Lecture Notes for ASTR 5622
Astrophysical Dynamics

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January 23, 2006

Perturbed Motion

These lectures will span 4 topics:

1. Gauss' planetary equations.
These are used to calculate how an orbit evolves over time ($\dot{a}$, $\dot{e}$, etc)
due to a perturbing force (like PR drag, aerodynamic drag, etc.).

2. Epicyclic motion in a non–keplerian potential.
These results can be used to describe the motion of a star orbiting in a very non–keplerian potential well (like a star orbiting a galaxy),
or in a slightly non–keplerian well (like a satellite orbiting an oblate planet).
Orbital precession will be assessed.

3. Resonances.
Lindblad resonances in a galaxy,
and secular & mean–motion resonances in a planetary system.

4. Resonant Trapping.
Use the preceding results to consider what happens when a drag force delivers particles to a perturber's resonances, and assess where it does (or does not) get trapped at resonance.

Gauss' Planetary Equations

These eqn's are very useful for calculating the orbital drifts that a particle would suffer due to a drag force:

- PR drag on a dust grain
- electromagnetic forces on a charged dust grain
- gravity from an extended object (circumstellar disk, ring, or shell)

Note that Gauss' eqn's are not that useful when the disturbing force is the gravity of another orbiting companion. But we will tackle that problem later using the methods of point 3 above.

From Section 2.9 of M&D:
Suppose m_2 is in an elliptic orbit about the primary m_1, and that m_2 is also subject to an additional acceleration

$$\mathbf{a} = a_r\hat{\mathbf{r}} + a_\theta\hat{\theta} + a_n\hat{\mathbf{n}} \tag{3.1}$$

where the a_r, a_θ components of acceleration in the radial and azimuthal direction (in m_2's orbit plane), and a_n is normal to the orbit plane.

This perturbation means that m_2's orbit elements are no longer constant; rather, they vary at rates $\dot{a}$, $\dot{e}$, that we will calculate.

Calculate $\dot{a}$:

The total *specific* work done by $\mathbf{a}$ on m_2 as it travels from $\mathbf{r_1} \rightarrow \mathbf{r_2}$ is

$$W = \int_{\mathbf{r_1}}^{\mathbf{r_2}} \mathbf{a} \cdot d\mathbf{r} \tag{3.2}$$

So $\Delta W = \mathbf{a} \cdot \mathbf{\Delta r}$ = the small differential work/mass that $\mathbf{a}$ does on m_2 as it is displaced the small distance $\mathbf{\Delta r}$. This work changes m_2's specific energy E by amount ΔE in time interval Δt, so

$$\frac{\Delta E}{\Delta t} = \mathbf{a} \cdot \frac{\mathbf{\Delta r}}{\Delta t} \tag{3.3}$$

$$\text{or} \quad \dot{E} = \mathbf{a} \cdot \dot{\mathbf{r}} \tag{3.4}$$

$$\text{where} \quad \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\theta\hat{\theta} \tag{3.5}$$

$$\text{so} \quad \dot{E} = a_r\dot{r} + a_\theta r\dot{\theta}. \tag{3.6}$$

$$\text{Since} \quad E = -\frac{\mu}{2a} \qquad \text{and} \qquad h = r^2\dot{\theta}, \tag{3.7}$$

$$\dot{E} = \frac{\mu}{2a^2}\dot{a} \tag{3.8}$$

Write the radial and tangential velocities $\dot{r}$ & $r\dot{\theta}$
in terms of *osculating orbit elements*:

Recall that these orbit elements are defined by *assuming* that m_2's motion is pure unperturbed 2–body motion. The velocities in turn can be obtained from osculating orbit elements that are presumed constant at time t.

To get radial velocity $\dot{r}$, start with the ellipse equation:

$$r(t) = \frac{p}{1 + e\cos f}, \quad \text{where} \tag{3.9}$$

$$\text{semilatus rectum} \quad p = a(1 - e^2) = h^2/\mu, \tag{3.10}$$

$$\text{true anomaly} \quad f(t) = \theta - \tilde{\omega} \tag{3.11}$$

$$\text{thus} \quad \dot{r} = \frac{pe\sin f\dot{f}}{(1 + e\cos f)^2} = \frac{r^2}{p}\frac{h}{r^2}e\sin f \tag{3.12}$$

$$\text{since} \quad \dot{f} = \dot{\theta} = h/r^2 \tag{3.13}$$

$$\text{so} \quad \dot{r} = \frac{eh}{p}\sin f = \frac{e\sin f}{\sqrt{1 - e^2}}an = \ m_2\text{'s radial vel'} \tag{3.14}$$

$$\text{where mean motion} \quad n = \sqrt{\frac{\mu}{a^{3/2}}}, \quad \text{and } \mu = n^2a^3 \tag{3.15}$$

The tangential velocity $r\dot{\theta}$ is obtained from

$$r\dot{\theta} = \frac{h}{r} = \frac{h}{p}(1 + e\cos f) = \frac{1 + e\cos f}{\sqrt{1 - e^2}}an \tag{3.16}$$

Plug these results into $\dot{E}$:

$$\dot{E} = \frac{1}{2}n^2a\dot{a} = \frac{an}{\sqrt{1 - e^2}}\left[a_r e\sin f + a_\theta(1 + e\cos f)\right] \tag{3.17}$$

$$\text{so} \quad \dot{a} = \frac{2}{n\sqrt{1 - e^2}}\left[a_r e\sin f + a_\theta(1 + e\cos f)\right] \tag{3.18}$$

Recall that the torque on m_2 is due to $\mathbf{a}$ is $|\mathbf{T}^\star| = m_2|\mathbf{r} \times \mathbf{a}| \simeq m_2 a a_\theta$ when the orbit is nearly circular, ie $e \ll 1$. In this case, $\dot{a} \simeq 2a_\theta/n = 2T^\star/m_2 an$, which recovers your earlier result, Eqn. (2.138).

$\Rightarrow$the tangential acceleration a_θ determines m_2's radial drift;
this is sometimes called the 'along–track acceleration'.

Calculate $\dot{e}$ and di/dt by considering m_2's specific angular momentum $\mathbf{h} = h\hat{\mathbf{n}}$, where $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ is normal to the orbit plane:

The *specific* torque on m_2 due to acceleration $\mathbf{a}$ is

$$\mathbf{T} = \frac{d\mathbf{h}}{dt} = \mathbf{r} \times \mathbf{a} = r\hat{\mathbf{r}} \times (a_r\hat{\mathbf{r}} + a_\theta\hat{\theta} + a_n\hat{\mathbf{n}}) = ra_\theta\hat{\mathbf{n}} - ra_n\hat{\theta} \quad (3.19)$$

$$= \dot{h}\hat{\mathbf{n}} + h\frac{d\hat{\mathbf{n}}}{dt} \quad (3.20)$$

$$\text{Thus} \quad \dot{h} = ra_\theta \quad \text{and} \quad \frac{d\hat{\mathbf{n}}}{dt} = -\frac{ra_n}{h}\hat{\theta} = \frac{ra_n}{h}\hat{\mathbf{r}} \times \hat{\mathbf{n}} \quad (3.21)$$

The left eqn' tells us that only the *magnitude* of $\mathbf{h}$ is altered by a_θ, and the right eqn' says that its *direction* is only altered by a_n.

Since $h = \sqrt{\mu a(1-e^2)}$,

$$\frac{dh}{dt} = \frac{1}{2}\sqrt{\frac{\mu(1-e^2)}{a}}\dot{a} - \sqrt{\frac{\mu a}{1-e^2}}e\dot{e} = ra_\theta \quad (3.22)$$

$$\text{so} \quad e\dot{e} = \frac{(1-e^2)}{2a}\dot{a} - \sqrt{\frac{1-e^2}{\mu a}}ra_\theta \quad (3.23)$$

$$= \frac{\sqrt{1-e^2}}{na}\left[a_r e\sin f + a_\theta(1+e\cos f)\right] - \frac{\sqrt{1-e^2}}{na}\frac{r}{a}a_\theta \quad (3.24)$$

$$= \frac{\sqrt{1-e^2}}{na}\left[a_r e\sin f + a_\theta\left(1+e\cos f - \frac{r}{a}\right)\right] \quad (3.25)$$

$$\text{Recall} \quad r = a(1-e\cos E) \qquad E = \text{eccentric anomaly, Eqn. (1.67)} \quad (3.26)$$

$$\text{Thus} \quad \dot{e} = \frac{\sqrt{1-e^2}}{na}\left[a_r\sin f + a_\theta\left(\cos f + \cos E\right)\right] \quad (3.27)$$

For a body in a nearly circular orbit,
$e \ll 1$, $\cos E \simeq \cos f$ (see Eqns 1.90–1.91), so

$$\dot{e} \simeq \frac{a_r \sin f + 2a_\theta \cos f}{an} \tag{3.28}$$

Now calculate di/dt from

$$\frac{d\hat{\mathbf{n}}}{dt} = \frac{ra_n}{h}\hat{\mathbf{r}} \times \hat{\mathbf{n}} \tag{3.29}$$

Note that i is the tilt of the orbital plane wrt the $\hat{\mathbf{x}}_{ref}$–$\hat{\mathbf{y}}_{ref}$ reference plane, which is fixed in space:

and note that $\hat{\mathbf{z}}_{ref} \cdot \hat{\mathbf{n}} = \cos i$.
The unit vector $\hat{\boldsymbol{\Omega}}$ points from m_1 towards the ascending node.

$$\text{so} \quad \frac{d}{dt}(\hat{\mathbf{z}}_{ref}\cdot\hat{\mathbf{n}}) \;=\; -\sin i\frac{di}{dt} = \frac{ra_n}{h}\hat{\mathbf{z}}_{ref}\cdot(\hat{\mathbf{r}}\times\hat{\mathbf{n}}) \tag{3.30}$$

and use the vector identity

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \;=\; \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) \tag{3.31}$$

$$\text{so} \quad \frac{di}{dt} \;=\; = -\frac{ra_n}{h\sin i}\hat{\mathbf{r}}\cdot(\hat{\mathbf{n}}\times\hat{\mathbf{z}})_{ref} \tag{3.32}$$

$$\tag{3.33}$$

The sketch shows that $\hat{\mathbf{n}}\times\hat{\mathbf{z}}_{ref} = -\sin i\hat{\mathbf{\Omega}}$,
and that $\hat{\mathbf{r}}\cdot\hat{\mathbf{\Omega}} = \cos(\omega+f)$, so

$$\frac{di}{dt} \;=\; \frac{ra_n}{h}\cos(\omega+f) \tag{3.34}$$

For nearly circular orbits, $h = r^2\dot{\theta} \simeq a^2 n$ and

$$\frac{di}{dt} \;\simeq\; \frac{a_n}{an}\cos\theta \tag{3.35}$$

where angle θ is measured from the ascending node.

