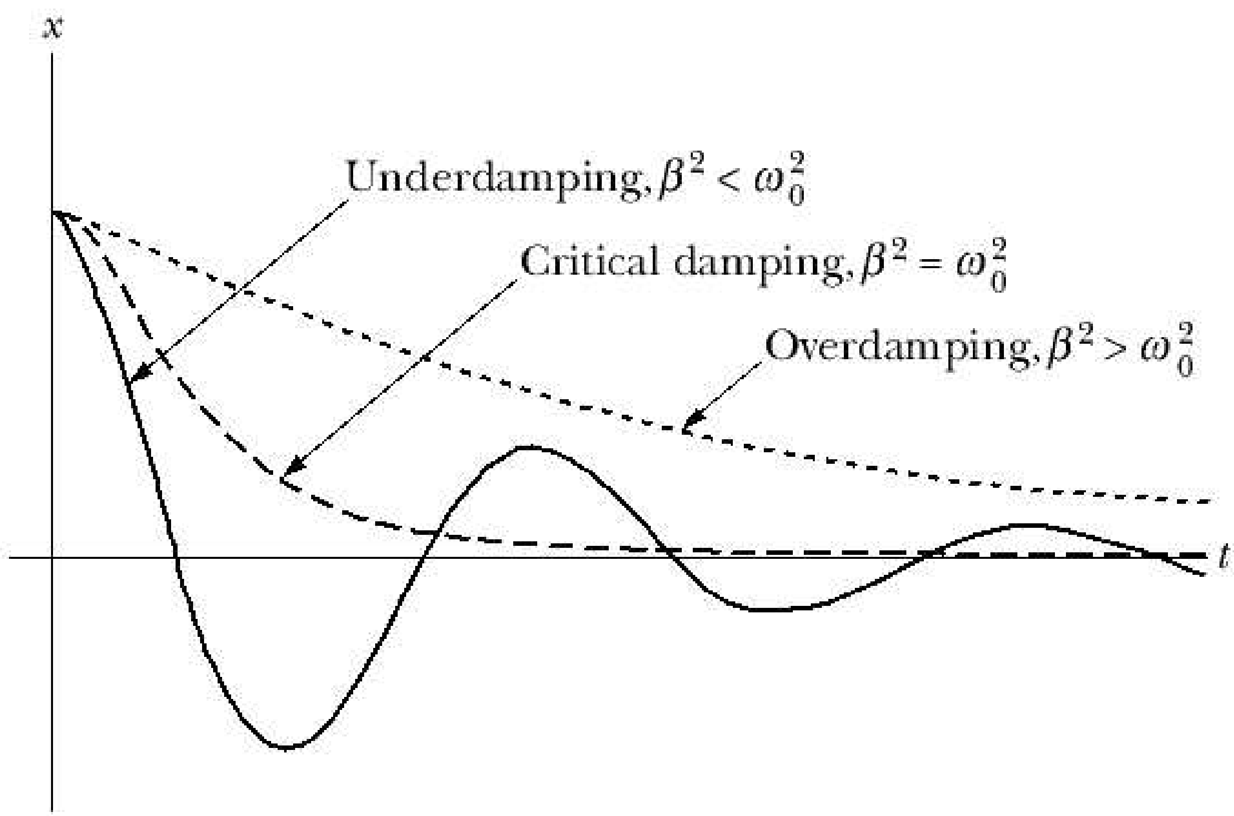


Fig. 3-6.

Problem Set #2
due Tuesday October 4
at start of class

Problems 1, 2A, 2B, & text problems 3–7, 3–11, 3–40, 3–42, 3–45.

Example 3.3

A pendulum of mass m and length ℓ moves through oil that resists the motion with force $F_{res} = -2m\sqrt{g/\ell}(\ell\dot{\theta})$. The bob is initially released from position $\theta = \theta_0$. Find the motion, $\theta(t), \dot{\theta}(t)$, and sketch the system's phase diagram.

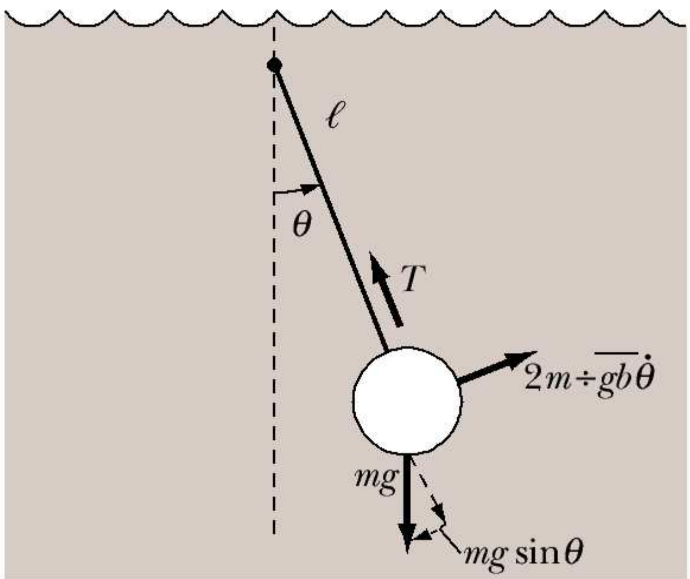


Fig. 3-13.

