CHAPTER 6

LINDBLAD RESONANCES

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6.1 FOURIER EXPANSION OF THE PERTURBING POTENTIAL

The following will calculate the motion of a massless test particle in orbit about a primary star, while the particle is also perturbed by a secondary planet's Lindblad resonance. The secondary's mass is m_s, so its gravitational potential is $\Phi_s = -Gm_s/\Delta$ where $\Delta = |\mathbf{r} - \mathbf{r}_s|$ is the particle's distance from the secondary, $\mathbf{r}$ is the particle's position relative to the primary, and $\mathbf{r}_s$ is the secondary's; see Fig. 6.1. The system is assumed coplanar, and this problem simplifies further when the secondary's gravitational potential Φ_s is Fourier-expanded in the particle's relative longitude $\varphi = \theta - \theta_s$, where θ and θ_s are the particle's and secondary's longitudes measured from the $\hat{\mathbf{x}}$ axis. That expansion is Eqn. (5.28), but note that the secondary's potential is even in φ, that is, $\Phi_s(r, \varphi) = \Phi_s(r, -\varphi)$. Consequently, the η_m coefficients in the odd part of Eqn. (5.28) must be zero, and so the Fourier expansion becomes

$$\Phi_s(r, \varphi) = \frac{1}{2}\phi_0(r) + \sum_{m=1}^{\infty} \phi_m(r)\cos(m\varphi). \tag{6.1}$$

Note that this is a fairly general expansion, since it could also apply to a particle that is instead orbiting in a disk galaxy while perturbed by a galactic bar, where φ would then be the particle's longitude relative to the bar's axis.

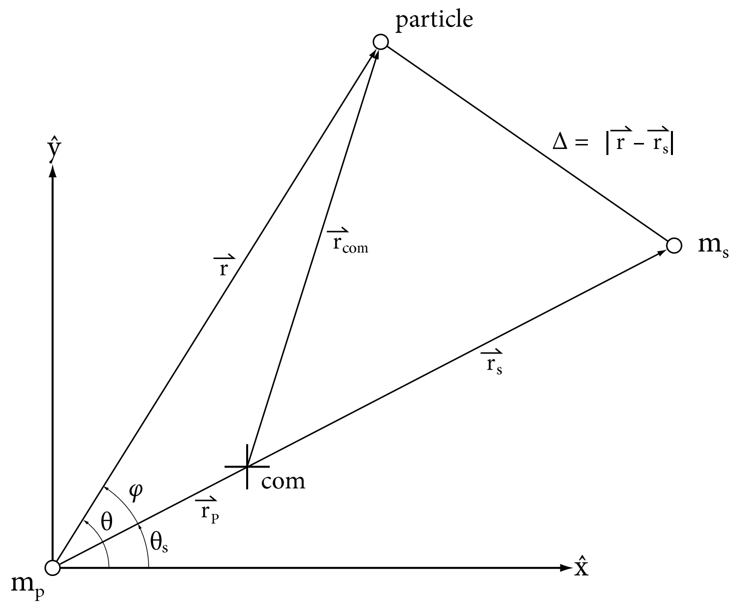

Figure 6.1 A massles particle is in orbit about a primary star of mass m_p and is perturbed by a secondary planet of mass m_s. The system is coplanar, where $\mathbf{r}$ is particle's position vector relative to the primary, and $\mathbf{r}_s$ is the secondary's position vector relative to the primary. The particle's relative longitude is $\varphi = \theta - \theta_s$, where θ and θ_s are the particle's and secondary's longitudes measured from the $\hat{\mathbf{x}}$ axis. The particle's position relative to the system's center-of-mass (COM) is $\mathbf{r}_{\rm com}$, and $\mathbf{r}_p$ is the primary's position relative to the COM.

To solve for the Fourier coefficient ϕ_m, multiply the above by $\cos(m'\varphi)$ and integrate over $-\pi \leq \varphi \leq \pi$, which yields $\pi\phi_{m'}(r)$ on the right-hand side, so the Fourier coefficient $\phi_{m'}(r)$ is

$$\phi_m(r) = \frac{2}{\pi}\int_0^\pi \Phi_s(r,\varphi)\cos(m\varphi)d\varphi, \tag{6.2}$$

where $\Phi_s = -Gm_s/\Delta$ and the separation $\Delta = |\mathbf{r} - \mathbf{r}_s| = (r^2 + r_2 - 2rr_s\cos\varphi)^{1/2} = r_s(1 + \beta^2 - 2\beta\cos\varphi)^{1/2}$ where $\beta = r/r_s$. Inserting this into Eqn. (6.2) then provides

$$\phi_m(r) = -\frac{Gm_s}{r_s} b_{1/2}^{(m)}(\beta) \tag{6.3}$$

where the Laplace coefficient is again

$$b_s^{(m)}(\beta) = \frac{2}{\pi}\int_0^\pi \frac{\cos(m\varphi)d\varphi}{(1+\beta^2 - 2\beta\cos\varphi)^s} \tag{6.4}$$

(e.g., Eqn. 5.30).

6.1.1 the indirect potential

Note that the coordinate system adopted in Fig. 6.1 is non-inertial, since the origin follows the primary that is in orbit about the system's COM. Consequently, this choice of origin also results in an additional acceleration that can be calculated from the gradient of what will be called the *indirect potential* Φ_i. The particle's equation of motion is Newton's second law of motion (Eqn. 1.18), $\ddot{\mathbf{r}}_{\text{com}} = -\nabla(\Phi_p + \Phi_s)$, where $\ddot{\mathbf{r}}_{\text{com}}$ is the particle's position relative to the system's center-of-mass (COM), which is an intertial coordinate system. However, it is often convenient to use a primary-centered coordinate system like that shown in Fig. 6.1, which is not inertial. Nonetheless, Newton's second law can still be adapted for use in the non-inertial frame, since $\mathbf{r} = \mathbf{r}_{\text{com}} - \mathbf{r}_p$, so $\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_{\text{com}} - \ddot{\mathbf{r}}_p = -\nabla(\Phi_p + \Phi_s) - \ddot{\mathbf{r}}_p$ where $\ddot{\mathbf{r}}_p = (Gm_s/r_s^3)\mathbf{r}_s$ is the primary's acceleration due to the secondary's gravity. That acceleration can be written as the gradient of the indirect potential $\Phi_i = (Gm_s/r_s^3)\mathbf{r}\cdot\mathbf{r}_s$, which will be confirmed in problem 6.1. The particle's equation of motion now resembles Newton's second law, $\ddot{\mathbf{r}} = -\nabla(\Phi_p + \Phi_s + \Phi_i)$ where

$$\Phi_i = \frac{Gm_s}{r_s^3}\mathbf{r}\cdot\mathbf{r}_s = \frac{Gm_s}{r_s}\beta\cos\varphi. \tag{6.5}$$

The indirect potential is usually combined with the secondary's potential, so that

$$\Phi_s \to \Phi_s + \Phi_i = -\frac{Gm_s}{\Delta} + \frac{Gm_s}{r_s}\beta\cos\varphi. \tag{6.6}$$

Note that the indirect potiential only contributes an $m = 1$ term in the Fourier expansion of Φ_s, so Eqn. (6.3) is then rewritten as

$$\phi_m(r) = -\frac{Gm_s}{r_s}\left[b_{1/2}^{(m)}(\beta) - \delta_{m1}\beta\right], \tag{6.7}$$

where the Kronecker delta δ_{m1} is 1 when $m = 1$ and zero otherwise. Lastly, it should be noted that the gradients in the above potentials should have been calculated with respect to the COM coordinate $\mathbf{r}_{\text{com}}$, but the same accelerations also result when calculating gradients with respect to the primary-centered coordinate $\mathbf{r}$.