Now calculate $\dot{\Omega}$.

Let $\mathbf{\Delta n}$ = the change in the orbit normal $\hat{\mathbf{n}}$ after small time interval Δt.

We are interested in $\Delta n_\Omega \equiv \hat{\mathbf{\Omega}}\cdot\mathbf{\Delta n}$, which is the component of $\mathbf{\Delta n}$ that is parallel to $\hat{\mathbf{\Omega}}$ that causes the node to rotate.

Thus $\Delta n_\Omega = \hat{\mathbf{\Omega}}\cdot\mathbf{\Delta n}$ is $\hat{\mathbf{n}}$'s change in the $\hat{\mathbf{x}}_{ref}$–$\hat{\mathbf{y}}_{ref}$ plane, and

$$\Delta\Omega \;=\; \frac{\Delta n_\Omega}{\sin i} \qquad \text{(see figure)} \tag{3.36}$$

is the total change in m_2's node after time Δt. Thus

$$\dot{\Omega} = \frac{\Delta\Omega}{\Delta t} = \frac{\hat{\mathbf{\Omega}}\cdot\mathbf{\Delta n}}{\sin i\Delta t} = \frac{1}{\sin i}\hat{\mathbf{\Omega}}\cdot\frac{d\hat{\mathbf{n}}}{dt} = -\frac{ra_n}{h\sin i}\hat{\mathbf{\Omega}}\cdot\hat{\theta} \tag{3.37}$$

$$= -\frac{ra_n\cos(\theta+\pi/2)}{h\sin i} \tag{3.38}$$

$$= \frac{ra_n}{h\sin i}\sin\theta \tag{3.39}$$

Assignment #4
due Tuesday February 28
at the start of class

1. a.) A particle in a low–e orbit is perturbed by acceleration $\mathbf{a}$, Eqn. (3.1). Show that its longitude of perihelion $\tilde{\omega}$ varies as

$$\dot{\tilde{\omega}} \simeq \frac{2a_\theta\sin f - a_r\cos f}{ean} + \mathcal{O}(e^0). \tag{3.40}$$

An easy way to obtain $\dot{\tilde{\omega}}$ is calculate the time derivative of $r(t)$ in the epicyclic approximation, $r \simeq a - ae\cos(\theta - \tilde{\omega})$ (see Eqn's 1.36, 1.37, 1.98). DO NOT quote M&D's exact (and more laborious) solution back at me...

b.) Show that Eqn' (3.40) agrees with M&D's exact calculation of $\dot{\omega}$ in the limit that $e \ll 1$.

2. a.) A particle orbit's is completely embedded within a circumstellar gas disk, with both orbiting a central star. The particle's orbit is inclined slightly relative to the disk's midplane, ie, $\sin i \ll 1$. Show that the particle's height z above/below the disk midplane is

$$z(\theta) \simeq a\sin i\sin\theta \tag{3.41}$$

where θ is the particle's longitude measure wrt its ascending node.

b.) The disk has a constant gas density ρ. Use Gauss' law to show that the vertical component of the disk's gravity is $a_n = -4\pi G\rho z$.

c.) Show that the disk's gravity causes the particle's node to precess at the time–averaged rate of

$$< \dot{\Omega} > \simeq -2\pi G\rho/n \qquad (3.42)$$

where n is its mean–motion, and $<>$ indicates time–averaging over an orbit.

3. a.) Use Eqn's (1.5) and (1.9) to show that the Laplace–Runge–Lenz vector,

$$\mathbf{A} = \dot{\mathbf{r}} \times \mathbf{h} - \mu\hat{\mathbf{r}}, \qquad (3.43)$$

is conserved (ie, $d\mathbf{A}/dt = 0$) for the two–body problem.

b.) Show that $\mathbf{A}$ points from m_1 towards m_2's periapse,
and that its magnitude is $A = \mu e$.

Gauss Law & Poisson's Eqn'

Now derive Gauss' Law:
Begin by placing an imaginary 'Gaussian surface' S around point mass m:

The gravitational acceleration as some point on the surface is

$$\mathbf{g} = -\frac{Gm}{r^2}\hat{\mathbf{r}} \tag{3.44}$$

Let $\mathbf{da} = da\hat{\mathbf{n}}$ represent a small patch on S of area da whose orientation is described by a unit vector $\hat{\mathbf{n}}$ that is normal to da.

Then $\mathbf{g}{\cdot}d\mathbf{a}$ is the 'gravitational flux' passing through area $\mathbf{da}$.
Think of this as a measure of the number of 'lines of force' passing thru $\mathbf{da}$; the larger m is, the larger the grav' flux.

The total gravitational flux Ψ that passes through surface S is then

$$\Psi = \int_S \mathbf{g}\cdot\hat{\mathbf{n}}da \tag{3.45}$$

$$\text{and since} \quad \mathbf{g}\cdot\hat{\mathbf{n}} = -\frac{Gm\cos\theta}{r^2}, \tag{3.46}$$

$$\Psi = -Gm\int_S \frac{\cos\theta da}{r^2} \tag{3.47}$$

Now note that $\cos\theta da$ is the projected area of da as seen by an observer sitting on m, so $d\Omega = \cos\theta da/r^2$ is the *solid angle* that da subtends, as seen by someone at m. Then

$$\Psi \;=\; -Gm\int_S d\Omega \tag{3.48}$$

What is $\int_S d\Omega$?

Thus $\Psi = -4\pi Gm$.

Obviously, if surface S contains several masses m_i then

$$\Psi \;=\; \int_S \mathbf{g}\cdot\hat{\mathbf{n}}da = -4\pi G\sum m_i = -4\pi G M_{enc} \tag{3.49}$$

where M_{enc} = total mass enclosed by surface S.

This is sometimes known as the integral form of Gauss' Law.

It is only useful for problems having a high degree of symmetry, such that the area integral is easily evaluated.

Trivial example: use Gauss's law to calculate the grav' acceleration $\mathbf{g}(\mathbf{r})$ inside & outside a sphere of radius R and constant density ρ.

Note that the body has spherical symmetry, so $\mathbf{g}(\mathbf{r}) = g(r)\hat{\mathbf{r}}$.

What kind of Gaussian surface should I use?
Use one that takes advantage of the problem's symmetry,
and makes the area integral easy.

Note that the normal to surface S is $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. Then the flux Ψ is

$$\Psi = \int_S \mathbf{g} \cdot \hat{\mathbf{r}} da = \int_S g(r) da = ? \tag{3.50}$$

$$= -4\pi G M_{enc} \tag{3.51}$$

What is M_{enc} ?

$$M_{enc}(r) = \begin{cases} \frac{4\pi}{3}\rho r^3 & r < R \\ \frac{4\pi}{3}\rho R^3 = M_{total} & r \geq R \end{cases} \tag{3.52}$$

So

$$\text{so} \quad g(r) = -\frac{G M_{enc}(r)}{r^2} = \begin{cases} -\frac{4\pi}{3} G\rho r & r < R \\ -G M_{total}/r^2 & r \geq R, \end{cases} \tag{3.53}$$

as expected.

You will do something similar to solve problem 2b.) of Assignment #4.

Poisson & Laplace Eqns.

Now derive Poisson's eqn. from Ψ:

$$\text{flux} \quad \Psi \ = \ \int_S \mathbf{g} \cdot \hat{\mathbf{n}} da = -4\pi G M_{enc} = -4\pi G \int_V \rho dV \tag{3.54}$$

where $\rho(\mathbf{r})$ is the matter density enclosed by volume V and surface S.

Next use the divergence theorem for the generic vector field $\mathbf{A}(\mathbf{r})$ that you derived in your vector calculus class:

$$\text{divergence theorem} \quad \int_S \mathbf{A} \cdot \hat{\mathbf{n}} da \ = \ \int_V \nabla \cdot \mathbf{A} dV \tag{3.55}$$

$$\text{so} \quad \Psi = \int_S \mathbf{g} \cdot \hat{\mathbf{n}} da \ = \ \int_V \nabla \cdot \mathbf{g} dV \tag{3.56}$$

Now recall the EOM for a particle that might be roaming about this system:

$$\ddot{\mathbf{r}} \ = \ -\nabla\Phi = \mathbf{g} \tag{3.57}$$

where $\Phi(\mathbf{r})$ is the system's gravitational potential, so

$$\Psi = -\int_V \nabla\Phi^2 dV = -4\pi G \int_V \rho dV, \tag{3.58}$$

$$\text{or} \quad \int_V (-\nabla^2\Phi + 4\pi G\rho) dV = 0 \tag{3.59}$$

This result holds for any arbitrary volume V.

What does this tell us about the integrand?

$$\nabla^2\Phi = 4\pi G\rho \tag{3.60}$$

This is *Poisson's eqn'*, which is the differential form of Gauss' Law.

This eqn' relates the mass density $\rho(\mathbf{r})$ to its gravitational potential $\Phi(\mathbf{r})$.

It is of fundamental importance to hydrodynamic studies of gravitating systems: galaxy formation, star formation, etc.

We use this equation later when we study gravitational instabilities, and spiral wave theory.

In free space where $\rho = 0$, you have *Laplace's eqn.*:

$$\nabla^2\Phi = 0 \tag{3.61}$$

Assignment #4
due Tuesday February 28
at the start of class

4. a.) Use Gauss' planetary equations to show that Poynting–Robertson (PR) drag causes an orbiting dust grain's semimajor axis a to shrink at the time–averaged rate

$$<\dot{a}> \;\simeq\; -2\beta\left(\frac{an}{c}\right)(1+3e^2)an + \mathcal{O}(e^3) \tag{3.62}$$

and that PR drag damps its eccentricity at the time–averaged rate

$$<\dot{e}> \;\simeq\; -\frac{5}{2}\beta\left(\frac{an}{c}\right)en + \mathcal{O}(e^3) \tag{3.63}$$

What is di/dt?

b.) A pair of Kuiper Belt Objects (KBOs) collide, generating a cloud of icy grains having radii $R = 10\mu$m a distance $r = 40$ AU away from the Sun. The parent KBOs had eccentricities of $e \sim 0.2$. What is the timescale for the grain's orbital evolution due to PR drag, $\tau_a \equiv |a/\dot{a}|$, in years? What is the timescale $\tau_e \equiv |e/\dot{e}|$ for e–damping due to PR drag?

5.) Solve Eqn. (3.62) for a grain's semimajor axis $a(t)$, assuming an initially circular orbit. Suppose an $R = 10\mu$m ice–grain were orbiting the star β Pictoris (which has an extensive dust–disk extending out to $r \sim 10^3$ AU) at $r = 100$ AU. How long until that grain spirals into β Pic?