What coordinate system should we use?

Asses the forces:
gravitational $\mathbf{F}_{grav} = mg\cos\theta\hat{\mathbf{r}} - mg\sin\theta\hat{\theta}$
frictional $\mathbf{F}_{res} = -2m\sqrt{g/\ell}(\ell\dot{\theta})\hat{\theta}$

Note that the radial forces balance: $T = mg\cos\theta$

$$\text{the } \hat{\theta} \text{ force:} \quad F_\theta = -mg\sin\theta - 2m\sqrt{g/\ell}(\ell\dot{\theta}) = ma_\theta$$

What is the acceleration a_θ in the $\hat{\theta}$ direction? $\ell\ddot{\theta}$,

$$\begin{aligned}
\text{so the EOM is} \quad \ddot{\theta} + 2\sqrt{\frac{g}{\ell}}\dot{\theta} + \frac{g}{\ell}\sin\theta &= 0 \\
\text{or} \quad \ddot{\theta} + 2\beta\dot{\theta} + \omega_0^2\sin\theta &= 0 \\
\text{where} \quad \beta = \sqrt{\frac{g}{\ell}} \quad \text{and } \omega_0^2 &= \frac{g}{\ell} \\
\text{For small displacements} \quad \sin\theta &\simeq \theta \\
\text{so} \quad \ddot{\theta} + 2\beta\dot{\theta} + \omega_0^2\theta &= 0
\end{aligned}$$

Is the bob's motion underdamped? Overdamped?

Since $\beta = \omega_0$, the motion is critically damped.

What is the solution for the bob's motion?

Recall that $\theta(t) = (A + Bt)e^{-\beta t}$ where constants A, B
are determined by initial conditions: $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$

$\Rightarrow A = \theta_0$.

Since $\dot{\theta} = [B - \beta(A + Bt)]e^{-\beta t} = 0$ when $t = 0 \Rightarrow B = \beta A$ so

$$\begin{aligned}
\theta(t) &= \theta_0(1 + \beta t)e^{-\beta t} \\
\dot{\theta}(t) &= -\theta_0\beta^2 t e^{-\beta t}
\end{aligned}$$

Note that $\theta \to 0$ when $t \gg T_{damping}$,
where $T_{damping} = 1/\beta$ is the damping timescale.

Also note that the angular velocity $\dot{\theta}$ initially grows
(in a negative sense) as $\dot{\theta} \simeq -\theta_0 \beta^2 t$ when $t \ll T_{damping}$,
but that $\dot{\theta} \to 0$ when $t \gg T_{damping}$.

Ultimately, all trajectories dive for the origin in the phase diagram:

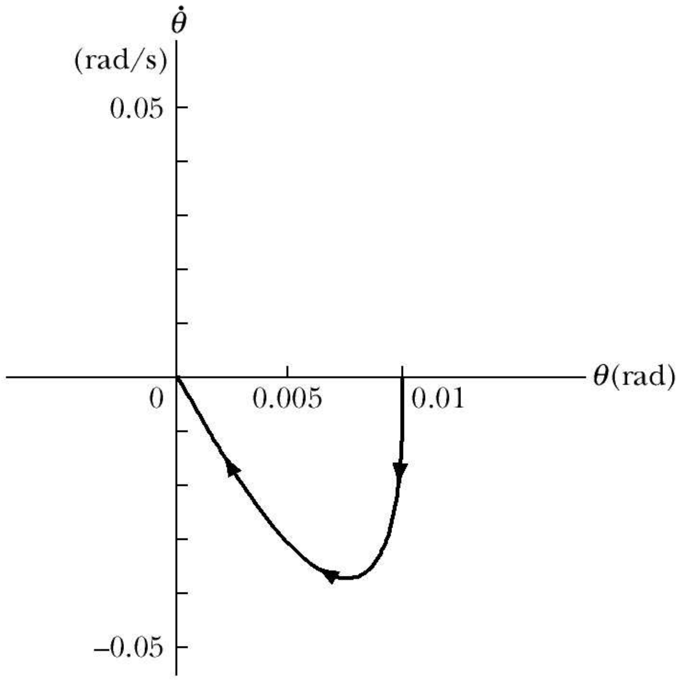


Fig. 3-14.

Are there any attractors in this phase diagram?

Sinusoidal Driving Forces

Again, lets consider mass m attached to spring k that rests on a place with friction coefficient b.

But now add a sinusoidal driving force $F_0 \cos(\omega t)$ due to a piston that is also pushing on m:

$$\begin{aligned} \text{force} \quad F(t) &= -kx - b\dot{x} + F_0 \cos(\omega t) = m\ddot{x} \\ \text{so} \quad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= A\cos(\omega t) \end{aligned}$$

where $A = F_0/m =$ the driving acceleration, and $\omega =$ driving frequency.