The next Section will then show that each term in the Fourier expansion of Eqn. (6.1) corresponds to a Lindblad resonance, which is a site where the particle's response to the secondary's perturbations is large. However, these Lindblad resonances are usually spatially segregated. When that is the case, then the particle behaves as if it were responding to a single m^{th} term in this Fourier expansion, which allows one to simplify Eqn. (6.1) as

$$\Phi_s(r,\varphi) \simeq \phi_m(r)\cos(m\varphi) = \Re e\left[\phi_m(r)e^{im(\theta-\theta_s)}\right] \tag{6.8}$$

since $\varphi = \theta - \theta_s$, and with complex notation invoked on the right-hand side of this equation for later convenience.

6.1.2 motion near a Lindblad resonance

The particle's equation of motion $\ddot{\mathbf{r}} = -\nabla(\Phi_p + \Phi_s)$, which has radial and tangential components (Eqn. 5.1)

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\partial}{\partial r}(\Phi_p + \Phi_s) \simeq -\frac{\partial}{\partial r}\left[\Phi_p + \phi_m e^{im(\theta-\theta_s)}\right] \tag{6.9a}$$

$$\text{and}\quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -\frac{1}{r}\frac{\partial}{\partial\theta}(\Phi_p + \Phi_s) \simeq -\frac{im}{r}\phi_m e^{im(\theta-\theta_s)}. \tag{6.9b}$$

Note that the $\Re e$ notation has been dropped from the above, so it is to be understood that only the real parts of the following equations are to be preserved.

The particle's undisturbed state is presumed to be a circular orbit, $r = r_0$. However the secondary's perturbation of the particle, which is assumed to be small such that $|\nabla\Phi_s| \ll |\nabla\Phi_p|$, will cause the particle's motion to deviate from a circular orbit such that

$$r(t) = r_0 + r_1(t) \tag{6.10a}$$

$$\theta(t) = \theta_0 + \Omega_0 t + \theta_1(t), \tag{6.10b}$$

where θ_0 is an arbitrary phase, Ω_0 is the particle's mean angular velocity, and $|r_1| \ll r_0$ and $|\theta_1| \ll 1$. The smallness of r_1 and θ_1 allows use to linearize the equations of motion, which makes an analytic solution possible. The secondary's orbit is assumed circular, so its semimajor axis is $r_s = a_s$, and its longitude is $\theta_s(t) = \Omega_s t$ where Ω_s is its orbital angular velocity, with time $t = 0$ taken to be the time when the secondary crosses the $\hat{\mathbf{x}}$ axis. Consequenty, the phase that appears in Eqn. (6.9) is $m(\theta - \theta_s) = m\theta_0 + \omega_m t + m\theta_1$, where $\omega_m = m(\Omega_0 - \Omega_s)$ is known as the Doppler-shifted forcing frequency. However we will only need this phase to lowest order in Eqn. (6.10), so

$$m(\theta - \theta_s) \simeq m\theta_0 + \omega_m t \tag{6.11}$$

in the above. If we then use the particle's specific angular momentum $h = r^2\dot{\theta}$ in Eqn. (6.9b), then

$$\frac{dh}{dt} \simeq -im\phi_m e^{i(m\theta_0 + \omega_m t)} \tag{6.12a}$$

$$\text{so} \quad h(t) \simeq h_0 - \frac{m}{\omega_m}\phi_m\, e^{i(m\theta_0 + \omega_m t)} \tag{6.12b}$$

when Eqn. (6.12a) is integrated with respect to time t, and h_0 is an integration constant. Since $h = r^2\dot{\theta} \simeq r_0^2\Omega_0 + 2r_0\Omega_0 r_1 + r_0^2\dot{\theta}_1$, this means than

$$\dot{\theta}_1 \simeq -\frac{2\Omega_0}{r_0}r_1 - \frac{m}{r_0^2\omega_m}\phi_m e^{i(m\theta_0 + \omega_m t)}. \tag{6.13}$$

Next, insert Eqns. (6.10) into the radial part of the equation of motion (6.9a), and expand to first order in the small quantities, which yields

$$\ddot{r}_0 + \ddot{r}_1 + \left(\left.\frac{\partial\Phi_p}{\partial r}\right|_{r_0} - r_0\Omega_0^2 \right) + \left(3\Omega_0^2 + \left.\frac{\partial^2\Phi_p}{\partial r^2}\right|_{r_0} \right) r_1 \simeq - \left.\left(\frac{\partial\phi_m}{\partial r} + \frac{2m\Omega_0}{r_0\omega_m}\phi_m \right)\right|_{r_0} e^{i(m\theta_0 + \omega_m t)} \tag{6.14}$$

when Eqn. (6.13) is used to eliminate $\dot{\theta}_1$ and $\partial\Phi_p/\partial_r$ is Eqn. (5.6a), and the $|_{r_0}$ indicates that all quantities are to be evaluated at the particle's mean distance r_0. The particle's time-averaged orbit is assumed to be static, so $\ddot{r}_0 = 0$. Also note that the constant in the first parenthesis must be zero, since all of the other terms in this equation are oscillatory. This is the requirement for centrifugal equilibrium, which again provides the particle's mean angular velocity,

$$\Omega^2(r) = \frac{1}{r}\frac{\partial\Phi_p}{\partial r}, \tag{6.15}$$

where $\Omega_0 = \Omega(r_0)$. The constant in the second parenthesis is

$$\kappa^2(r) = 3\Omega^2 + \frac{\partial^2 \Phi_p}{\partial r^2} = 4\Omega^2 + r\frac{\partial \Omega^2}{\partial r}, \tag{6.16}$$

where $\kappa_0 = \kappa(r_0)$ is again the particle's epicyclic frequency from Eqn. (5.13). The coefficient in the right parenthesis is known as the secondary's *forcing function*,

$$\Psi_m(r) = -\frac{\partial \phi_m}{\partial r} - \frac{2m\Omega_0}{r_0\omega_m}\phi_m. \tag{6.17}$$

With these definitions, the particle's equation of motion simplifies to

$$\ddot{r}_1 + \kappa_0^2 r_1 = \Psi_m(r_0)e^{i(m\theta_0+\omega_m t)}, \tag{6.18}$$

which is the equation for a forced simple harmonic oscillator.

The solution to Eqn. (6.18) is the sum of two parts, $r_1 = r_{\text{free}} + r_{\text{forced}}$. The free part satisfies Eqn. (6.18) with the right-hand set to zero, which according to Eqns. (5.14–5.15) is $r_{\text{free}} = -Re^{i(\kappa_0 t+\varphi)}$ with an arbitrary phase φ now included in the solution. If there are many particles orbiting at the resonance, then a convenient measure of their free motions is their *dispersion velocity* c, which is their mean velocity relative to the local circular velocity $r_0\Omega_0$. Problem 6.2 shows that these quantities are related via $c = fR\Omega$ where $f \sim 1$.

The particle's forced solution to Eqn. (6.18) has the form $r_{\text{forced}} = -\mathcal{R}e^{i(m\theta_0+\omega_m t)}$; inserting this into Eqn. (6.18) shows that the amplitude of the particle's forced motion is

$$\mathcal{R}(r) = -\frac{\Psi_m}{D(r_0)} \tag{6.19}$$

where

$$D(r) = \kappa^2 - \omega_m^2 \tag{6.20}$$

is the particle's distance from resonance in frequency-squared units. The remainder will assume that the particle is so close to the resonance that its forced motion dominates over the free part, *i.e.* $|\mathcal{R}| \gg |R|$, so that the free part may be neglected.