Radiation Forces

Radiation forces are often the dominant perturbing force affecting small grains (ie, dust).

Radiation pressure from luminous objects (evolved stars, accretion disks around stars/black holes/galactic nuclei, etc) can be sufficient to drive out any dust in the vicinity.

Radiation forces are relevant to star formation. Observations of recently–formed O and B stars show these bright stars can blow away the residual gas & dust, due to radiation pressure & stellar winds. The formation of O & B stars in a star–forming region can in fact terminate subsequent formation.

These forces also are important to planetary dynamics. Comets & asteroids continually generate dust, due to sublimation and/or collisions, and solar radiation pressure will drive dust smaller than $R \lesssim 1\mu$m out of the Solar System; these are the so–called β meteoroids that spacecraft sometimes detect streaming away from the Sun.

The shape of a comet's dust tail (straight & narrow versus a broad fan) is entirely controlled by solar radiation pressure.

Interplanetary dust larger than $R \gtrsim 1\mu$m also suffer PR drag, which results in slow orbital decay (see problems 4–5, above), which tends to deplete planetary systems of their dust.

Circumstellar dust disks are routinely observed orbiting other stars. Since PR drag destroys such disks, their existence is usually interpreted as evidence for replenishment via colliding asteroids or comets.

PR drag can deliver dust to resonances with planets and cause dust to accumulate there. Dust trapped at a resonance can result in a clumpy dust ring (e.g., ϵ Eridani).

PR drag coupled with *satellite* resonances control the dust rings of Jupiter.

The Yarkovsky effect (YE) is a more subtle radiation force; it is a consequence of a body's anisotropic thermal emission that occurs when its surface temperature is uneven. The YE is believed to play a role in the delivery of $R \sim 10$cm meteorites to Earth.

The YE also plays a role in the delivery of $R \lesssim 1$ km Near–Earth Objects (NEOs), and thus is a factor when considering the impact hazard of NEOs.

Over billions of years, the YE can alter the spins of asteroids and the orbits of asteroid satellites.

We will assess radiation pressure (which can alter the *shape* of an orbit) and PR drag (which causes orbits to slowly drift over time).

Radiation pressure and PR drag

Lets consider a small dust grain that is orbiting a star.
The geometry in the star's rest frame is

Now consider this system in dust rest frame;
in this frame, the star orbits about the dust.

this is obtained via the Galilean transformation
$\mathbf{r} \to \mathbf{r} - \mathbf{r_d}$ and $\dot{\mathbf{r}} \to \dot{\mathbf{r}} - \dot{\mathbf{r}}_\mathbf{d}$.

(Technically, we should be doing a Lorentz transformation,
but that is Galilean when $|\dot{\mathbf{r}}| \ll c$.)

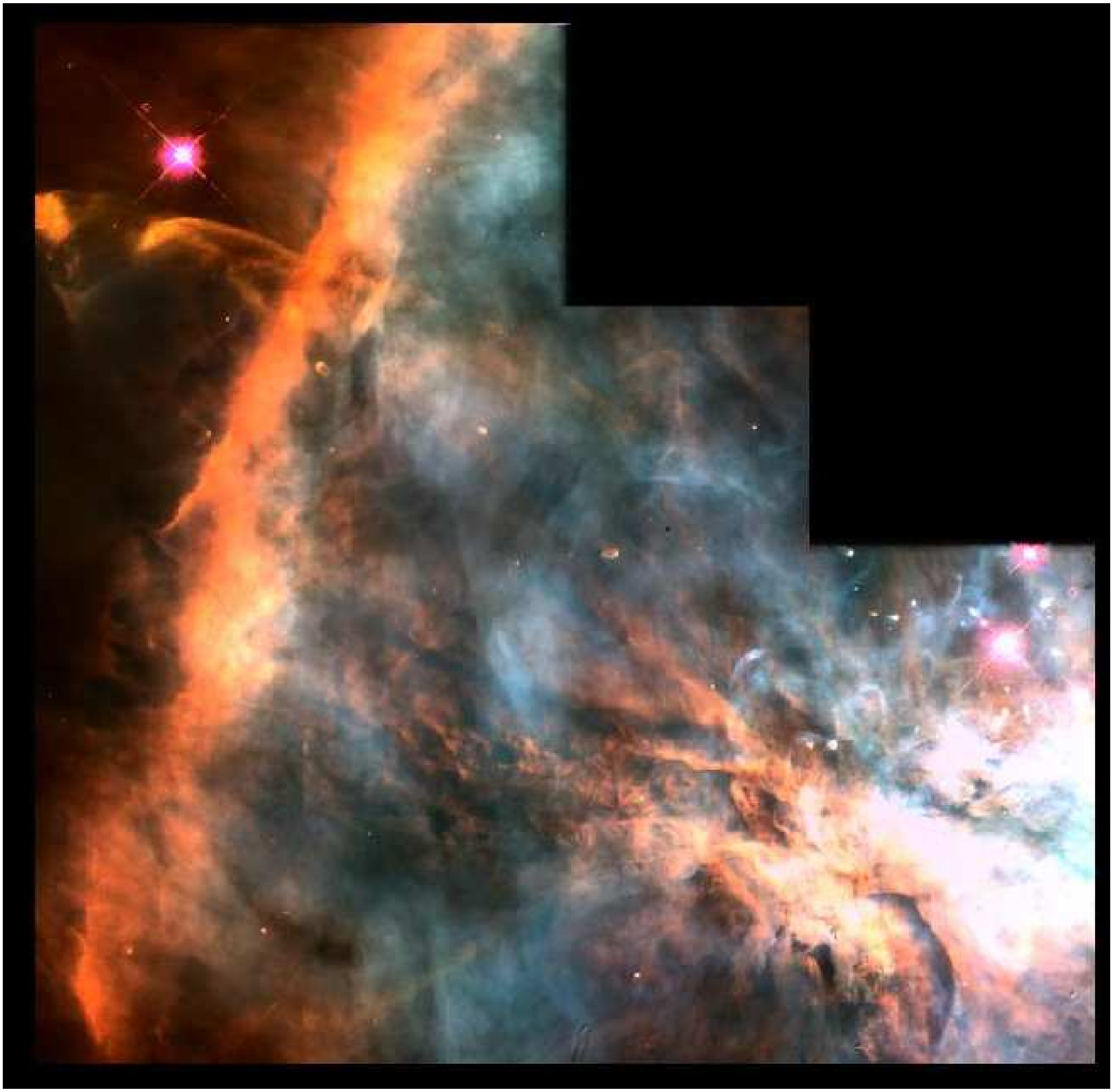


Figure 3.1: Orion nebula imaged by HST

Figure 3.2: Comet Hale Bopp, imaged by Dave Schleicher.