The general solution is

$$\begin{aligned} x(t) &= x_c(t) + x_p(t) \\ \text{where} \quad x_c(t) &= \textit{complimentary} \text{ solution, ie, the solution for when } A = 0 \\ \text{and} \quad x_p(t) &= \textit{particular} \text{ solution, ie, the solution for when } A \neq 0 \end{aligned}$$

Note that we have already obtained the complimentary solution:

$$\begin{aligned} x_c(t) &= e^{-\beta t}(A_1 e^{\gamma t} + A_2 e^{-\gamma t}) \\ \text{where} \quad \gamma &= \sqrt{\beta^2 - \omega_0^2} \end{aligned}$$

x_c is called the *transient* part of the solution since it decays away as $\sim e^{-\beta t}$.

The constants A_1 and A_2 will depend on initial conditions $x(0)$ and $\dot{x}(0)$.

However the system eventually 'loses memory' of initial conditions (eg, A_1, A_2) later when $t \gg T_{damping} = 1/\beta$.

Now get the particular solution x_p:

Insert $x = x_c + x_p$ into the EOM.
Obviously, only the x_p terms survive:

$$\ddot{x}_p \ + \ 2\beta \dot{x}_p + \omega_0^2 x_p = A \cos(\omega t)$$

Since the driving force $\propto \cos(\omega t)$,
we likely need a similar term in the particular solution. Try

$$x_p(t) \ = \ D \cos(\omega t - \delta) \quad \text{where amplitude } D \text{ and phase-shift } \delta \text{ are constants}$$

It will be convenient to expand x_p using trig identity D.12:

$$\begin{aligned}
\cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\
\text{so} \quad x_p &= D \cos(\omega t - \delta) = D(\cos \omega t \cos \delta + \sin \omega t \sin \delta) \\
\text{and} \quad \dot{x}_p &= \omega D(-\sin \omega t \cos \delta + \cos \omega t \sin \delta) \\
\text{and} \quad \ddot{x}_p &= \omega^2 D(-\cos \omega t \cos \delta - \sin \omega t \sin \delta)
\end{aligned}$$

Insert these into equation of motion and collect $\sin \omega t$ and $\cos \omega t$ terms:

$$\begin{aligned}
\cos \omega t\{-\omega^2 D \cos \delta + 2\beta\omega D \sin \delta + \omega_0^2 D \cos \delta - A]+ & \\
\sin \omega t[-\omega^2 D \sin \delta - 2\beta\omega D \cos \delta + \omega_0^2 D \sin \delta\} = & \quad 0 \\
\text{or} \quad \cos \omega t\{-A + D[(\omega_0^2 - \omega^2) \cos \delta + 2\beta\omega \sin \delta]\}+ & \\
\sin \omega t D\{(\omega_0^2 - \omega^2) \sin \delta - 2\beta\omega \cos \delta\} = & \quad 0
\end{aligned}$$

Note that the LHS must be zero for any time t.
What does this say about the coefficients in the $\{\}$?

the $\sin\omega t$ coefficient yields

$$\begin{aligned}
(\omega_0^2 - \omega^2)\sin\delta &= 2\beta\omega\cos\delta \\
\text{so} \quad \frac{\sin\delta}{\cos\delta} = \tan\delta &= \frac{2\beta\omega}{\omega_0^2 - \omega^2} \equiv \frac{Y}{X} \\
\text{thus} \quad \sin\delta = \frac{Y}{\sqrt{X^2+Y^2}} &= \frac{2\beta\omega}{\sqrt{(\omega_0^2-\omega^2)^2 + (2\beta\omega)^2}} \\
\text{and} \quad \cos\delta = \frac{X}{\sqrt{X^2+Y^2}} &= \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2-\omega^2)^2 + (2\beta\omega)^2}}
\end{aligned}$$

Note that the driving force has a phase ωt,
while the system's response has the phase $\omega t - \delta$,
so δ is the *phase difference* that exists between the driving force and the response of the system; it is a consequence of friction, β.

Now get the amplitude D from the coefficient of the $\cos\omega t$ term:

$$\begin{aligned}
D[(\omega_0^2 - \omega^2)\cos\delta \; + \; 2\beta\omega\sin\delta] = A \\
\text{so} \quad D &= \frac{A}{(\omega_0^2 - \omega^2)\cos\delta + 2\beta\omega\sin\delta} \\
&= A/[(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2]/\sqrt{(\omega_0^2-\omega^2)^2 + (2\beta\omega)^2} \\
&= \frac{A}{\sqrt{(\omega_0^2-\omega^2)^2 + (2\beta\omega)^2}}
\end{aligned}$$

D is often referred to as the system's *forced* amplitude.

It is usually convenient to think of $D(\omega)$ and $\delta(\omega)$
as functions of the driving frequency ω.

The complete solution for the motion of the damped/driven oscillator is

$$x(t) = x_c + x_p = x_c + D(\omega)\cos[\omega t - \delta(\omega)]$$

where the complimentary (ie, transient) solution x_c decays away as $e^{-\beta t}$.

The *steady–state* part, x_p, is usually the more interesting part of the solution.

Note that all of the information about the system's initial conditions, $x(0)$ and $\dot{x}(0)$, are contained in the transient part x_c, so the system eventually loses memory of its initial state as $x_c \to 0$.

Resonances

When building a driven oscillator, the driving frequency ω is usually tunable.