6.1.3 resonance location

A *Lindblad resonance* is a site where $|D| \ll \Omega^2$, which makes the particle's response $|\mathcal{R}|$ large. Exact resonance $r = r_r$ is where $D(r_r) = 0$, which is where $\kappa = \epsilon\omega_m$ where $\epsilon = \pm 1$, or

$$\kappa(r_r) = \epsilon m[\Omega(r_r) - \Omega_s] \tag{6.21}$$

where Ω_s is the satellite's orbital angular velocity. When the primary's gravitational potential is Keplerian, $\kappa = \Omega$ so $\Omega/\Omega_s = m/(m-\epsilon) = (a_s/r_0)^{3/2}$ and

$$r_r = \left(1 - \frac{\epsilon}{m}\right)^{2/3} a_s \tag{6.22}$$

is the radius of the m^{th} Lindblad resonance. But keep in mind that this resonance position would be altered if the primary's gravitational potential where non-Keplerian, such as for an oblate planet; see problem 6.11.

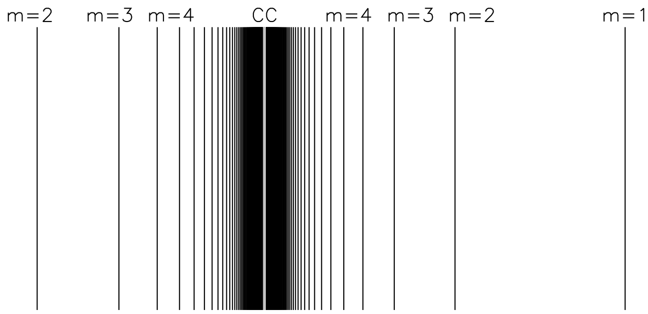

Figure 6.2 Vertical lines indicate the radii of the secondary's m^{th} inner and outer Lindblad resonances (Eqn. 6.22) shown relative to the satellite's orbit, which is also known at the corotation circle (CC). Radial distance r increases to the right, and the primary is far to the left. Only the $m \leq 4$ Lindblad resonances are labelled. Not shown is the $m = 1$ inner Lindblad resonance that lies at $r_r = 0$ according to Eqn. (6.22).

Resonances with $\epsilon = +1$ are *inner* Lindblad resonances, since they lie interior to the secondary's orbit, while those with $\epsilon = -1$ are *outer* Lindblad resonances. Figure 6.2 shows the relative radii of these resonances. Note that the $m \gg 1$ Lindblad resonances tend to accumulate at the satellite's orbit $r = a_s$. That site is also known as the corotation circle (CC), since a particle orbiting there will corotate with the secondary. Note, though, that the results obtained above do not apply to a particle orbiting near the CC, since the assumption of segregated resonances, e.g., Eqn. (6.8), does not apply there; see Fig. 6.2. Note that the corotation circle is also a resonance, since that is where $\omega_m = m(\Omega - \Omega_s)$ is small, which causes Ψ_m and thus $\mathcal{R}$ to be large. This resonance is known as the *corotation resonance*.

6.1.4 forced eccentricity

The particle's response $\mathcal{R}$ to the secondary's resonant perturbations is only large when orbiting near the resonance. This then allows one to Taylor expand $D(r)$ about the resonance so that

$$D(r_0) \simeq (r_0 - a_r)\frac{dR}{dr_0} = x\mathcal{D} \tag{6.23}$$

where

$$x = \frac{r_0 - a_r}{a_r} \tag{6.24}$$

is the particle's fractional distance from resonance, and the constant

$$\mathcal{D} = \left(r\frac{dD}{dR}\right)\bigg|_{r_0} = 3\epsilon(m - \epsilon)\Omega_0^2 \tag{6.25}$$

(see problem 6.4). Inserting Eqn. (6.7) into Eqn. (6.17) evaluated at resonance $r = r_0$ then provides the secondary's forcing function

$$\Psi_m = \epsilon f_m^\epsilon \mu_s r_0 \Omega_0^2 \tag{6.26}$$

(see problem 6.5), where $\mu_s = m_s/m_p$ is the secondary's fractional mass and

$$f_m^\epsilon = \epsilon\beta^2\frac{\partial b_{1/2}^{(m)}}{\partial\beta} + 2m\beta b_{1/2}^{(m)} - (2m+\epsilon)\beta^2\delta_{m1} \tag{6.27}$$

is a positive coefficient that depends on the resonance in question. The Laplace coefficient $b_{1/2}^{(m)}(\beta)$ must be evaluated numerically, but that is easily done using the method described in example 5.3, and with the derivative calculated via Eqn. (5.40a). Several of the f_m^ϵ coefficients are also tabulated in Table 6.1. as part of problem 6.6.

The particle's *forced eccentricity* is Eqn. (6.19) divided by the particle's mean distance from the primary, which is

$$e_f = \left|\frac{\mathcal{R}}{r_0}\right| = \left|\frac{\psi_m}{x}\right| \tag{6.28}$$

where

$$\psi_m = \frac{\Psi_m}{r\mathcal{D}} = \frac{f_m^\epsilon \mu_s}{3(m-\epsilon)} \tag{6.29}$$

is a dimensionless version of the secondary's forcing function. Keep in mind that the particle's forced eccentricity is quite distinct from the particle's osculating eccentricity, which is defined in Chapter 3.

6.1.5 streamlines, and the nonlinear zone

Now consider a scenario where there are many particles orbiting in a disk about the primary, with that disk being perturbed by the secondary's Lindblad resonance. Inserting Eqns. (6.19), (6.11) (6.23) and (6.26) into (6.10a) also allows us to write a particle's trajectory as a function of longitude θ instead of time,

$$r(\theta) = a + \mathrm{sgn}(x)|\mathcal{R}|\cos m(\theta - \theta_s), \tag{6.30}$$

where $\mathrm{sgn}(x) = \pm 1$ indicates the sign of x. The path traced by that particle is a *streamline*, so the other particles' streamlines thus describe that disk's forced motions. A particle is said to be in *conjuction* when it is at the secondary's longitude $\theta = \theta_s$. So a particle orbiting exterior to the Lindblad resonance, where $x > 0$, would also experience its maximal radial displacement there since $r(\theta_s) = a + |\mathcal{R}|$, so the particle is also at its apoapse when in conjunction. That particle's streamline is then said to be *apo-aligned* with the seconday. Similarly, a particle orbiting interior to the Lindblad resonance has $r(\theta_s) = a - |\mathcal{R}|$ when in conjunction, so its streamline is *peri-aligned*.

However these signs reverse at longitudes $\theta = \theta_c$ where $m(\theta_c - \theta_s) = \pm 180°$ and $r(\theta_c) = a - \mathrm{sgn}(x)|\mathcal{R}|$. Thus a particle orbiting interior to the Lindblad resonance ($x < 0$) will be at apoapse while a particle exterior to the resonance ($x > 0$) would be at periapse. Consequently, those particles orbiting closer than one epicyclic distance from the resonance, where $|r(\theta_c) - a| < |\mathcal{R}(x)|$, will experience radial excursions that carry them across the resonance at certain longitudues, where they can collide or interact with particles on the other side that are also trying to cross the resonance. The region around the resonance where these interactions occur is known as the *nonlinear zone*, and the radial half-width of this zone,

$$x_{\mathrm{NL}} = \sqrt{|\psi_m|}, \tag{6.31}$$

is obtained by requiring $|r(\theta_c) - a| = |\mathcal{R}(x_{\rm NL})| = r_0 e(x_{\rm NL})$ at the zone's edge. This region is called the nonlinear zone because Eqn. (6.28) predicts that the particles in the disk will have an eccentricity gradient, $|ade/da| = |de/dx|$, that is not small, which in turn will result in large variations in the disk's surface density; see problem 6.7 and Eqns. (6.70).