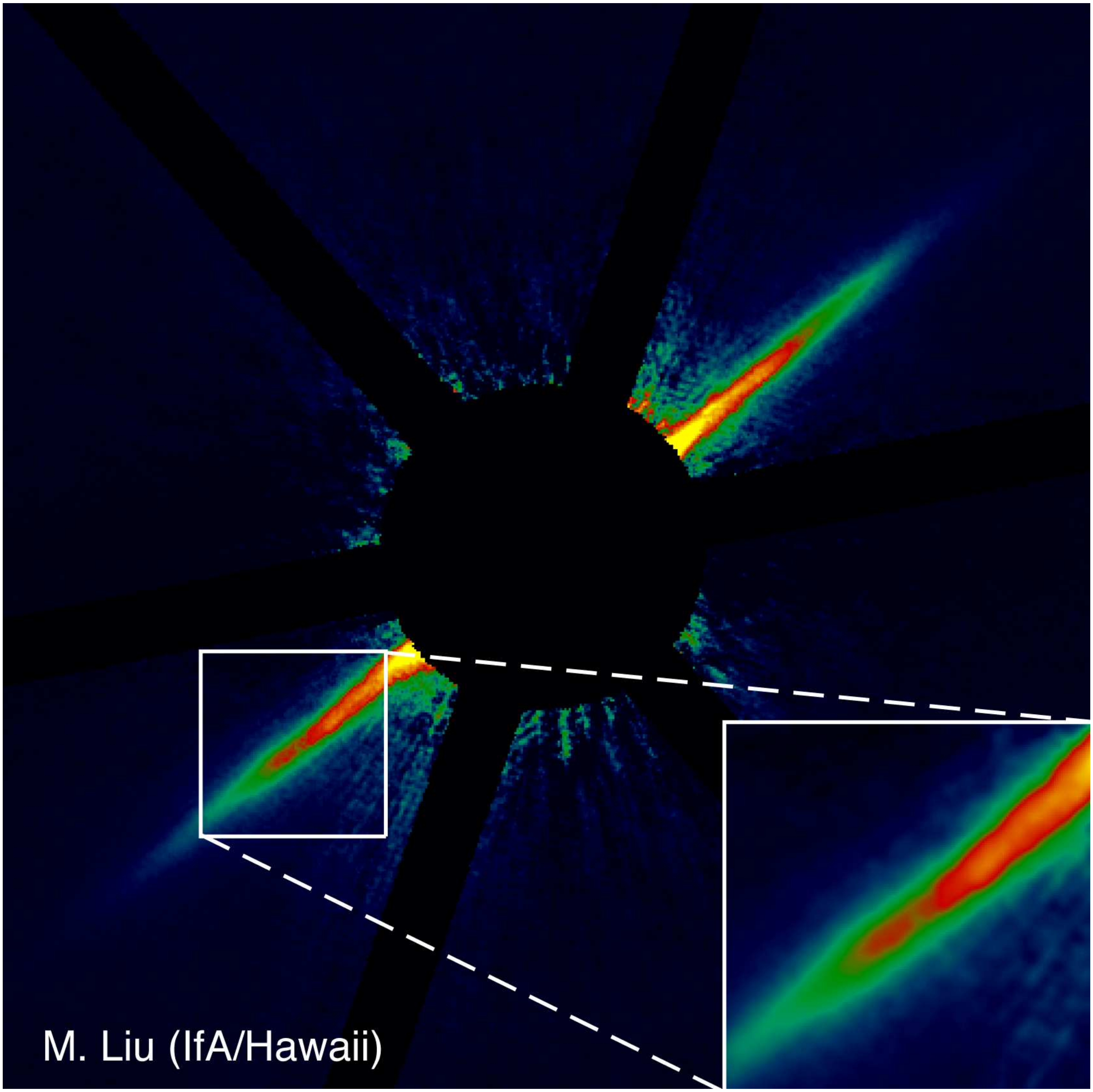


Figure 3.3: AU Microscopi, imaged by HST

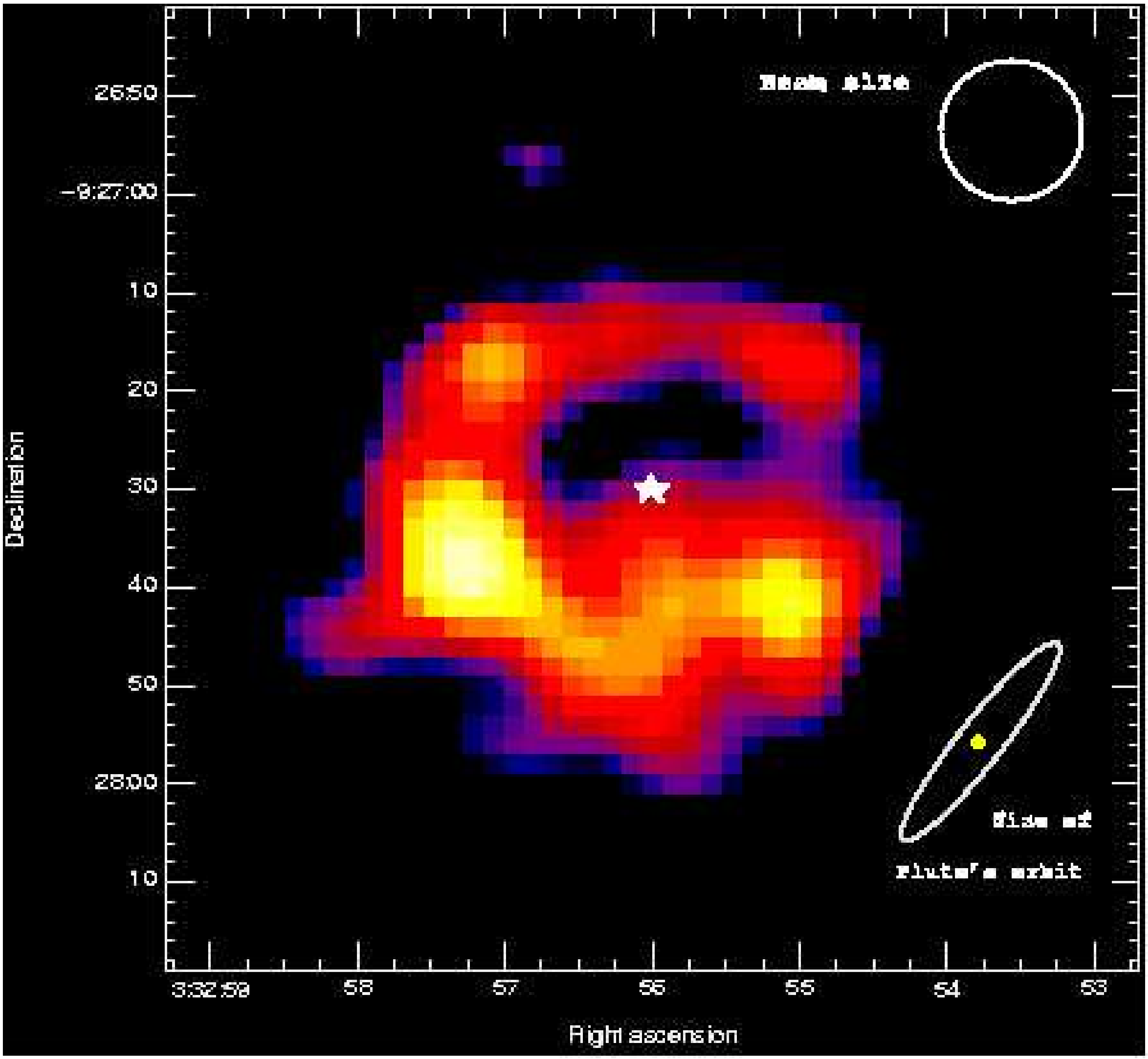


Figure 3.4: ϵ Eridani at submillimeter wavelengths.

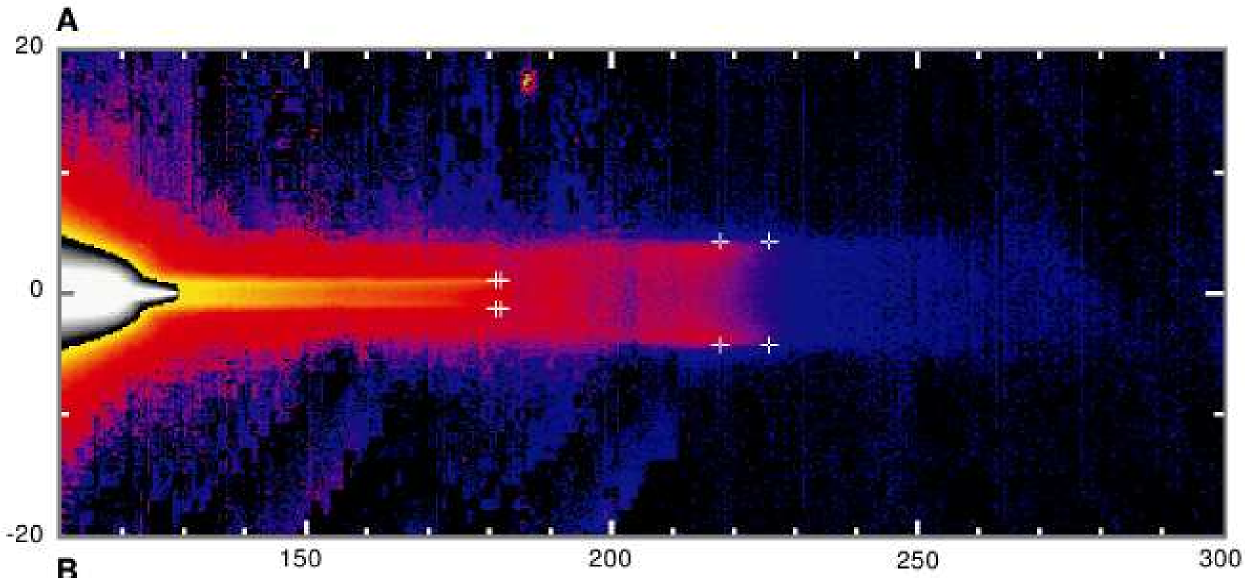


Figure 3.5: Jupiter's dust rings, imaged by Galileo spacecraft

Figure 3.6: Zodiacal light.

The star is in the $-\hat{\mathbf{r}}$ direction.
Since the speed of light $c =$ constant, the sketch shows that a *moving* grain is bombarded by stellar photons that were emitted from a star that *appears* to be displaced slightly in the $+\hat{\theta}$ direction by angle $\phi = |\dot{\mathbf{r}}|/c$.

Since these photons have some momentum in the $+\hat{\theta}$ direction,
they act like a drag force—PR drag—when the grain intercepts the photons.

This drag goes away when the grain is motionless.

This effect is analogous to driving in the rain on a windless day—
driving faster makes the raindrops trajectories more slanted.

This effect is also related to *stellar aberration*, which is the apparent displacement of a star (up to 20") due to the Earth's orbital motion.

Quantify this:
$F = L_\star/4\pi r^2 =$ energy flux incident upon a *stationary* grain;
$L_\star =$ star's luminosity (units=energy/time),
so flux has units energy/area/time.

If the grain is moving with radial velocity $\dot{r}$, it will intercept photons Doppler–shifted to shorter/longer wavelength's & higher/lower energies, so
$F_{ds} = F(1 - \dot{r}/c) =$ Doppler–shifted flux.

Then $\dot{E} = F_{ds}A =$ rate energy is delivered to grain of cross–section A.

Recall a photon carries energy $E = pc$ where $p =$ photon momentum, so

$$\dot{p} \;=\; \frac{\dot{E}}{c} = \frac{FA}{c}\left(1 - \frac{\dot{r}}{c}\right) \equiv \dot{p}_{inc} \tag{3.64}$$

is the rate at which stellar photons deliver momentum to the grain;
this is the incident momentum transfer rate, $\dot{p}_{inc}$.

Keep in mind that momentum is a vector:
$\dot{\mathbf{p}}_{inc} = \dot{p}_{inc}\hat{\mathbf{p}}$ where direction $\hat{\mathbf{p}} = \cos\phi\hat{\mathbf{r}} - \sin\phi\hat{\theta} \simeq \hat{\mathbf{r}} - \dot{\mathbf{r}}/c$
to first order in $|\dot{\mathbf{r}}/c| \ll 1$ (see figure).

The dust grain can do several things with an incident photon:

- it can absorb it (warming the grain)
- it can scatter it (ie, deflect it)
- and it can emit another thermal photon (cooling the grain)

If we average all possibilities over all photons, and invoke Newton's 2nd law, then the force on a grain of mass m is

$$m\ddot{\mathbf{r}} \;=\; \dot{\mathbf{p}}_{inc} + \dot{\mathbf{p}}_{scat} + \dot{\mathbf{p}}_{therm} \tag{3.65}$$

where the terms are understood as averages over many photons, and thus represent the net effects due to the absorption of some incident photons, the scattering of others, and thermal emission.

Small dust grains are certain to have a uniform surface temperature, so $\dot{\mathbf{p}}_{therm} = 0$ on average. *Why?*

For simplicity, we will assume that dust grains are *isotropic* light scatters, so $\dot{\mathbf{p}}_{scat} = 0$ on average. *Why?*

Actually, observations of the zodiacal light (sunlight scattered by interplanetary dust) shows that these dust are *not* isotropic light–scatterers.

However our faulty assumption merely introduces some uncertainty that is probably no more than a factor of 2. Why? To verify this, consider two extreme cases: a grain that is a perfect forward–scatterer, and then a perfect back–scatterer.

To account for this uncertainty, introduce a fudge–factor
Q_{pr} = radiation pressure efficiency factor, so $m\ddot{\mathbf{r}} = Q_{pr}\dot{\mathbf{p}}_{inc}$.

Lastly, I note that if the "grain's" surface temperature is not uniform, then $\dot{\mathbf{p}}_{therm} \neq 0$, and the body suffers a thermal force known as the Yarkovsky effect (YE), which afflicts bodies having sizes 10cm$\lesssim R \lesssim$ 1km.

Why the upper & lower limits on the YE?