Often you are interested in determining the driving frequency ω that maximizes the system's response $D(\omega)$.

This occurs are the resonant frequency $\omega = \omega_R$:

$$\begin{aligned} \text{write} \quad D(\omega) &= \frac{A}{\sqrt{f(\omega)}} \\ \text{where} \quad f(\omega) &= (\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2 \end{aligned}$$

The resonance frequency ω_R is obtained by setting $df/d\omega|_{\omega_R} = 0$, ie

$$\begin{aligned} 2(\omega_0^2 - \omega_R^2)(-2\omega_R) + 8\beta^2\omega_R &= 0 \\ \text{so} \quad \omega_0^2 - \omega_R^2 &= 2\beta^2 \quad \text{or} \quad \omega_R = \sqrt{\omega_0^2 - 2\beta^2} \end{aligned}$$

Note that for a lightly damped system with $\beta \ll \omega_0$, the resonance frequency is nearly the system's naturally oscillation frequency, $\omega_R \simeq \omega_0$.

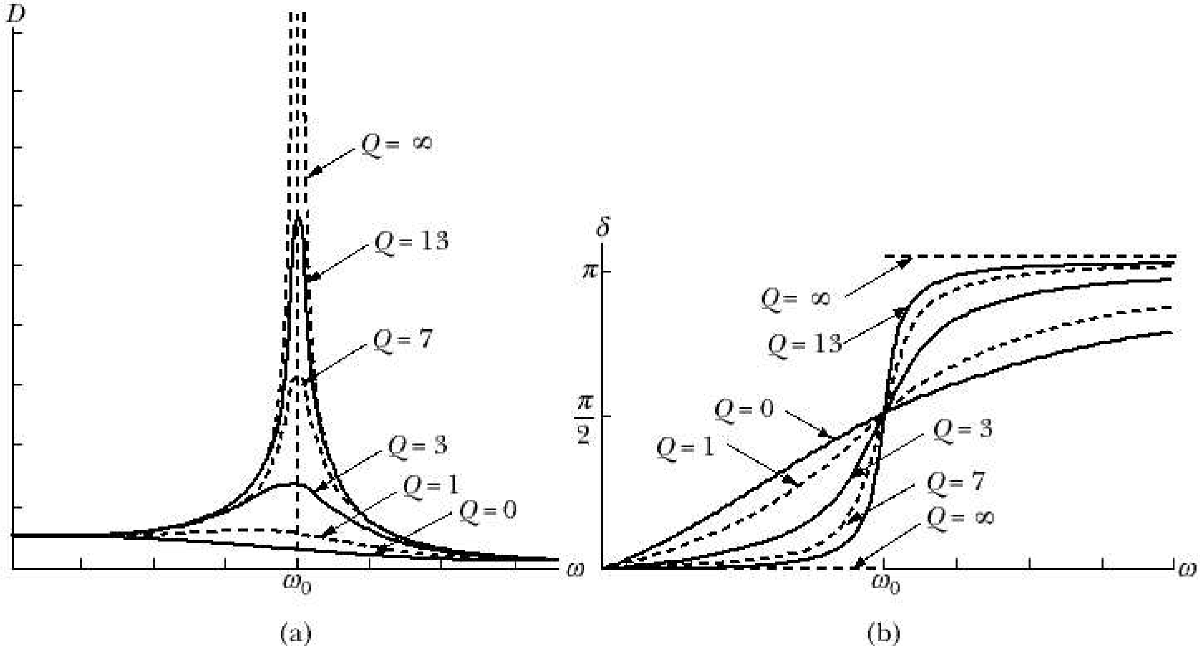


Fig. 3-16.

Recall that

$$D(\omega) = \frac{A}{\sqrt{(\omega_0^2-\omega^2)^2+(2\beta\omega)^2}} = \text{ response to driving force}$$

and $$\sin\delta = \frac{2\beta\omega}{\sqrt{(\omega_0^2-\omega^2)^2+(2\beta\omega)^2}} \quad \text{and} \quad \cos\delta = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+(2\beta\omega)^2}}$$

Another characteristic of a resonance is that the system's response is *in–phase* when $\omega \ll \omega_R$. For a lightly damped system ($\beta \ll \omega_0$) $\sin\delta \simeq 2\beta\omega/\omega_0^2 \ll 1 \Rightarrow \delta \simeq 0$.

But when $\omega \gg \omega_R$, $\cos\delta \simeq -1 \Rightarrow \delta \simeq \pi$

Plots of $D(\omega)$ and $\delta(\omega)$ can all be parameterized by a dimensionless 'quality factor' Q:

$$Q = \frac{\omega_R}{2\beta}$$

Q is a dimensionless measure of the friction in your oscillator.
It is also an indicator of the 'width' of the resonance,
Example: an audio speaker is a driven/damped oscillator, typically has $Q \sim 100$.

Electronic circuits, like a stereo receiver tuned to a radio station, has $Q \sim 10^4$, while lasers can have $Q \sim 10^{10}$.

The Fig. shows that a high–Q system (ie. one with small friction β) has as large response (ie. large $D(\omega)$) at the resonant frequency $\omega = \omega_R$.