6.2 HIGHER ORDER LINDBLAD RESONANCES

The preceeding derivation assumed that the perturbing secondary's orbit is circular. However, when the secondary's orbit is instead eccentric, an additional suite of resonances also appear in the system. To calculate a particle's response to these higher-order resonances, assume that the secondary's eccentricity $e_s \ll 1$ is also small. Since the secundary's orbit is unperturbed, its motion is epicyclic, and its polar coordinates as a function of time are Eqns. (5.16),

$$r_s(t) = a_s(1 - e_s \cos \kappa_s t) \tag{6.32a}$$

$$\theta_s(t) = \Omega_s t + 2e_s \frac{\Omega_s}{\kappa_s} \sin \kappa_s t \tag{6.32b}$$

where time $t = 0$ is again chosen to be the moment when the satellite crosses the $\hat{\mathbf{x}}$ axis. Inserting these into the $m^{\rm th}$ Fourier component of the secondary's gravitational potential, Eqns. (6.7–6.8) and Taylor-expanding to first order in e_s (see problem 6.9), will show that the satellite's nonzero eccentricity splits the $m^{\rm th}$ potential into three distinct components such that

$$\Phi_s(r, \theta) \to \sum_{k=-1}^{+1} \phi_m^k(r_0) \cos m(\theta - \Omega_{mk} t) \tag{6.33}$$

where the $k = 0$ component of ϕ_m^k is Eqn. (6.7) while the $k = \pm 1$ coefficients are

$$\phi_m^k = -\frac{Gm_s}{a_s} e_s \left[\frac{1}{2} \left(\beta \frac{\partial}{\partial \beta} + 1 + 2km \frac{\Omega_s}{\kappa_s} \right) b_{1/2}^{(m)}(\beta) - \beta \left(1 + km \frac{\Omega_s}{\kappa_s} \right) \delta_{m1} \right] \tag{6.34}$$

with $r_s \to a_s$ and $\beta \to r_0/a_s$, where r_0 is the perturbed particle's mean orbit radius. The potential components in Eqn. (6.33) rotate with an angular velocity

$$\Omega_{mk} = \Omega_s + \frac{k}{m} \kappa_s \tag{6.35}$$

that is known as the *pattern speed*. According to Eqn. (6.34), the strength of the $|k| = 1$ component of the secondary's potential is weaker than its $k = 0$ component by a factor e_s. And if the secondary's potential Φ_s had instead been expanded to all higher orders in e_s, we would have found that $\phi_m^k \propto e_s^{|k|} \cos m(\theta - \Omega_{mk} t)$, with the summation in Eqn. (6.34) also extended over $\pm\infty$ [1]. The index k will be used to distinguish between all of these many potential components, with $|k|$ also known as the *order* of the Lindblad resonance.

Comparing Eqn. (6.33) to Eqn. (6.8) shows that the perturbing potentials have the same form when ω_m is replaced by $m(\Omega - \Omega_{mk})$; see Eqn. (6.11). Thus the solution for the particle's motion at a $|k| = 1$ resonance has the same form as the $k = 0$ solution, Eqn. (6.19), provided $\omega_m \to m(\Omega - \Omega_{mk})$. The condition for exact resonance is then $D(r) = \kappa^2 - m^2(\Omega - \Omega_{mk})^2 = 0$, so the resonance is the site where $\kappa = \epsilon m(\Omega - \Omega_{mk}) =$

$\epsilon m(\Omega - \Omega_s) - \epsilon k\kappa_s$ is satisfied, where $\epsilon = \pm 1$ again destinguishes between an inner and an outer Lindblad resonance. If the system's gravitational potential is Keplerian, then $\kappa = \Omega$ and $\Omega_s/\Omega = (m - \epsilon)/(m + k) = (a_r/a_s)^{3/2}$, so the resonance radius a_r is

$$a_r = \left(\frac{1 - \epsilon/m}{1 + k/m}\right)^{2/3} a_s. \tag{6.36}$$

Evidently, every zeroth-order ($k = 0$) Lindblad resonance is also straddled by a pair of first-order ($k = \pm 1$) resonances whose magnitude is weaker by a factor of e_s; see Fig. 6.3. As Eqn. (6.36) shows, there is one first-order resonance having $k = \epsilon$ that lies further away from the corotation circle (CC) than the $k = 0$ Lindblad resonance; that site is known as an *external* Lindblad resonance. Equation (6.36) also shows that there is a $k = -\epsilon$ resonance, which is a *coorbital* Lindblad resonance since it lies on the corotation circle. Also note that if the particle is orbiting at an $m \gg 1$ Lindblad resonance, then the resonance lies at $a_r/a_s \simeq 1 - 2(\epsilon + k)/3m$, so that resonance's fractional distance from the secondary is

$$x_L = \frac{a_r - a_s}{a_s} \simeq -\frac{2(\epsilon + k)}{3m}. \tag{6.37}$$

However the zeroeth-order and first-order external Lindblad resonances are usually spatially segregated, which then allows one to write Φ_s in Eqn. (6.33) as being due to a single component such that

$$\Phi_s(r, \theta) \simeq \phi_m^k(r_0) \cos m(\theta - \Omega_{mk} t) \tag{6.38}$$

where $\theta = \theta_0 + \Omega_0 t$ is the longitude of the particle's guiding center. Lastly, note that we could also have expanded the secondary's potential Φ_s to first order in the particle's forced eccentricity e_f, which would then have resulted in an additional suite of first-order Lindblad resonances that are described in problem 6.12.

As Fig. 6.2 shows, the high-m resonances lie very close to the secondary's orbit. And if the secondary is orbiting in or near a broad disk of matter, such as a satellite orbiting near a planetary ring, or a young planet that is still orbiting within a planet-forming circumstellar disk, then it is these nearby high-m resonances that dominate the secondary's interaction with the disk. As Section 6.3.3 will show, the secondary's zeroeth-order Lindblad resonances also facilitates an exchange of angular momentum and energy between the secondary and the resonant disk-matter in a way that alters the secondary's semimajor axis a that can, for instance, drive the planet-migration that will be examined later in Chapter 13. Other studies have also shown that the disk matter orbiting at a secondary's external Lindblad resonances tends to excite the secondary's eccentricity, while matter orbiting at the secondary's external corotation resonances tends to damp its eccentricity, all at comparable rates [1]. So it is these resonant disk-secondary interactions that are very important to studies of ring-satellite interactions, as well as the orbital evolution of young extra-solar planets that are still embedded in their planet-forming circumstellar disk.

6.2.1 Corotation resonances

A *corotation resonance* is a site where the secondary's forcing function, Eqn. (6.17), is singular, which occurs where $\omega_m \to m(\Omega - \Omega_{mk}) = 0$. The particle's forced response, Eqn. (6.19) gets large there because this potential component corotates with the particle's mean motion. When the system is Keplerian, $\Omega = \kappa$, so the resonance condition $\Omega = \Omega_{mk}$

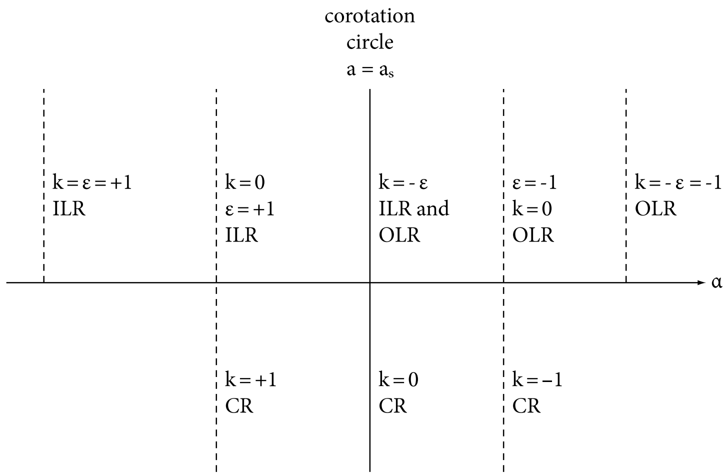

Figure 6.3 This schematic shows the locations of the zeroeth ($k = 0$) and first-order ($|k| = 1$) Lindblad resonances (LRs) and corotation resonances (CR), relative to the corotation circle (*i.e.*, the secondary's orbit); see Eqns. (6.36) and (6.39). The primary is far to the left and the secondary lies on the corotation circle CC, and semimajor axis a increases to the right.