Thus our dust grain experiences an acceleration that is,
to first order in $|\dot{\mathbf{r}}/c| \ll 1$:

$$\ddot{\mathbf{r}} \simeq \frac{FAQ_{pr}}{mc}\left(1 - \frac{\dot{r}}{c}\right)\left(\hat{\mathbf{r}} - \frac{\dot{\mathbf{r}}}{c}\right) \quad (3.66)$$

$$\simeq \frac{FAQ_{pr}}{mc}\left[\left(1 - \frac{2\dot{r}}{c}\right)\hat{\mathbf{r}} - \frac{r\dot{\theta}}{c}\hat{\theta}\right] \quad (3.67)$$

$$\equiv \mathbf{a}_{rad} + \mathbf{a}_{PR} \quad (3.68)$$

where the velocity independent acceleration $\mathbf{a}_{rad}$ called radiation pressure, and the velocity–dependent acceleration $\mathbf{a}_{RP}$ is called PR drag:

$$\mathbf{a}_{rad} \equiv a_{rad}\hat{\mathbf{r}} \quad \text{where } a_{rad} = \frac{FAQ_{pr}}{mc}, \quad (3.69)$$

$$\text{and} \quad \mathbf{a}_{PR} \equiv -a_{rad}\left(\frac{2\dot{r}}{c}\hat{\mathbf{r}} + \frac{r\dot{\theta}}{c}\hat{\theta}\right) \quad (3.70)$$

Use this result to solve problems 4 & 5 of Assignment #4.

Note that radiation pressure is an $a_rad \propto r^{-2}$ acceleration, outwards:

$$a_{rad} = \frac{L_\star Q_{pr}}{4\pi r^2 mc} = \frac{3L_\star Q_{pr}}{16\pi\rho r^2 Rc} \quad (3.71)$$

for spherical grains of radius R, density ρ.

Thus radiation pressure merely opposes the star's gravity $g(r) = -GM_\star/r^2$ by the fractional amount

$$\beta = \frac{a_{rad}}{|g|} = \frac{3L_\star Q_{pr}}{16\pi GM_\star \rho Rc} \quad (3.72)$$

$$\text{so the EOM is} \quad \ddot{\mathbf{r}} = -\frac{GM_\star}{r^2}(1-\beta)\hat{\mathbf{r}} \quad (3.73)$$

so a dust grain behaves as if orbiting a star of mass smaller by factor $1-\beta$.

For a $\rho \sim 3$ gm/cm^3 grain orbiting the Sun, $\beta \sim 0.2(R/1\mu\text{m})^{-1}$, assuming $Q_{pr} \sim 1$.

Radiation pressure is most relevant to studies of cometary dust tails, since radiation pressure + centrifugal force (aka, keplerian sheer) determine the tail curvature.

Technically, our results are valid in the *geometric optics* limit, which occurs for grains larger than the typical wavelength of starlight (ie, $R \gg 1\mu$m for dust orbiting the Sun).

Assessing radiation forces for smaller grains requires a theory for how E&M waves interact small bodies having $R \lesssim \lambda$, such as *Mie theory.*

What happens to small grains having $\beta > 1$? (β *meteoroids*).

Epicyclic motion in a non–keplerian potential

'Til now we have been studying the motion of a secondary orbiting a point–mass whose gravity $\mathbf{g}$ & potential $\Phi(\mathbf{r})$ are *keplerian*, which has the form

$$\mathbf{g}(\mathbf{r}) \;=\; -\nabla\Phi = -\frac{\mu}{r^2}\hat{\mathbf{r}} \qquad \text{where} \qquad \Phi = -\frac{\mu}{r}. \tag{3.74}$$

The solution for m_2's motion are elliptical or hyperbolic orbits, and low e & i orbits are *epicyclic.*

Now lets solve a slightly more general problem: for the motion of a body orbiting in a potential well that is non–keplerian, where Φ is *not* $\propto r^{-1}$.

We will assume that the system's vertical gravity is antisymmetric about the $z = 0$ plane, ie, $\mathbf{g}(r, \theta, -z) = -\mathbf{g}(r, \theta, z)$.

A flattened system, like a disk galaxy or an oblate planet, has this property

Initially, assume that the potential is independent of θ (ie, azimuthally symmetric): $\Phi = \Phi(r, z)$.

Examples:

- star orbiting a featureless disk galaxy (a very non–keplerian system)
- a satellite orbiting an oblate gas giant planet (slightly non–keplerian)

We will find that nearly circular orbits in this system are also epicyclic.

Once we have mastered this problem, we will then introduce a weak perturbation that breaks the problem's azimuthal symmetry. Examples include:

- a rotating bar in the center of a disk galaxy
- another satellite

These perturbations are periodic, and *resonances* can result.

Resonances in a galaxy they are called Lindblad resonances,
in planetary environments they are called mean motion & secular resonances.

The EOM

The EOM for particle P orbiting in this potential is $\ddot{\mathbf{r}} = -\nabla\Phi$, where

$$\begin{aligned} \mathbf{r} &= r\hat{\mathbf{r}} + z\hat{\mathbf{z}} & (3.75) \\ \dot{\mathbf{r}} &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{\mathbf{z}} & (3.76) \\ \ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\theta} + \ddot{z}\hat{\mathbf{z}} & (3.77) \end{aligned}$$

in cylindrical coordinates.

Keep in mind that r is P's *in–plane* distance from the origin;
P's total distance is $|\mathbf{r}| = \sqrt{r^2 + z^2}$.

The EOM has components

$$\begin{aligned} \hat{\mathbf{r}} &: \quad \ddot{r} - r\dot{\theta}^2 = -\frac{\partial\Phi}{\partial r} & (3.78) \\ \hat{\theta} &: \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = 0 & (3.79) \\ \hat{\mathbf{z}} &: \quad \ddot{z} = -\frac{\partial\Phi}{\partial z} & (3.80) \end{aligned}$$

thus the z–component of angular momentum, $h_z = r^2\dot{\theta}$, is conserved.

The zeroth order solution: circular, coplanar orbits

First consider the simplest orbit—a circular orbit in the system's midplane:

$$r(t) = r_0 \tag{3.81}$$
$$\theta(t) = \theta_0 + \theta'(t) \tag{3.82}$$
$$z(t) = 0 \tag{3.83}$$

The $\hat{\mathbf{r}}$ eqn tells us that

$$\dot{\theta}'^2 = \frac{1}{r_0}\frac{\partial \Phi}{\partial r}\bigg|_{\mathbf{r_0}} \equiv \Omega_0^2(r_0) = \text{ constant} \tag{3.84}$$

where the $|_{\mathbf{r_0}}$ is a reminder to evaluate quantities at $r = r_0$ and $z = 0$.

Thus P orbits with a constant angular velocity: $\theta(t) = \theta_0 + \Omega_0 t$, and its position vector is $\mathbf{r_0} = (r_0, \theta_0 + \Omega_0 t, 0)$.

Note that when the potential is keplerian, $\Phi = -\mu/r$ and we recover $\Omega_0^2 = \mu/r_0^3 = n^2$, the mean motion^2.

First order solution: nearly circular, almost coplanar orbits

Now assume P's orbit deviates only slightly from a circular orbit in the midplane:

$$r(t) = r_0 + r_1(t) \tag{3.85}$$
$$\theta(t) = \theta_0 + \Omega_0 t + \theta_1(t) \tag{3.86}$$
$$z(t) = z_1(t) \tag{3.87}$$

where the deviations are small: $|r_1| \ll r_0$, $|\theta_1| \ll 1$, and $|z_1| \ll r_0$.

The EOM are

$$\ddot{r}_1 - (r_0 + r_1)(\Omega_0 + \dot{\theta}_1)^2 = -\frac{\partial \Phi}{\partial r} \tag{3.88}$$
$$h_z = (r_0 + r_1)^2(\Omega_0 + \dot{\theta}_1) = \text{ integration constant} \tag{3.89}$$
$$\ddot{z}_1 = -\frac{\partial \Phi}{\partial z} \tag{3.90}$$

Next, *linearize* the EOM.

This means to Taylor expand about P's unperturbed orbit $\mathbf{r} = \mathbf{r_0}$ to first–order in the small quantities:

$$\frac{\partial \Phi}{\partial r} \simeq \left.\frac{\partial \Phi}{\partial r}\right|_{\mathbf{r_0}} + r_1 \left.\frac{\partial^2 \Phi}{\partial r^2}\right|_{\mathbf{r_0}} = r_0\Omega_0^2 + \left.\frac{\partial^2 \Phi}{\partial r^2}\right|_{\mathbf{r_0}} r_1 \tag{3.91}$$

$$\frac{\partial \Phi}{\partial z} \simeq \left.\frac{\partial \Phi}{\partial z}\right|_{\mathbf{r_0}} + z_1 \left.\frac{\partial^2 \Phi}{\partial z^2}\right|_{\mathbf{r_0}} \tag{3.92}$$

what is $\partial\Phi/\partial z|_{\mathbf{r_0}}$?

$$\text{thus} \quad \frac{\partial \Phi}{\partial z} \simeq \nu_0^2 z_1 \tag{3.93}$$

$$\text{where} \quad \nu_0^2 \equiv \left.\frac{\partial^2 \Phi}{\partial z^2}\right|_{\mathbf{r_0}} = \text{constant} \tag{3.94}$$

We linearize the EOM by dropping second–order small terms (r_1^2, $r_1\dot{\theta}_1$, etc):

$$\ddot{r}_1 - 2r_0\Omega_0\dot{\theta}_1 - \left(\Omega_0^2 - \left.\frac{\partial^2 \Phi}{\partial r^2}\right|_{\mathbf{r_0}}\right) r_1 \simeq 0 \tag{3.95}$$

$$\Omega_0 + \dot{\theta}_1 = \frac{h_z}{r_0^2}(1 + r_1/r_0)^{-2} \simeq \frac{h_z}{r_0^2} - \frac{2h_z}{r_0^3}r_1 \tag{3.96}$$

$$\text{so} \quad \dot{\theta}_1 = \frac{h_z}{r_0^2} - \Omega_0 - \frac{2h_z}{r_0^3}r_1 \tag{3.97}$$

$$\text{and} \quad \ddot{z}_1 + \nu_0^2 z_1 \simeq 0 \tag{3.98}$$

We still have some freedom in choosing the meaning of the constant h_z.

To simplify things, set $h_z = r_0^2\Omega_0$,
which is equivalent to setting the the radius of the star's *guiding center* r_0 such that that orbit has specific angular momentum h_z. Then

$$\dot{\theta}_1 = -\frac{2h_z}{r_0^3}r_1 \;=\; -2\Omega_0\frac{r_1}{r_0} \tag{3.99}$$

$$\text{so} \quad \ddot{r}_1 + \left(3\Omega_0^2 + \left.\frac{\partial^2\Phi}{\partial r^2}\right|_{\mathbf{r_0}}\right) r_1 \;\simeq\; 0 \tag{3.100}$$

If we set the constant

$$\kappa_0^2 \;\equiv\; 3\Omega_0^2 + \left.\frac{\partial^2\Phi}{\partial r^2}\right|_{\mathbf{r_0}} = 4\Omega_0^2 + r_0 \left.\frac{\partial\Omega^2}{\partial r}\right|_{r_0} \tag{3.101}$$

the EOM are simply

$$\ddot{r}_1 + \kappa_0^2 r_1 \;\simeq\; 0 \tag{3.102}$$

$$\dot{\theta}_1 \;\simeq\; -2\Omega_0\frac{r_1}{r_0} \tag{3.103}$$

$$\ddot{z}_1 + \nu_0^2 z_1 \;\simeq\; 0 \tag{3.104}$$

What is the solution to these EOM?