Simple example

Springs k_1 and $k_2 = 2k_1$ are attached to mass m which slides with friction coefficient b or $\beta = b/2m = \sqrt{k_1/100m}$. What is this system's Q?

$$
\begin{aligned}
\text{the EOM is} \quad F = -k_1 x - k_2 x - b\dot{x} &= m\ddot{x} \\
\text{so} \quad \ddot{x} + \omega_0^2 x + 2\beta\dot{x} &= 0 \\
\text{where} \quad \omega_0^2 = (k_1 + k_2)/m = 3k_1/m &= \text{natural osc' freq'}^2 \\
\text{the resonant frequency is } \omega_R &= \sqrt{\omega_0^2 - 2\beta^2} \\
&= \sqrt{\frac{3k_1}{m} - \frac{2k_1}{100m}} = \sqrt{\frac{298k_1}{100m}} \\
\text{and} \quad Q &= \frac{\omega_R}{2\beta} = \frac{\sqrt{298k_1/100m}}{2\sqrt{k_1/100m}} = \frac{\sqrt{298}}{2} \simeq 8.6
\end{aligned}
$$

Not an especially high–quality oscillator...

Superposition of Discrete Driving Forces

Thus far we have considered a damped SHO driven by a single sinusoidal force $F(t) = mA\cos(\omega t)$:

$$\begin{aligned} \text{the EOM is} \quad \ddot{x} + \omega_0^2 x + 2\beta\dot{x} &= A\cos(\omega t) \\ \text{and its solution is} \quad x(t) &= x_c(t) + x_p(t) \end{aligned}$$

where $x_c =$ complimentary (or transient) part, is damped by friction,
and $x_p = D\cos(\omega t - \delta) =$ particular solution (eg, the forced response).

What if there two distinct driving forces operating with different frequencies: $F(t)/m = A_1\cos(\omega_1 t - \phi_1) + A_2\cos(\omega_2 t - \phi_2)$?

How will the system's forced solution x_p change?

(Note that the driving force has acquired a phase ϕ;
to account for this, simply replace $\omega t \to \omega t - \phi$ in the solution.)

We anticipate that $x_p = D_1\cos(\omega_1 t - \phi_1 - \delta_1) + D_2\cos(\omega_2 t - \phi_2 - \delta_2)$
$\Rightarrow$ for each driving acceleration A_n operating at frequency ω_n,
the system responds with a displacement $x_n(t) = D_n\cos(\omega_n t - \phi_n - \delta_n)$.

And if there are N distinct driving forces

$$F(t) = \sum_{n=1}^{N} F_n(t) = m\sum_{n=1}^{N} A_n\cos(\omega_n t - \phi_n)$$

we anticipate the solution is $x(t) = x_c(t) + x_p(t) = x_c + \sum x_n(t)$.

To verify this solution, insert $x = x_c + x_p$ into the EOM:
$\ddot{x}_p + \omega_0^2 x_p + 2\beta\dot{x}_p = F/m$.

Recall that the contribution from the transient part, x_c, sum to zero, so

$$\sum_{n=1}^{N} \left[\ddot{x}_n + \omega_0^2 x_n + 2\beta\dot{x}_n - A_n \cos(\omega_n t - \phi_n)\right] \;=\; 0$$

Note that the x_n and its derivatives are sinusoids in $\omega_n t$, which means that they are drawn from a complete set of orthogonal functions.

What does this say about the quantity in the []?

$$\Rightarrow \ddot{x}_n + \omega_0^2 x_n + 2\beta\dot{x}_n = A_n \cos(\omega_n t - \phi_n)$$

and we already know the solution to this eqn:

$$\begin{aligned}
x_n(t) &= D_n \cos(\omega_n t - \phi_n - \delta_n) \\
\text{where} \quad D_n(\omega_n) &= \frac{A_n}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + (2\beta\omega_n)^2}} = \text{amplitude of } n^{th} \text{ response} \\
\text{and} \tan\delta_n &= \frac{2\beta\omega_n}{\omega_0^2 - \omega_n^2} = n^{th} \text{ phase shift}
\end{aligned}$$

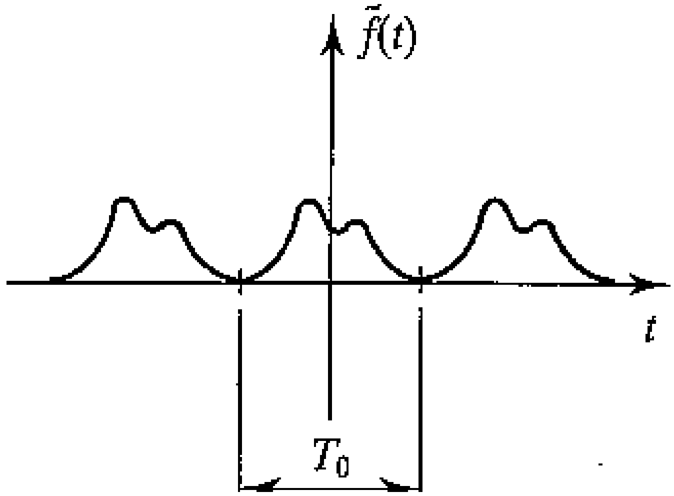
The total solution is $x(t) = x_c(t) + \sum x_n(t)$ where x_c is the transient response, determined by initial conditions.