becomes $\Omega_s/\Omega = m/(m+k) = (a_r/a_s)^{3/2}$, so the resonance has a semimajor axis

$$a_r = \left(1 + \frac{k}{m}\right)^{-2/3} a_s. \tag{6.39}$$

Evidently there are an infinite number of zeroeth-order ($k = 0$) corotation resonances that reside at the secondary's orbit; these are also known as the coorbital corotation resonances. But there is also an infinite number of high-order ($k \neq 0$) corotation resonance that lie off the corotation radius, known as external corotation resonances. Also note that when $m \gg 1$, the corotation resonance lies a fractional distance

$$x_C = \frac{a_r - a_s}{a_s} \simeq -\frac{2k}{3m} \tag{6.40}$$

away from the secondary. Comparison to Eqn. (6.37) shows that the secondary's first-order $|k| = 1$ corotation resonances also overlap with its zeroeth order ($k = 0$) Lindblad resonances when $m \gg 1$, which is also illustrated in Fig. 6.3.

6.2.2 Vertical resonances

Now calculate the particle's linearized response to the vertical component of the secondary's gravity. As the following will show, the particle's response to the vertical resonances has many similarities to its response to the first order horizontal Lindblad resonances.

The equation for the particle's vertical motion is $\ddot{z} = -\frac{\partial}{\partial z}(\Phi_p + \Phi_s)$. Assume that the particle's vertical distance from the secondary's orbit plane are small, which then allows this to be Taylor-expanded in z, so $\partial\Phi_p/\partial z \simeq \nu^2 z$ where ν is the particle's vertical oscillation frequence from Eqn. (5.8); this frequency is simply the particle's mean motion when the primary is spherical or slightly faster when oblate (see Eqns. 5.18 and 5.52). If the particle lies a horizontal distance Δ away from the secondary whose polar coordinates are

r_s, θ_s, z_s, then the secondary's gravitational potential is $\Phi_s = -Gm_s/\sqrt{\Delta^2 + (z - z_s)^2}$ and its vertical accelerations is $-\partial\Phi_s/\partial z \simeq -Gm_s(z - z_s)/\Delta^3$ when their vertical separation is small compared to their horizontal separation, $|z - z_s| \ll \Delta$, so

$$\ddot{z} = -(\nu^2 + Gm_s/\Delta^3)z - \frac{Gm_s z_s}{\Delta^3} \tag{6.41}$$

but note that $\nu^2 \simeq Gm_p/r^3 \gg Gm_s/\Delta^3$ when the separation $\Delta \sim \mathcal{O}(r)$ so the second term in the parenthesis is negligible.

The secondary's vertical motion is $z_s(t) = a_s \sin i_s \sin(\nu_s t)$ where a_s, i_s, ν_s are its semimajor axis, inclination, and vertical oscillation frequency, with time $t = 0$ chosen to be when the secondary is at its ascending node, Eqn. (5.16c). To calculate the particle's response to the secondary's resonant forcing, Fourier expand the vertical acceleration

$$-\frac{Gm_s z_s}{\Delta^3} = -\frac{Gm_s a_s \sin i_s \sin(\nu_s t)}{(r^2 + a_s^2 - 2ra_s\cos\varphi)^{3/2}} = \frac{1}{2}f_0(r,t) + \sum_{m=1}^{\infty} f_m(r,t)\cos(m\varphi) \tag{6.42}$$

where $\varphi = \theta - \theta_s$ is the particle's longitude relative to the secondary's. Multiplying by $\cos(m\varphi)$ and integrating over all φ then yields the Fourier coefficient $f_m(r,t)$:

$$\begin{aligned} f_m(r,t) = \frac{2}{\pi}\int_0^{\pi}\left(-\frac{Gm_s z_s}{\Delta^3}\right)\cos(m\varphi) &= -\frac{Gm_s \sin i_s}{a^2} b_{3/2}^{(m)}(\beta)\sin(\nu_s t) \\ &= -2A_m \sin(\nu_s t), \end{aligned} \tag{6.43}$$

where the factor $A_m = (Gm_s \sin i_s/2a^2)b_{3/2}^{(m)}(\beta)$ depends on the Laplace coefficient $b_{3/2}^{(m)}(\beta)$ that is a function of $\beta = r/a_s$. So the secondary's acceleration is

$$\begin{aligned} -\frac{Gm_s z_s}{\Delta^3} &= -\sum_{m=0}^{\infty} 2A_m \sin(\nu_s t)\cos(m\varphi) \\ &= -\sum_{m=0}^{\infty} A_m\{\sin m[\theta - (\Omega_s - \nu_s/m)t] - \sin m[\theta - (\Omega_s + \nu_s/m)t]\} \end{aligned} \tag{6.44}$$

where the above assumes that the coordinate system is oriented so that the secondary's longitude of ascending node is zero. With $\bar{\Omega}_{mk} = \Omega_s + k\nu_s/m$ being the vertical pattern speed, the above can be written more compactly as

$$-\frac{Gm_s z_s}{\Delta^3} = \sum_{k=-1}^{k=1}\sum_{m=0}^{\infty} kA_m \sin m(\theta - \bar{\Omega}_{mk}t), \tag{6.45}$$

which is the vertical analog of Eqns. (6.33–6.35). To lowest order the particle's longitude is $\theta \simeq \theta_0 + \Omega t$ so the argument of the sinusoid can be written $m(\theta - \bar{\Omega}_{mk}t) \simeq m\theta_0 + \bar{\omega}_{mk}t$ where $\bar{\omega}_{mk} = m(\Omega - \bar{\Omega}_{mk})$ is the vertical Doppler-shifted forcing frequency.

Each term in the above sum corresponds to a vertical resonance, each of which are again spatially segregated, so we only need to consider the particle's response to a single resonant term, and in this approximation Eqn. (6.41) for the particle's vertical motion becomes

$$\ddot{z} \simeq -\nu^2 z + kA_m \sin(m\theta_0 + \bar{\omega}_{mk}t). \tag{6.46}$$

The solution is again the sum of free and forced components,

$$z(t) = a \sin i \sin(\nu t + \phi_0) + Z_f \sin(m\theta_0 + \bar{\omega}_{mk} t) \tag{6.47}$$

where a, i, ν, ϕ_0 are constants that describe the particle's free motion, Eqn. (5.16c). Inserting this into Eqn. (6.46) then yield's the amplitude of the particle's forced motion,

$$Z_f = \frac{kA_m}{\nu^2 - \bar{\omega}_{mk}^2}, \tag{6.48}$$

so the vertical resonance is the site that satisfies $\bar{\omega}_{mk} = m\Omega - m\Omega_s - k\nu_s = \epsilon\nu$ where as usual $\epsilon = \pm 1$. If the system is Keplerian then $\nu = \Omega$ and the resonance condition becomes $\Omega_s/\Omega(r_V) = (m - \epsilon)/(m + k)$ so the vertical resonance at $r = r_V$ is at

$$r_V = \left(\frac{1 - \epsilon/m}{1 + k/m}\right)^{2/3} a_s \tag{6.49}$$

which is also where the first order horizontal resonances lie, Eqn. (6.36). Again, the $k = \epsilon$ resonances are *external* vertical resonances since they lie interior ($\epsilon = +1$) or exterior ($\epsilon = -1$) to the corotation circle, while the $k = -\epsilon$ vertical resonances are *coorbital* and lie on the corotation circle at $r_V = a_s$.