Evidently, P behaves as if it were a coupled 3D oscillator:

$$r_1(t) \;=\; -R\cos\kappa_0 t \tag{3.105}$$

$$\dot{\theta}_1(t) \;=\; \frac{2R}{r_0}\Omega_0\cos\kappa_0 t \tag{3.106}$$

$$\text{so} \quad \theta_1(t) \;=\; \frac{2R}{r_0}\frac{\Omega_0}{\kappa_0}\sin\kappa_0 t \tag{3.107}$$

$$z_1(t) \;=\; Z\sin(\nu_0 t + \phi_0) \tag{3.108}$$

Where constants R and Z are the *epicyclic amplitudes*.
Note that time $t = 0$ corresponds to periapse passage.

If we identify the star's epicyclic amplitudes $R = er_0$ and $Z = r_0 \sin i$ with the star's orbital eccentricity e, inclination i, and semimajor axis $a = r_0$, mean motion $n = \Omega_0$, and $\theta = f + \tilde{\omega}$ where $f =$ true anomaly, $\tilde{\omega} = \theta_0 =$ longitude of periapse, $\phi_0 = -\Omega =$ then we recover the epicyclic motion of the 2–body problem (see eqns 1.98, 1.99, & 3.41):

$$\begin{aligned} r(t) &= r_0 - er_0 \cos \kappa_0 t \\ \theta(t) &= \theta_0 + \Omega_0 t + 2e\frac{\Omega_0}{\kappa_0} \sin \kappa_0 t \\ z(t) &= r_0 \sin i \sin(\nu_0 t + \phi_0) \end{aligned} \tag{3.109}$$

This described the epicyclic motion of a particle that is orbiting in a non–keplerian potential $\Phi(r, z)$.

The frequency κ_0 is known as the *epicyclic* frequency, which is the frequency of P's *radial & transverse* oscillations about a guiding center of radius r_0,

while ν is P's *vertical* oscillation frequency.

Assignment #4
due Tuesday February 28
at the start of class

6. a.) If particle P were orbiting in a keplerian potential, $\Phi = -\mu/r$, then Eqn's (3.109) should be equivalent to Eqn's (1.98, 1.99, & 3.41), which in turn requires $\kappa_0 = \nu_0 = \Omega_0 = n = \sqrt{\mu/r_0^3}$. Show that this is indeed the case.

b.) Show that the angle ϕ_0 in Eqn (3.109) can also be identified with P's argument of perihelion, ω.

Additional problems pending...

orbital precession in a non–keplerian potential

The fact that $\Omega_0 = \kappa_0 = \nu_0$ for the kepler problem means that orbits are *closed*, ie, the P's motion repeats after each orbital period $T_{orb} = 2\pi/\Omega_0$.

However the typical galactic potential is not keplerian;
our Galaxy has $\nu_0 > \kappa_0 > \Omega_0$,
which also implies that stellar orbits are not closed, ie, they precess:

The figure shows the longitude of periapse $\tilde{\omega}$ advances $\Delta\tilde{\omega} = 2\pi\Omega_0/\kappa_0 - 2\pi$
after time $\Delta t = 2\pi/\kappa_0 =$ time between periapse passage,
so the periapse longitude *precesses* at the rate

$$\dot{\tilde{\omega}} = \frac{\Delta\tilde{\omega}}{\Delta t} = \Omega_0 - \kappa_0. \tag{3.110}$$

Consideration of P's vertical motions will show that its longitude of ascending node, Ω_{node}, also precesses at the rate

$$\dot{\Omega}_{node} = \Omega_0 - \nu_0 \tag{3.111}$$

Note that in most planetary environments, $\nu_0 > \Omega_0 > \kappa_0$,
so the longitude of periapse typically advances $\dot{\tilde{\omega}} > 0$,
while the longitude of ascending node $\dot{\Omega}_{node} < 0$ usually regresses.

Assignment #4
due Tuesday February 28
at the start of class

7.) a.) The centrifugal force on a rotating star or planet tends to counterbalances gravity slightly, making the body oblate, or slightly fatter at its equator. The gravitational potential of an oblate primary of mass M_p and mean radius R_p can be written as a sum over Legendre polynomials P_i:

$$\Phi(r) = \frac{-GM_p}{r}\left[1 - \sum_{n=2}^{\infty}\left(\frac{R_p}{r}\right)^n P_n(\sin\alpha)\right] \tag{3.112}$$

where r is the distance of particle P from the primary, α is P's angular distance above/below the primary's equatorial $z = 0$ plane, and the constant coefficients J_i are the primary's *zonal harmonics*. See page 138 of M&D for a table of the P_i. For a distant particle orbiting at $r \gg R_p$, we usually need only terms up to the $n = 2$ harmonic in the above potential. Use this approximation to calculate the frequencies κ_0 and ν_0, and then show that P's orbit precesses at the rates

$$\dot{\tilde{\omega}} \simeq \frac{3}{2}J_2\left(\frac{R_p}{r_0}\right)^2\Omega_0 \tag{3.113}$$

$$\dot{\Omega}_{node} \simeq -\frac{3}{2}J_2\left(\frac{R_p}{r_0}\right)^2\Omega_0 \tag{3.114}$$

b.) Use Gauss' eqn' (3.39) to confirm Eqn' (3.114).

c.) Punch some numbers. Saturn is the most oblate planet in the Solar System, having $J_2 = 0.0163$. What is the orbital period for a particle orbiting in the middle of the A ring? What is its precession period, $P_{\tilde{\omega}} = 2\pi/\dot{\tilde{\omega}}$ in units of orbit periods?

The Epicyclic frequency κ, the Oort Constants A & B, and the Galactic Potential

Recall the epicyclic frequency κ for a star orbiting in the Galaxy is

$$\kappa^2 \equiv 4\Omega^2 + r\frac{\partial \Omega^2}{\partial r} = 4\Omega^2 + 2r\Omega\frac{\partial \Omega}{\partial r} \quad (3.115)$$

Knowledge of κ is useful in Galactic studies, since it is sensitive to the gradients in the Galactic potential, and thus a Galaxy's mass distribution and its rotation curve $v_c = r\Omega$, which is the speed of a star in a circular orbit.

You can glean this information by measuring the so–called Oort constants $A(r)$ & $B(r)$ (which are actually functions of Galactic radius r):

Chapter 22 of the text by Carroll & Ostlie (C&O) shows that the A & B 'constants' are related to a star's radial and transverse velocities as measured by an observer at the Sun:

$$v_r = dA\sin(2\ell) \quad (3.116)$$

$$v_t = d[A\cos(2\ell) + B]. \quad (3.117)$$

If you know the star's distance d and its longitude ℓ from the Galactic Center, then you can infer local values of $A(r)$ & $B(r)$ from observations of many nearby stars.

C&O derive the relation between A & B and
a star's circular velocity $v_c = r\Omega$ about the Galactic center:

$$A(r) = \frac{1}{2}\left(\frac{v_c}{r} - \frac{dv_c}{dr}\right) = -\frac{r}{2}\frac{d\Omega}{dr} \tag{3.118}$$

$$\text{and} \quad B(r) = A - \Omega \tag{3.119}$$

Since

$$\frac{d\Omega}{dr} = -\frac{2A}{r}, \tag{3.120}$$

$$\kappa^2 = -4B\Omega_0 \tag{3.121}$$

Thus a determination of the Oort constant for nearby stars provides the Sun's orbital angular velocity Ω and epicyclic frequency κ.

Table 1–2 in Binney & Tremaine (B&T) quotes
$A \simeq 15$ km/sec/kpc and $B \simeq -12$ km/sec/kpc
for stars in the solar neighborhood, so the Sun's angular velocity about the Galaxy is $\Omega = A - B = 27$ km/sec/kpc $\simeq 3 \times 10^{-8}$ radians/year,
so the Sun's orbital period is $T_{orb} = 2\pi/\Omega \simeq 2 \times 10^8$ years.

The Sun's epicyclic frequency is $\kappa = \sqrt{-4B\Omega} = 36$ km/sec/kpc,
so its precession rate is $\dot{\tilde{\omega}} = \Omega - \kappa = -9$ km/sec/kpc $= -\frac{1}{3}\Omega$,
so the Sun's precession period is $T_{\tilde{\omega}} = 2\pi/|\dot{\tilde{\omega}}| = 3T_{orb}$.

This illustrates on of the main differences between orbits in a planetary system versus a galactic system—orbital precession in a galaxy occurs on a timescale comparable to the orbit period, while precession in planetary system occurs on much longer timescales.

Due to this rapid precession, the star trajectory eventually fills a torus about the Galactic center—see figures in Section 3.3 of B&T.

Perturbed epicyclic motion, and resonances

(From Section 3.3 of B&T.) Consider particle P in a nearly circular orbit in an axially symmetric potential $\Phi = \Phi_0(r, z)$ that in general in non–keplerian.

If P is unperturbed, its motion is epicyclic, ala Eqn's (3.105).

Now add an additional disturbance to this system: $\Phi \rightarrow \Phi_0 + \Phi_1$
where $\Phi_0(r, z)$ = axially symmetric potential, as before
and $\Phi_1(r, \theta, z, t)$ = time–dependent, non–axially symm' perturbing potential.

Our results will be quite general, since particle P could be

- an asteroid that is perturbed by a planet having a potential Φ_1,
- a star orbiting in a galaxy that also has a rotating central bar.
- Φ_1 could also represent the disturbance due to a spiral wave that is propagating in a galaxy, a circumstellar gas disk, or a planetary ring.

These results have many applications in the dynamics of planetary systems & galaxies, and will lead us to the concept of *orbital resonances*.

To keep this discussion simple, we will assume that P is confined to the system's midplane, and that there are no vertical perturbations,
so $\Phi_1 = \Phi_1(r, \theta, t)$.

However our results are easily generalized to handle vertical perturbations (like, say, a spiral *bending* wave).

It will also be convenient to Fourier expand Φ_1 as in the time–series

$$\Phi_1(r, \theta, t) = \sum_{m=0}^{\infty} \phi_m(r) \cos[m(\theta - \Omega_{ps} t)] \qquad (3.122)$$

where $m =$ the *azimuthal wavenumber*,
$\phi_m(r)$ is the amplitude of the m^{th} perturbation,
and Ω_{ps} = the *pattern speed*, which is the angular velocity at which our disturbance rotates (like, say, a galactic bar, or an orbiting planet).