Continuous Spectrum of Driving Forces

We just solve for the motion of a damped SHO when it is driven by a combination of N discrete forces $F(t) = \sum F_n$
that operate at N discrete frequencies ω_n.

However this solution might not appear to be very useful, since it might not be widely applicable.

We would much rather have the solution to the more general problem where $F(t) =$ any arbitrary driving force that is periodic over time $T = 2\pi/\omega$:



How can we use the preceding results to solve this more general problem?

We would like to replace $F(t)$ with $\sum F_n(t)$,
since we know the solution to the EOM in this case.

To do this, *Fourier expand* the driving force $F(t)$, that is, assume that

$$\frac{F(t)}{m} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

which can also be written as $$= \sum_{n=0}^{\infty} A_n \cos(n\omega t - \phi_n) \text{ via trig identity}$$

Note we have replaced the n^{th} driving frequency $\omega_n \rightarrow n\omega$, where $\omega = 2\pi/T$.

The solution to the EOM has the usual form $x(t) = x_c + x_p$ where

$$\begin{aligned} x_p &= \sum_{n=0}^{\infty} x_n(t) \\ \text{where} \quad x_n(t) &= D_n \cos(n\omega t - \phi_n - \delta_n) \end{aligned}$$

where D_n and δ_n are (since $\omega_n \to n\omega$):

$$\begin{aligned} D_n(\omega) &= \frac{A_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2\beta n\omega)^2}} \\ \text{and} \quad \tan\delta_n &= \frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2} \end{aligned}$$

Applying these results to the problem of a damped SHO that is driven by an arbitrary force $F(t)$ that is periodic in time $T = 2\pi/\omega$ requires doing the following calculations:

1. Fourier expand $F(t)$ to get the a_n, b_n coefficients

2. relating the a_n, b_n to the A_n, ϕ_n

3. Forming the solution $x(t) = x_c + \sum x_n$

4. Satisfying initial conditions $x(0)$ and $\dot{x}(0)$,
which yields the A_1, A_2 appearing in $x_c = e^{-\beta t}(A_1 e^{\gamma t} + A_2 e^{-\gamma t})$

Example: a damped SHO driven by a sawtooth function

This calculation come from example 3.6, and will illustrate the above 4 steps.

A SHO having natural oscillation frequency ω_0 and a damping frequency β is driven by a sawtooth function:

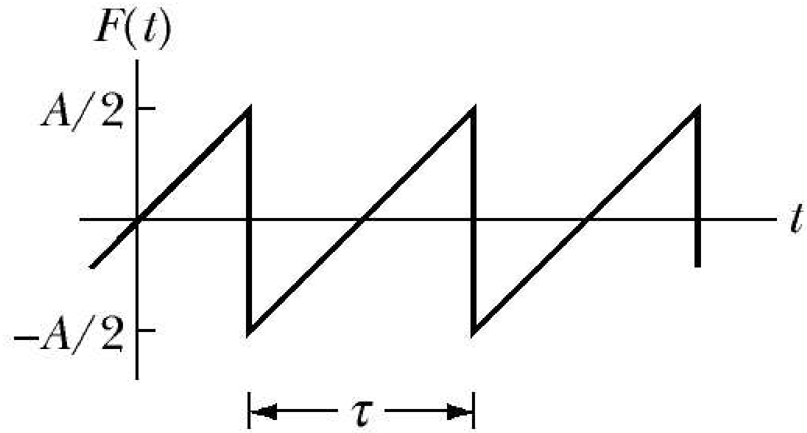


Fig. 3-19.

$$\frac{F(t)}{m} = \frac{At}{\tau} \quad \text{over time } -\tau/2 < t < \tau/2,$$

and repeating over other times.

What is the driving frequency ω?

Step 1: Fourier expand $F(t)$

Assume that $F(t)$ can be written as

$$\frac{F(t)}{m} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

To get the a_n coefficient, multiply by $\cos(n'\omega t)$ where n' = unspecified int', and integrate over one forcing cycle $\tau = 2\pi/\omega$:

$$\frac{1}{m}\int_{-\tau/2}^{\tau/2} F(t)\cos(n'\omega t)dt = \sum_{n=1}^{\infty} a_n \int_{-\tau/2}^{\tau/2} \cos(n\omega t)\cos(n'\omega t)dt$$

use trig identity, from D.12 $\quad \cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$

$$\text{so} \quad \cos(n\omega t)\cos(n'\omega t) = \frac{1}{2}\cos[(n+n')\omega t] + \frac{1}{2}\cos[(n-n')\omega t]$$

$$\text{and} \quad \text{RHS} = \sum_{n=1}^{\infty} \frac{1}{2}a_n \int_{-\tau/2}^{\tau/2} \{\cos[(n+n')\omega t] + \cos[(n-n')\omega t]\}dt$$

Consider only single n^{th} term in this sum.

When $n \neq n'$, what do these integrals evaluate to?