6.3 RESONANCE TRAPPING

When an orbiting particle is also subject to a dissipative force, its orbit will decay, and that radial motion can deliver the particle to a secondary's Lindblad resonance where it might get trapped at the resonance if the dissipative force is sufficiently weak. For example, when the orbit of a dust grain decays due to Poynting Robertson drag, trapping occurs when the grain encounters a planet's resonance whose perturbations also supplies the grain with enough orbital energy and angular momentum to offset its losses due to drag, which stabilizes and traps the grain at the resonance. And to illustrate the resonance trapping phenomenon, the following will consider a related problem, a planetesimal whose orbit decays due to aerodynamic drag with the solar nebula gas, which was considered earlier in Section 3.2.2.

6.3.1 orbit decay due to nebula drag

The acceleration on the planetesimal due to nebula gas drag is $\mathbf{a}_d = -|\mathbf{u}|\mathbf{u}/\lambda_d$ where $\mathbf{u} = \dot{\mathbf{r}} - \mathbf{v}_{\text{gas}}$ is the particle's velocity $\dot{\mathbf{r}}$ relative to the gas $\mathbf{v}_{\text{gas}}$; see Section 3.2.2. But this particular drag law is not linear in velocity, which makes the mathematics a bit more difficult, so to sidestep that the following will replace the speed $|\mathbf{u}|$ with a constant value $\langle u \rangle$ that is averaged over the planetesimal's orbit. Then $\mathbf{a}_d \to -\langle u \rangle \mathbf{u}/\lambda_d = -k_d \Omega \mathbf{u}$ where the small dimensionless drag parameter $k_d = \langle u \rangle/\lambda_d \Omega \ll 1$. This minor substitution simplifies the mathematics without changing the nature of the problem.

The particle is also being perturbed by an orbiting secondary, so the particle's polar coordinates are written $r(t) = r_0 + r_1(t) + r_d(t)$ and $\theta(t) = \theta_0 + \Omega t + \theta_1(t) + \theta_d(t)$ where r_0, θ_0 are its polar coordinates at time $t = 0$, r_1 and θ_1 are the particle's oscillatory displacements due to the secondary's resonant forcing, and r_d and θ_d account for its secular drift due to the drag. The following will assume in advance that the planetesimal is already trapped in a static orbit at the secondary's resonance, so $r_d = 0 = \theta_d$. And since the nebula gas' circular velocity is $\mathbf{v}_{\text{gas}} = (1 - \eta) r \Omega \,\hat{\boldsymbol{\theta}}$, from Eqn. (3.39), the acceleration of the planetesimal due to drag is $\mathbf{a}_d = -k_d \Omega \dot{r}_1 \,\hat{\mathbf{r}} - k_d r_0 \Omega(\dot{\theta}_1 + \eta\Omega)\,\hat{\boldsymbol{\theta}}$.

6.3.2 equations of motion

Equations (6.9) describe the particle's motion in a gas-free environment, so adding $\mathbf{a}_d$ to the right hand side will account for the gas drag. The particle's specific angular momentum is $h = r^2\dot{\theta} \simeq r_0^2\Omega + 2r_0\Omega r_1 + r_0^2\dot{\theta}_1 = h_0 + h_1(t) + h_d(t)$ where $h_0 = r_0^2\Omega$ is the initial angular momentum while $h_1 = 2r_0\Omega r_1 + r_0^2\dot{\theta}_1$ accounts for the oscillations that are due to the secondary's resonant forcing while h_d accounts for the losses due to gas drag. Inserting h into the angular equation of motion (6.9b) yields

$$\frac{dh}{dt} = \dot{h}_1 + \dot{h}_d = -im\phi_m e^{i(m\theta_0+\omega_m t)} - k_d r_0^2\Omega(\dot{\theta}_1 + \eta\Omega). \tag{6.50}$$

The oscillatory and secular portions of the above equation must be satisfied separately, so $\dot{h}_1 = -im\phi_m e^{i(m\theta_0+\omega_m t)} - k_d r_0^2\Omega\dot{\theta}_1$, which can be integrated in time so

$$h_1(t) = -\frac{m}{\omega_m}\phi_m e^{i(m\theta_0+\omega_m t)} - k_d r_0^2\Omega\theta_1 = 2r_0\Omega r_1 + r_0^2\dot{\theta}_1. \tag{6.51}$$

The particle's oscillatory response has the form

$$r_1(t) = -Re\left[\mathcal{R}e^{i(m\theta_0+\omega_m t)}\right] \tag{6.52a}$$

$$\theta_1(t) = Re\left[\Theta e^{i(m\theta_0+\omega_m t)}\right] \tag{6.52b}$$

so $\dot{\theta}_1 = i\omega_m\theta_1$ and Eqn. (6.51) can then be solved for θ_1:

$$\theta_1(t) \simeq \frac{i}{r_0^2\omega_m}\left(1 + \frac{ik_d\Omega}{\omega_m}\right)\left[\frac{m}{\omega_m}\phi_m e^{i(m\theta_0+\omega_m t)} + 2r_0\Omega r_1\right]. \tag{6.53}$$

Adding gas drag to the right hand side of Eqn. (6.9a) and expanding to first order in the small quantities yields

$$\ddot{r}_1 - r_0\Omega^2 - 2r_0\Omega\dot{\theta}_1 - \Omega^2 r_1 = -r_0\Omega^2 - (\kappa^2 - 3\Omega^2)r_1 - \frac{\partial\phi_m}{\partial r}e^{i(m\theta_0+\omega_m t)} - k_d\Omega\dot{r}_1, \tag{6.54}$$

and inserting Eqns. (6.52–6.53) into the above and solving for $\mathcal{R}$ yields the amplitude of the particle's forced radial motion,

$$\mathcal{R} = \frac{-\Psi_m + 2ik_d m\Omega^2\phi_m/r_0\omega_m^2}{\kappa^2 - \omega_m^2 + ik_d(1 + 4\Omega^2/\omega_m^2)\Omega\omega_m} \simeq \frac{-\Psi_m}{\kappa^2 - \omega_m^2 + 5i\epsilon k_d\Omega^2} \tag{6.55}$$

since the particle is near the resonance where $\omega_m \simeq \epsilon\kappa \simeq \epsilon\Omega$ and drag is weak, $k_d \ll 1$. The following will also need the real and imaginary parts of $\mathcal{R}$, so

$$\mathcal{R} = \frac{-\Psi_m D + 5i\epsilon k_d\Omega^2\Psi_m}{D^2 + (5k_d\Omega^2)^2} \tag{6.56}$$

where $D = \kappa^2 - \omega_m^2$ is the frequency distance from resonance. So drag introduces an imaginary component into the particle's motion $\mathcal{R}$. As the following will show, this causes the particle's response to be out of phase with the secondary's forcing, which allows the secondary to exert a net torque on the particle that can, if a certain criterion is satisfied, trap the particle at the resonance.

6.3.2.1 torque balance The particle's orbit decay will be halted, and resonant trapping occurs, when the torque that the secondary exerts on the particle compensates for the angular momentum losses that are due to gas drag. Equation (6.50) provides the rate at which the particle's specific angular momentum h evolves, and its secular part is $\dot{h}_d = -k_d\eta(r_0\Omega)^2$, which by the way appears unbalanced since there is no positive torque here to trap the particle at the resonance. That is because the torque that the secondary exerts on the particle is a second-order effect and is why that term is absent from these equations. Nonetheless one can still calculate that torque from these first-order results.