If the perturber is a spiral density wave that rotates with a constant angular velocity Ω_{ps}, then its potential Φ_1 would be represented as a single m^{th} term in the above sum, where $m =$ number of spiral arms in the wave.

If the perturber is a galactic bar, then it too would be represented via a single m^{th} term in Φ...but which one—what is m for a rotating bar?
(*Hint: consider the bar's density-variations as you go* 360° *in azimuth.*).

If the perturber is an orbiting planet,
then *all* the Fourier terms in Φ_1 are present.

However each term usually responsible for exciting a large response by particle P at discrete, narrow sites in your system—at *resonances*.

Figure 3.7: Whirlpool Galaxy, imaged by HST

Figure 3.8: Barred galaxy NGC 1300

These resonances are often (but not always) spatially segregated;
in that case we need only consider P's response
to a single m^{th} term in the Fourier expansion of Φ_1:

$$\Phi_1(r, \theta, t) \simeq \phi_m(r) \cos[m(\theta - \Omega_{ps} t)] \tag{3.123}$$

We justify the above approach more formally by noting that Φ_1 is expanded in terms of orthogonal functions. Consequently, our EOM must be satisfied *individually* for each m^{th} term in Φ_1. If that solution is R_m, then the general solution for P's motion is thus the summed response $\sum R_m$.

The EOM for particle P is now $\ddot{\mathbf{r}} = -\nabla\Phi$, so

$$\hat{\mathbf{r}}: \qquad \ddot{r} - r\dot{\theta}^2 = -\frac{\partial \Phi}{\partial r} \tag{3.124}$$

$$\hat{\theta}: \qquad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} \tag{3.125}$$

Again, assume that P's motion deviates only slightly from a
guiding center $\mathbf{r_0} = (r_0, \theta_0 + \Omega_0 t, 0)$ that travels in a circular orbit:

$$r(t) = r_0 + r_1(t) \tag{3.126}$$

$$\theta(t) = \theta_0 + \Omega_0 t + \theta_1(t) \tag{3.127}$$

$$\text{where} \quad \Omega_0^2 = \frac{1}{r}\left.\frac{\partial \Phi_0}{\partial r}\right|_{\mathbf{r_0}} \tag{3.128}$$

$$\text{and} \quad \Phi(r, \theta, z, t) = \Phi_0(r, z) + \phi_m(r)\cos[m(\theta - \Omega_{ps} t)] \tag{3.129}$$

Next, linearize the EOM, which means keeping terms to an accuracy that is first–order in the small quantities $r_1, \dot{\theta}_1, \nabla\Phi_1$, etc

Also, evaluate the perturbing accelerations $\nabla\Phi_1$ assuming
P's motion is *undisturbed*, ie, that it lies at the guiding center $\mathbf{r_0}$.

Then the $\hat{\theta}$ Eqn' becomes

$$
\begin{aligned}
\frac{d}{dt}(r^2\dot{\theta}) &\simeq m\phi_m(r)\sin m(\theta - \Omega_{ps}t)|_{\mathbf{r_0}} && (3.130)\\
&= m\phi_m(r_0)\sin(m\theta_0 + \omega_m t) && (3.131)\\
\text{where} \quad \omega_m &\equiv m(\Omega_0 - \Omega_{ps}) = \text{ Doppler–shifted forcing freq'} && (3.132)
\end{aligned}
$$

Integrate the $\hat{\theta}$ EOM:

$$
r^2\dot{\theta} \simeq (r_0^2 + 2r_0r_1)(\Omega_0 + \dot{\theta}_1) = -\frac{m\phi_m}{\omega_m}\cos(m\theta_0 + \omega_m t) + h_z \quad (3.133)
$$

and again set the integration constant $h_z = r_0^2\Omega_0$, so

$$
\dot{\theta}_1 \simeq -\frac{m\phi_m}{\omega_m r_0^2}\cos(m\theta_0 + \omega_m t) - \frac{2\Omega_0}{r_0}r_1 \quad (3.134)
$$

The $\hat{\mathbf{r}}$ EOM is

$$
\ddot{r}_1 - (r_0 + r_1)(\Omega_0 + \dot{\theta}_1)^2 \simeq \ddot{r}_1 - r_0\Omega_0^2 - 2r_0\Omega_0\dot{\theta}_1 - \Omega_0^2 r_1 \quad (3.135)
$$

$$
\begin{aligned}
&\simeq -\left.\frac{\partial\Phi_0}{\partial r}\right|_{r_0} - r_1\left.\frac{\partial^2\Phi_0}{\partial r^2}\right|_{r_0} - \left.\frac{\partial\phi_m}{\partial r}\right|_{r_0}\cos(m\theta_0 + \omega_m t)\\
\text{and note} \quad \partial\Phi_0/\partial r|_{r_0} &= r_0\Omega_0^2\\
\text{so} \quad \ddot{r}_1 - 2r_0\Omega_0\dot{\theta}_1 - \Omega_0^2 r_1 &= -\frac{\partial}{\partial r}(r\Omega^2)_{r_0}r_1 - \left.\frac{\partial\phi_m}{\partial r}\right|_{r_0}\cos(m\theta_0 + \omega_m t)
\end{aligned}
$$

Now insert $\dot{\theta}_1$ and collect terms:

$$
\begin{aligned}
\ddot{r}_1 + \kappa_0^2 r_1 &= -\psi_m(r_0)\cos(m\theta_0 + \omega_m t) && (3.140)\\
\text{where again} \quad \kappa_0^2 &= 4\Omega_0^2 + r\frac{\partial\Omega^2}{\partial r} && (3.141)\\
\text{and} \quad \psi_m(r) &\equiv \frac{\partial\phi_m}{\partial r} + \frac{2m\Omega}{\omega_m}\frac{\phi_m}{r} && (3.142)
\end{aligned}
$$

where ψ_m is also known as the *forcing function*.

Now we have an EOM for a simple harmonic oscillator that is *driven* by the ψ_m term, which is the amplitude of the driving acceleration,
with a driving frequency is $\omega_m(r_0) = m(\Omega_0 - \Omega_{ps})$,
also known as the 'Doppler–shifted' forcing frequency,
since it is the frequency of perturbations seen in the reference frame that corotates P's angular velocity Ω_0.

Note that P's natural frequency for radial oscillations is κ_0.

What happens when the driving frequency $|\omega_m|$ matches
P's natural oscillation frequency?

The solution to this inhomogeneous differential eqn' is $r_1(t) = r_e(t) + r_f(t)$,
where r_e satisfies the homogeneous EOM (no driving forces, ie $\psi_m = 0$),
and r_f is P's *forced response* due to ψ_m.

The homogeneous part of the solution is of course is familiar epicyclic motion, $r_e = -R\cos(\kappa_0 t) = -e_e r_0 \cos(\kappa_0 t)$;
this part of P's motion is sometimes called its *free* motion,
with $e_e =$ its *free* eccentricity, determined by P's initial conditions.

The solution for P's forced motion is

$$r_f(t) \;=\; -\frac{\psi_m(r_0)}{D(r_0)}\cos(m\theta_0 + \omega_m t) \tag{3.143}$$

$$\text{where} \quad D(r) \;\equiv\; \kappa^2 - \omega_m^2 \tag{3.144}$$

Plug r_1 into eqn' (3.140) and confirm that it is indeed a solution.

Assignment #5
due ?
at the start of class

1. Solve for P's azimuthal motion $\theta_1(t)$.
Insert $r_1 = r_e + r_f$ into eqn (3.134) and integrate to obtain

$$\begin{aligned} \theta_1(t) &= \theta_e(t) + \theta_f(t) && (3.145) \\ \text{where} \quad \theta_e(t) &= \frac{2R\Omega_0}{r_0\kappa_0}\sin(\kappa_0 t) && (3.146) \\ \text{and} \quad \theta_f(t) &= -\left(\frac{m\phi_m}{\omega_m^2 r_0^2} + \frac{2\Omega_0\Phi_m}{r\omega_m D}\right)\sin(m\theta_0 + \omega_m t), && (3.147) \end{aligned}$$

which are P's free epicyclic motion θ_e, and its forced motion θ_f.

2. Treat the Galaxy as a slab of matter of density $\rho(r)$, where r is the distance from the galactic core. The Sun is in a nearly circular orbit of radius r_0 about this core. Show that its vertical oscillation frequency is $\nu_0 \simeq \sqrt{4\pi G\rho_0}$ where $\rho_0 = \rho(r_0) \simeq 0.18\ \mathrm{M}_\odot/\mathrm{pc}^3$ (see Table 1-1 of B&T). What is the Sun's nodal precession period, $T_\Omega = 2\pi/|\dot{\Omega}_{node}|$, in units of its orbital period T_{orb}?

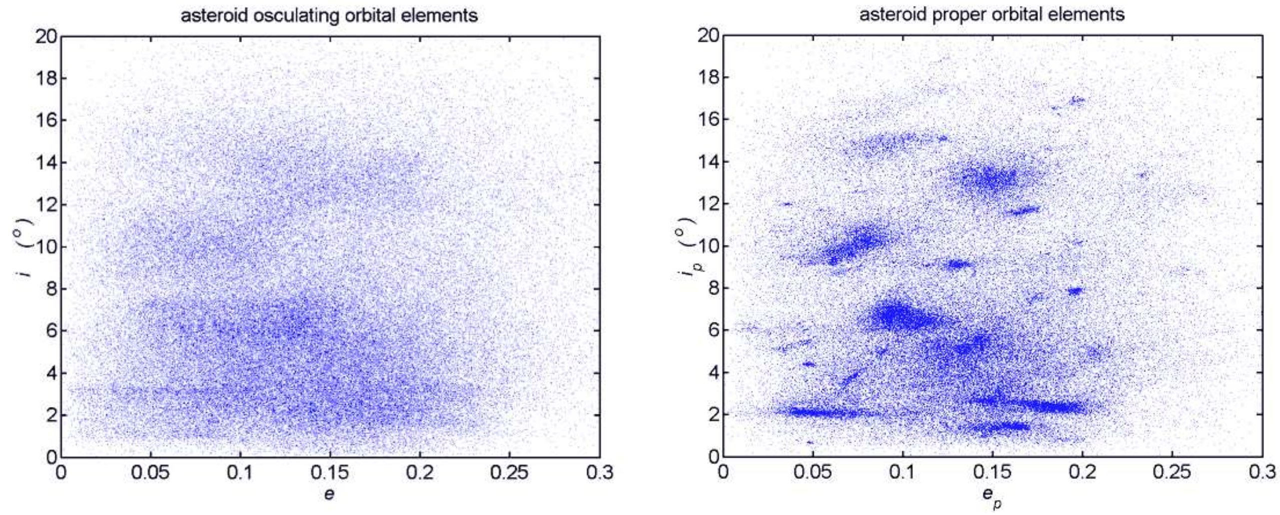


Figure 3.9: from Wikipedia

Free, forced, and proper orbit elements

In studies of asteroid orbits, the free eccentricity e_e is also called an asteroid's *proper* eccentricity (e_p in the above figure), which is the amplitude of the asteroid's motion that is *not* due to Jupiter's perturbations.