What about the $n = n'$ term?

$$\begin{aligned}
\Rightarrow \text{RHS} &= \frac{1}{2}a_{n'} \int_{-\tau/2}^{\tau/2} dt \\
&= \frac{\tau}{2}a_{n'} = \frac{\pi}{\omega}a_{n'} \\
\Rightarrow a_n &= \frac{\omega}{\pi}\int_{-\tau/2}^{\tau/2} \frac{F(t)}{m}\cos(n\omega t)dt
\end{aligned}$$

How do you solve for the b_n?

$$b_n = \frac{\omega}{\pi}\int_{-\tau/2}^{\tau/2} \frac{F(t)}{m}\sin(n\omega t)dt$$

These are the general formulae for the a_n, b_n.

$$
\begin{aligned}
\frac{F(t)}{m} &= \frac{At}{\tau} \\
\text{so} \quad a_n &= \frac{A\omega}{\pi\tau} \int_{-\tau/2}^{\tau/2} t \cos(n\omega t) dt
\end{aligned}
$$

Note that the integrand $f(t) = t\cos(n\omega t)$ is *odd*, ie $f(-t) = -f(t)$.

What does this say about the a_n?

An even function has $f(-t) = f(t)$.

$$
\begin{aligned}
\text{so} \quad b_n &= \frac{A\omega}{\pi\tau} \int_{-\tau/2}^{\tau/2} t \sin(n\omega t) dt \\
u\text{–substitution:} \quad u &= n\omega t \quad \text{so} \quad du = n\omega dt \\
\text{and} \quad b_n &= \frac{A\omega}{\pi\tau(n\omega)^2} \int_{-n\pi}^{+n\omega\tau/2=+n\pi} u \sin(u) du \\
&= \frac{A}{2\pi^2 n^2} (\sin u - u \cos u)|_{-n\pi}^{+n\pi} \\
&= \frac{A}{2\pi^2 n^2} [\sin(n\pi) - \sin(-n\pi) - n\pi\cos(n\pi) - n\pi\cos(-n\pi)] \\
&= -\frac{A\cos(n\pi)}{\pi n} = -\frac{A(-1)^n}{\pi n} = \frac{A(-1)^{n+1}}{\pi n}
\end{aligned}
$$

sawtooth example

$$
\text{Thus}\quad \frac{F(t)}{m} = \frac{At}{\tau} = \sum_{n=1}^{\infty} b_n \sin(n\omega t)
$$

$$
= \frac{A}{\pi}\left[\sin(\omega t) - \frac{1}{2}\sin(2\omega t) + \frac{1}{3}\sin(3\omega t)\cdots\right]
$$