The specific torque that the secondary exerts on the particle is

$$T_s = -(\mathbf{r} \times \nabla\Phi_s) \cdot \hat{\mathbf{z}} = -\frac{\partial\Phi_s}{\partial\theta} = -Re\left[im\phi_m(r)e^{m(\theta-\theta_s)}\right] = m\phi_m(r)\sin m(\theta-\theta_s). \tag{6.57}$$

Insert $r = r_0 + r_1$ and $\theta = \theta_0 + \Omega t + \theta_1$ into the above and Taylor expand to first order, so

$$T_s \simeq m\phi_m \sin\varphi + m^2\theta_1\phi_m\cos\varphi + mr_1\frac{\partial\phi_m}{\partial r}\sin\varphi \tag{6.58}$$

where angle $\varphi = m\theta_0 + \omega_m t$. The particle's displacement from circular motion is Eqn. (6.52), so $r_1 = -Re(\mathcal{R}e^{i\varphi}) = -\mathcal{R}_r\cos\varphi + \mathcal{R}_i\sin\varphi$ and $\theta_1 = Re(\Theta e^{i\varphi}) = \Theta_r\cos\varphi - \Theta_i\sin\varphi$ where the r and i subscripts indicate the real and imaginary parts of $\mathcal{R}$ and Θ. But we are interested in the time-averaged specific torque, which is obtained by averaging Eqn. (6.58) over one forcing period $2\pi/|\omega_m|$, which is

$$\langle T_s\rangle = \frac{1}{2}m^2\phi_m\Theta_r + \frac{1}{2}m\frac{\partial\phi_m}{\partial r}\mathcal{R}_i \tag{6.59}$$

since the only nonzero terms in the time-averaged torque are proportional to $\langle\cos^2\varphi\rangle = \frac{1}{2} = \langle\sin^2\varphi\rangle$.

Now note that Eqn. (6.55) implies $|r_1| \gg |\phi_m|/r\Omega^2$ when the particle is near a resonance so the r_1 term in Eqn. (6.53) dominates and $\theta_1 \simeq 2i\Omega r_1/r_0\omega_m$ so $\Theta \simeq -2i\Omega\mathcal{R}/r_0\omega_m$ and $\Theta_r = 2\Omega\mathcal{R}_i/r_0\omega_m$ and Eqn. (6.59) becomes

$$\langle T_s\rangle = -\frac{1}{2}m\Psi_m\mathcal{R}_i \tag{6.60}$$

The imaginary part of $\mathcal{R}$ is from Eqn. (6.56), $\mathcal{R}_i = 5\epsilon k_d\Omega^2\Psi_m/[D^2 + (5k_d\Omega^2)^2]$, so the specific torque that the satellite exerts on the particle is

$$\langle T_s\rangle = -\frac{5\epsilon m k_d\Omega^2\Psi_m^2/2}{D^2 + (5k_d\Omega^2)^2}, \tag{6.61}$$

which is maximal at resonance where $D = 0$. Note that the torque is positive and hence resonance trapping is only possible at the $\epsilon = -1$ outer Lindblad resonance.

The net time-averaged specific torque on the particle is $\langle\dot{h}\rangle = \dot{h}_d + \langle T_s\rangle$, where the torque from the gas drag is $\dot{h}_d = -k_d\eta(r\Omega)^2$. These torques sum to zero when the particle is trapped at the resonance, which provides an equation for the distance D from resonance where the particle gets trapped,

$$D^2 = \frac{5m\Psi_m^2}{2\eta r^2} - (5k_d\Omega^2)^2. \tag{6.62}$$

Inserting D^2 into Eqn. (6.56) then yields the trapped particle's forced eccentricity (see problem 6.13),

$$e = \frac{|\mathcal{R}|}{r} = \sqrt{\frac{2\eta}{5m}}. \tag{6.63}$$

Interestingly, the trapped particle's forced eccentricity is insensitive to quantities like the satellite's mass or the drag parameter k_d.

6.3.2.2 trapping threshold In order for resonant trapping to occur, the D^2 inferred from Eqn. (6.62) must be positive so the drag parameter k_d must be smaller than the critical value, $k_d < k_{\text{crit}}$, where

$$k_{\text{crit}} = \sqrt{\frac{m}{10\eta}\frac{|\Psi_m|}{r\Omega^2}}. \tag{6.64}$$

Note that the drag parameter $k_d \propto \lambda_{\text{gd}}^{-1}$ where $\lambda_{\text{gd}} \propto R_p$ with R_p the planetesimal radius (Eqn. 3.43), so the resonance trapping threshold $k_d < k_{\text{crit}}$ is equivalent to $R_p > R_{\text{crit}}$ where

$$R_{\text{crit}} \simeq \frac{\eta C_d}{2m^2\mu_s}\left(\frac{\rho_g}{\rho_p}\right)r \tag{6.65}$$

where μ_s is the planet's mass in units of the central star's, ρ_g is the nebula gas density, and ρ_p is the planetesimal's density, and C_d is the dimensionless drag coefficient of Section 3.2.2.2; see problem 6.14. If one adopts the nebula parameters described in Section 3.2.2.3 for an Earth-mass planet orbiting in the solar nebula at $r = 1$ AU, then the threshold for resonance trapping is $R_{\text{crit}} \sim 6$ km$/m^2$. So if the planetesimal has a size smaller than R_{crit}, it will drift inwards and across the m^{th} Lindblad resonance because the resonant torque is weaker than the torque from nebula drag. But that planetesimal will still encounter a sequence of higher m resonances that get progressively stronger, so resonance trapping seems assured. But these results only apply when resonances are spatially segregated, whereas Fig. 6.2 shows that the higher m resonances are very dense near the planet's orbit, and those overlapping resonances will result in chaotic orbital motion that in time will kick the planetesimal into an orbit that crosses the planet's orbit. Which would then result in the planetesimal being accreted or else scattered away by the planet. The upshot is that resonance trapping is only effective at trapping a planetesimal in a long-term stable orbit when the planetesimal is trapped at the more distant low-m outer Lindblad resonances.

6.3.3 the secondary's resonant torque on a disk

Equation (6.61) is the time-averaged torque per mass that the secondary exerts on a single particle orbiting near its m^{th} Lindblad resonance. Now assume that there are instead numerous particles present, spread out across a disk that has a mass surface density σ, with all particles subject to the drag acceleration $\mathbf{a}_d$. The resonant torque that the secondary exerts on an narrow annulus within that disk is $dT_m = \langle T_s\rangle 2\pi\sigma r dr = -\epsilon m 5 k_d \Omega^2 \Psi_m^2 \pi\sigma r^2 dx/[(x\mathcal{D})^2 + (5k_d\Omega^2)^2]$ where r is the radius of the annulus, $dr = rdx$ its radial width, and $D(x) = x\mathcal{D}$ where x is the fractional distance from resonance. The total integrated torque is

$$T_m = \int dT_m = -\frac{2\alpha_d m\pi\sigma r^2\Psi_m^2}{\mathcal{D}}\int_0^\infty \frac{dx}{x^2 + \alpha_d^2} \tag{6.66}$$

where $\alpha_d = 5k_d\Omega^2/|\mathcal{D}|$ is another small dimensionless drag parameter. The integral evaluates to $\pi/2\alpha_d$, so

$$T_m = -\frac{m\pi^2\sigma r^2\Psi_m^2}{\mathcal{D}}. \tag{6.67}$$

This is the torque that the secondary exerts at its m^{th} Lindblad resonance in a disk of particles that are subject to a weak drag force having $\alpha_d \ll 1$. Interestingly, this formula will be recovered again in Section 12.4.4, which calculates the torque that a secondary exerts when it launches a spiral density wave at its m^{th} Lindblad resonance in the a disk; see Eqns. (12.73–12.74).

And in problem 6.15 you will show that, when Eqn. (6.67) is summed over all of the secondary's resonances in the disk, you then recover the shepherding torque of Eqn. (4.42).