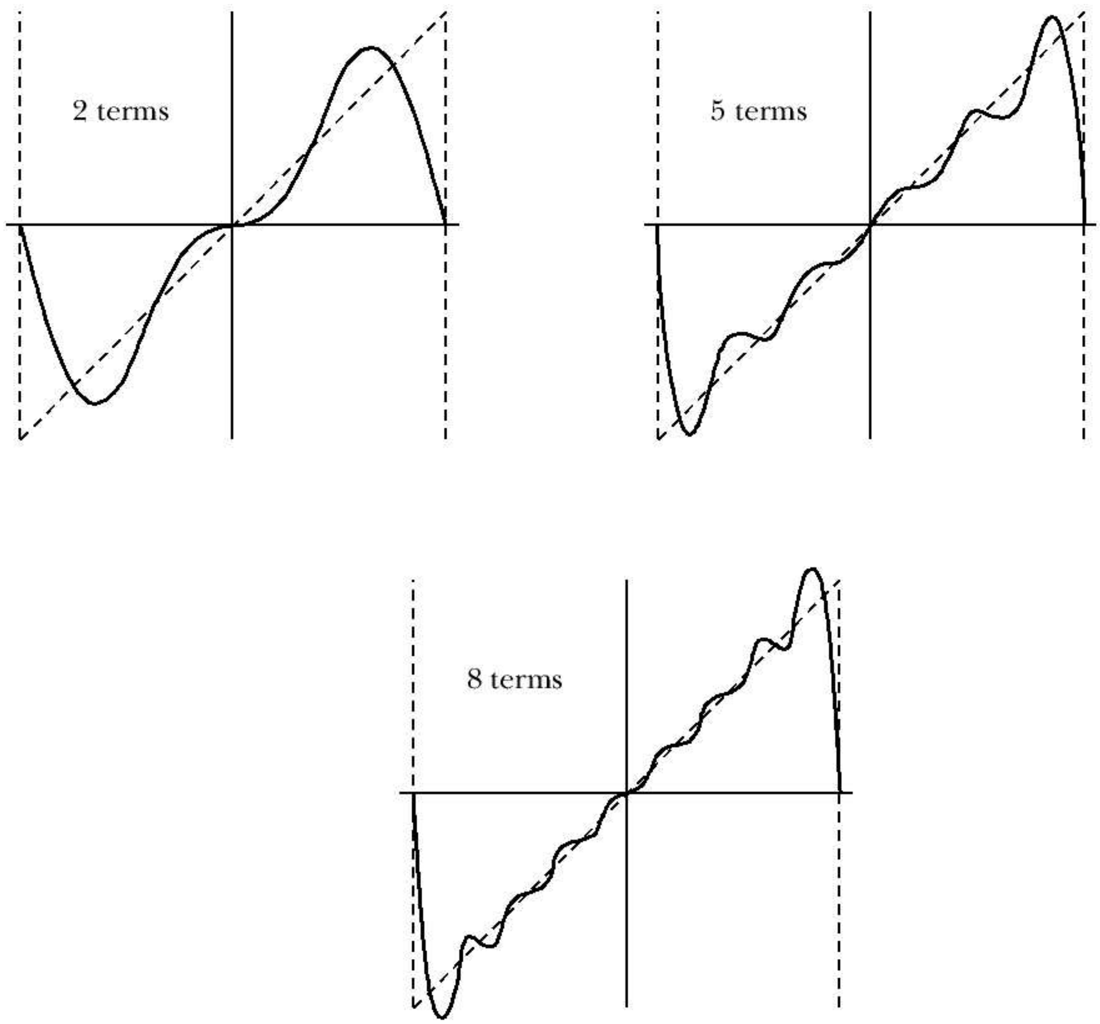


Fig. 3-20.

In practice, one keeps simply sums enough terms in the infinite series that your representation of the force $F(t)$ is sufficiently accurate.

Step 2: Get the A_n, ϕ_n from the a_n, b_n

Recall that in general,

$$\begin{aligned}
\frac{F(t)}{m} &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right] = \sum_{n=0}^{\infty} A_n\cos(n\omega t - \phi_n) \\
&= \sum_{n=0}^{\infty}\left[A_n\cos(n\omega t)\cos\phi_n + A_n\sin(n\omega t)\sin\phi_n\right] \quad \text{by (D.12)} \\
&\Rightarrow a_n = A_n\cos\phi_n \quad \text{and} \quad b_n = A_n\sin\phi_n \\
\text{so} \quad A_n &= \sqrt{a_n^2 + b_n^2} \quad \text{and} \quad \tan\phi_n = b_n/a_n
\end{aligned}$$

back to sawtooth example

In this case, $A_n = |b_n| = A/\pi n$,

while $\tan\phi_n = b_n/a_n$ is problematic for this example.

$$\begin{aligned}
\text{Instead use} \quad b_n &= A_n\sin\phi_n = |b_n|\sin\phi_n \\
\Rightarrow \sin\phi_n &= \mathrm{sgn}(b_n) = (-1)^{n+1} = \pm 1 \\
\text{so} \quad \phi_n &= (-1)^{n+1}\frac{\pi}{2}
\end{aligned}$$

Step 3: form the solution

$$\begin{aligned}
\text{form the solution} \quad x(t) &= x_c + \sum_{n=0}^{\infty} x_n \\
\text{where} \quad x_c(t) &= e^{-\beta t}(A_1 e^{\gamma t} + A_2 e^{-\gamma t}) = \text{ transient solution} \\
x_n(t) &= D_n \cos(n\omega t - \phi_n - \delta_n) = \ n^{th} \text{ forced solution} \\
\text{where} \quad D_n(\omega) &= \frac{A_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2\beta n\omega)^2}} \\
\text{and} \quad \tan\delta_n &= \frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2}
\end{aligned}$$

for the sawtooth problem

$$\begin{aligned}
A_n &= \frac{A}{\pi n} \\
\text{so} \quad D_n(\omega) &= \frac{A/\pi n}{\omega_0^2 - n^2\omega^2}
\end{aligned}$$

What is δ_n for this system?

$$\begin{aligned}
\text{and} \quad x &= x_c + x_p = x_c + x_1 + x_2 + x_3 \cdots \\
&= x_c + \frac{A\cos(\omega t - \pi/2)}{\pi(\omega_0^2 - \omega^2)} + \frac{A\cos(2\omega t + \pi/2)}{2\pi(\omega_0^2 - 4\omega^2)} + \frac{A\cos(3\omega t - \pi/2)}{3\pi(\omega_0^2 - 9\omega^2)} \cdots
\end{aligned}$$

Step 4: determine initial conditions

ie, adjust the A_1 and A_2 in the transient solution $x_c = e^{-\beta t}(A_1 e^{\gamma t} + A_2 e^{-\gamma t})$ so that initial conditions can be satisfied.

sawtooth

Suppose initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$

$$\begin{aligned} \text{note that} \quad x_p(0) &= 0 \quad \Rightarrow \; x_c = 0 \\ \text{so} \quad A_1 &= -A_2 \end{aligned}$$

also $\quad \dot{x}_c = \gamma A_1 - \gamma A_2 = 0 \quad$ at time $t = 0$

$$\Rightarrow A_1 = A_2$$

What does this say about the sawtooth oscillator's transient motion?

What are the resonant frequencies for the sawtooth oscillator?

What is the system's response when driven near its n^{th} resonance?

In this case, the system's response is dominated by the n^{th} resonant term:

$$x(t) \simeq x_n = \frac{A\cos[n\omega t + (-1)^n \pi/2]}{n\pi(\omega_0^2 - n^2\omega^2)}$$