Problems

6.1 Confirm that the indirect potential satisfies $\nabla\Phi_i = (Gm_s/r_s^3)\mathbf{r}_s$.

6.2 Show that $c = fR\Omega$.

6.3 Show that a planet's m^{th} Lindblad resonance lies a fractional distance

$$x = \frac{r_0 - r_s}{r_s} \simeq -\frac{2\epsilon}{3m} \tag{6.68}$$

away from the planet's orbit when $m \gg 1$, and that the distance between adjacent resonances is

$$\Delta x \simeq \frac{2}{3m^2}. \tag{6.69}$$

6.4 See Eqn. (6.25), and show that $\mathcal{D} = 3\epsilon(m - \epsilon)\Omega_0^2$

6.5 Insert Eqn. (6.7) into Eqn. (6.17) and evaluate it at resonance to obtain the forcing function that is given by Eqns. (6.26) and (6.27).

6.6 Write a computer program to evaluate the coefficients f_m^ϵ numerically using Eqn. (6.27), and calculate the $m \geq 8$ values that are missing in Table 6.1.

6.7 A disk is perturbed by a secondary's m^{th} Linblad resonance, which causes streamlines in the disk to spread or contract radially. Use a mass-conservation arguement to show that the perturbed disk's surface density σ varies as

$$\sigma(a, \theta) = \frac{\sigma_0}{J(a, \theta)} \tag{6.70a}$$

$$\text{where} \quad J(a, \theta) = \frac{\partial r}{\partial a} = 1 + \text{sgn}(x)(\partial e/\partial x)\cos m(\theta - \theta_s), \tag{6.70b}$$

where σ_0 would be the disk's undisturbed surface density, and where the right-hand side of Eqn. (6.70b) assumes that the disk's streamlines are either peri- or apo-aligned. Then show that the fractional variations in the disk's surface density are small in regions far from the resonance's nonlinear zone, where $|x| \gg x_{\text{NL}}$, and not necessarily small where $|x| < x_{\text{NL}}$.

6.8 a.) A particle orbits near a secondary's m^{th} Lindblad resonance, where $m \gg 1$. The secondary's function function, Eqns. (6.26–6.27), depends on the Laplace coefficient

Table 6.1. coefficients f_m^ϵ evaluated via Eqn. (6.27)

m	$\epsilon = +1$ inner LR	$\epsilon = -1$ outer LR
1	—	0.857
2	1.500	4.968
3	3.091	6.567
4	4.689	8.167
5	6.290	9.769
6	7.892	11.37
7	9.495	12.97
8		
9		
10		

$b_{1/2}^{(m)}(\beta)$. Show that

$$b_{1/2}^{(m)}(\beta) \simeq \frac{2}{\pi} K_0(m|x|) = \frac{2}{\pi} K_0(2/3) \simeq 0.4436, \tag{6.71}$$

when $m \gg 1$, where $\beta = 1 + x$ is the particle semimajor axis in units of the secondary's, x is the particle's fractional distance from the secondary's orbit, and K_0 is the modified Bessel function of Eqn. (A.25a). *Hint*: Note that β is nearly unity when $m \gg 1$, so most of the contribution to the integrand in Eqn. (6.4) occurs at angles where $\varphi \ll 1$. This allows one to extend the upper integrand to infinity, to replace β with $1 + x$ where $|x| \ll 1$, and $\cos\varphi \simeq 1 - \varphi^2/2$, which then yields Eqn. (6.71).

b.) Show that

$$\frac{\partial b_{1/2}^{(m)}}{\partial \beta} \simeq \frac{2\epsilon m}{\pi} K_1(2/3) \simeq 0.7168\epsilon m \tag{6.72}$$

for a particle orbiting near the secondary's $m \gg 1$ Lindblad resonance, where K_1 is the modified Bessel function of Eqn. (A.25b).

c.) Show that

$$f_m^\epsilon \simeq \frac{2m}{\pi} \left[2K_0(2/3) + K_1(2/3)\right] = \frac{2km}{\pi} \tag{6.73}$$

in this case, where k is the constant that appears in Eqn. (4.29).

6.9 Derive Eqns. (6.33–6.35).

6.10 A particle like that described in Section 6.1.2 orbits near a secondary's Lindblad resonance. Show that the particle's forced tangential displacement θ_1 is

$$\theta_1 \simeq 2\epsilon \frac{\Omega_0}{\kappa_0} e_f \sin m(\theta - \theta_s) \tag{6.74}$$

where e_f is the particle's forced eccentricity.

6.11 A particle orbits a slightly oblate planet of radius R_p, and is perturbed by a satellite. Show that position of the satellite's zeroeth-order Lindblad resonance, Eqn. (6.22), is shifted radially outwards by

$$\Delta r_r \simeq \frac{J_2}{2}\left(\frac{m+\epsilon}{m-\epsilon}\right)\frac{R_p^2}{r_r} \tag{6.75}$$

where J_2 is the planet's zonal harmonic.

6.12 a.) A particle is perturbed by a secondary whose orbit is circular. Taylor-expand the m^{th} Fourier component of the secondary's gravitational potential Φ_s to first order in the particle's forced eccentricity e_f, and show that Φ_s also includes a first-order term whose Doppler-shifted forcing frequency is $\omega'_m = 2m'(\Omega - \Omega_s)$ where m' is a positive integer. Then show that this term results in a first-order Lindblad resonance with semimajor axis is

$$a_r = \left(1 - \frac{\epsilon}{2m'}\right)^{2/3} a_s, \tag{6.76}$$

assuming that the primary's potential is Keplerian.

b.) Comparing Eqn. (6.76) to (6.22) shows that this first-order Lindblad resonance can overlap the secondary's zeroeth-order resonance when the resonance indices obey $2m' = m$. However, this would only occurs when the primary's gravitational potential is truly Keplerian. If the primary were an oblate planet, then these $2m' = m$ resonances would be separated by a radial distance Δa_r. Derive an approximate expression for Δa_r in terms of the resonance indices, J_2, R_p, and a_r.

6.13 Use Eqns. (6.56) and (6.62) to obtain the resonantly trapped particle's eccentricity, Eqn. (6.63).

6.14 For the resonance trapping scenario of Section 6.3, show that the planetesimal's speed relative to the gas is $\langle \mathbf{u} \rangle \simeq er\Omega$ when trapped at a low m Lindblad resonance, where e is Eqn. (6.63). Then insert Eqns. (6.26) and (6.73) into (6.64) to obtain the planetesimal size threshold for resonance trapping, Eqn. (6.65).

6.15 a.) Show that Eqn. (6.67) becomes

$$T_m \simeq -\frac{4\epsilon m^2 k^2}{3\pi}\mu_s^2 \sigma r^4 \Omega^2 \tag{6.77}$$

for the $m \gg 1$ Lindblad resonances that lie close to the secondary, where k in the above is from Eqn. (6.73).

b.) Now consider a secondary that orbits a radial distance Δr exterior to a disk of surface density σ. Show that the total torque that the secondary exerts on the disk is

$$T = \sum_{m=1}^{m_{\text{max}}} T_m \simeq -\frac{32k^2}{243\pi}\left|\frac{r}{\Delta r}\right|^3 \mu_s^2 \sigma r^4 \Omega^2 \tag{6.78}$$

where m_{max} is the highest-m resonance in the disk. Then convince yourself that this is $-1\times$ the shepherding torque that the disk exerts on the secondary, Eqn. (4.42), as expected.

REFERENCES

1. Goldreich, P. & S. Tremaine, 1980, Disk–Satellite Interactions, *ApJ*, **241**, 425. See this paper for a detailed analysis of the various resonant interactions that occur when an orbiting secondary perturbs a broad disk of matter.