Similarly, the *proper inclination* $i_e = i_p$
describes an asteroid's 'free' vertical motions.

What are the clusters seen among the asteroids' proper elements?
Why don't you see clusters in osculating orbit elements?

Proper elements are of interest since they are a consequence of all the other (non–Jovian) forces that have since perturbed asteroids over their 4.5Gyr history: collisions with other asteroids, stirring by long-gone protoplanets that may have roamed the early Belt, sweeping secular resonances due to dispersal of solar nebula, and radial drift due to the YE.

One way to test of models of terrestrial planet formation is to see if they produce as asteroid belt of debris that
(i.) appears sufficiently depleted (by a factor of ~ 1000),
and (ii.) has the right distribution of proper orbit elements.

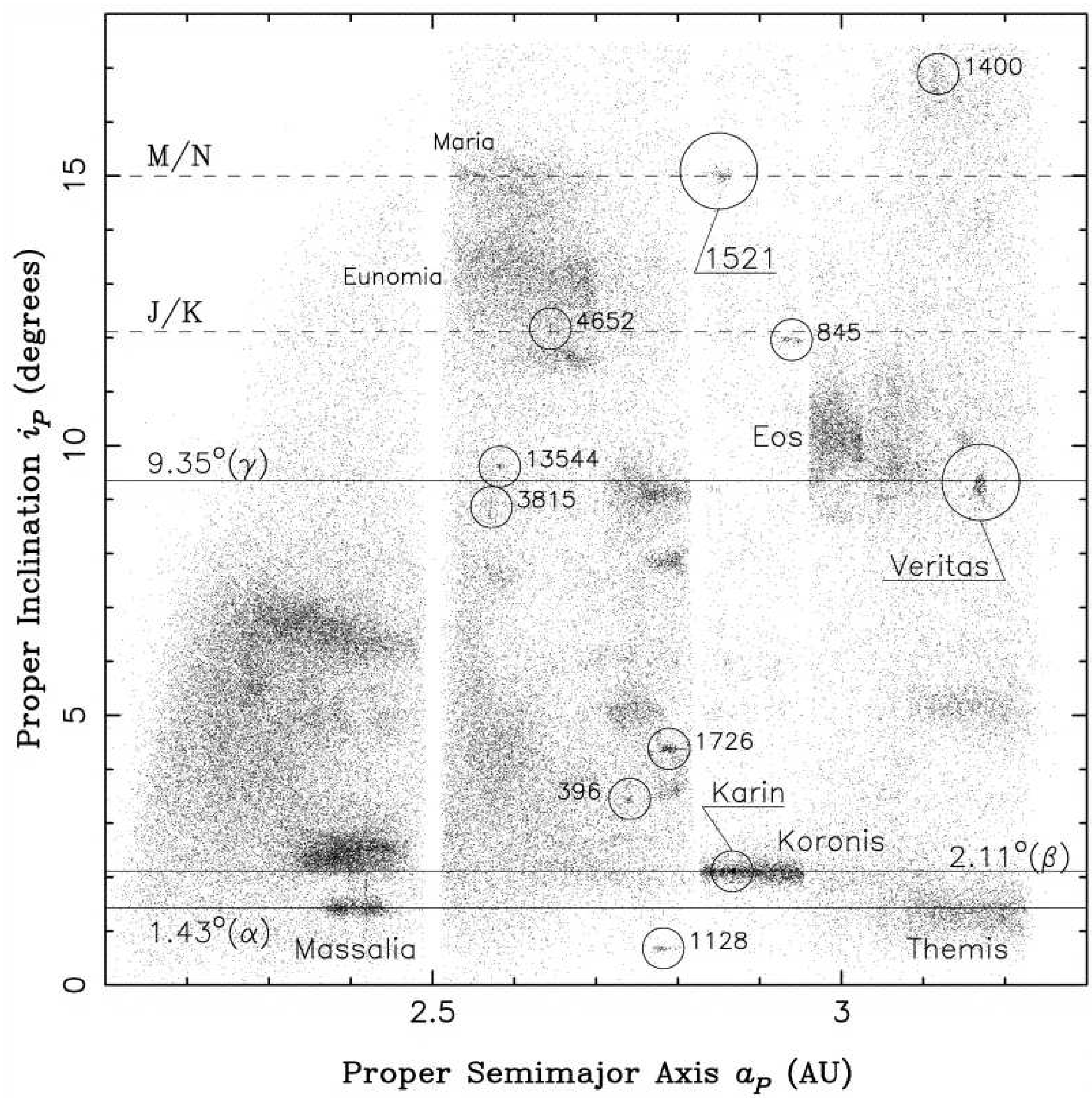


Figure 3.10: from Nesvory et al 2003.

An asteroid *family* results when asteroids collide and generate fragments.
The fragment's free/proper orbit elements are initially similar to the parent,
but subsequent collisions & YE causes those orbits to drift,
smearing the family out in orbit element space.

Which asteroid families are younger? which are older?
Why are there gaps in this figure?

Eos, Themis, and Koronis families formed ~ 1 Gyr ago,
while Karis *cluster* formed w/in Koronis family $\sim 10^7$ yrs ago.

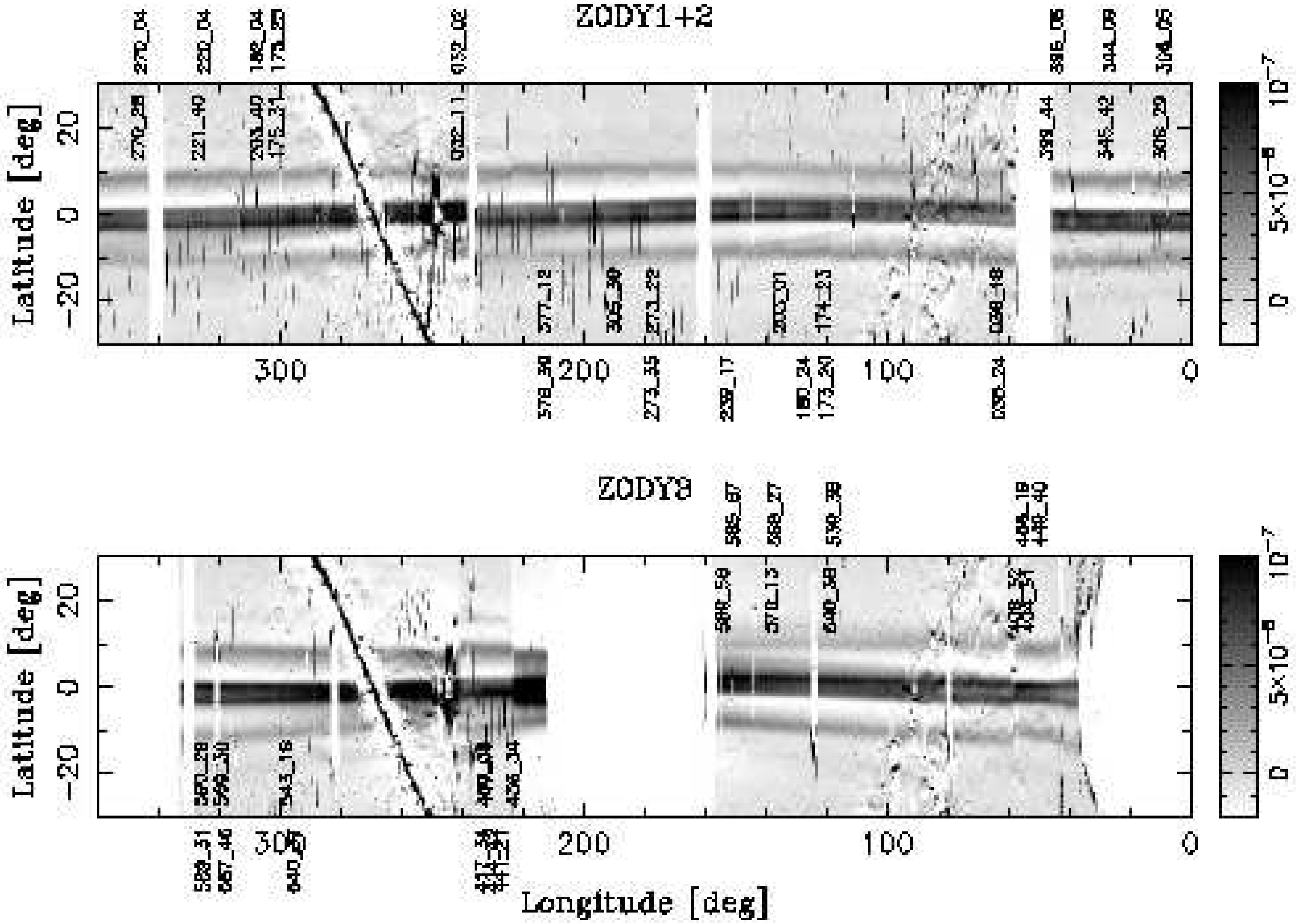


Figure 3.11: IRAS IR map, from Nesvory et al 2006.

The figure shows an IR map of the sky as seen by IRAS. This map has been filtered to remove the broad & featureless IR emission from interplanetary dust that extends $\sim \pm 30°$ above/below the ecliptic;

What are the tilted slashes in this figure?

This filtering enhances the weak zodiacal dust bands that circle the entire sky; Note that these bands have latitudes $\pm i_\beta = \pm 2.1°$ and $\pm i_\gamma = \pm 9.4°$ which happen to be the inclinations of the Karin cluster which lives inside the Koronis family) and the Veritas family. Why?

Why does each cluster produce a *pair* of bands that peak at latitudes = ± inclination of the source?

Hint: recall Eqn. (3.41): $z(\theta) = a \sin i \sin(\theta - \Omega_{node})$ where θ = longitude.

Why are these dust bands *bent*?

Lindblad and corotation resonances

Recall that particle P's forced motion is

$$r_f(t) = -\frac{\psi_m(r_0)}{D(r_0)} \cos(m\theta_0 + \omega_m t) \tag{3.148}$$

$$\text{where} \quad \psi_m(r) \equiv \frac{\partial \phi_m}{\partial r} + \frac{2m\Omega}{\omega_m}\frac{\phi_m}{r} \tag{3.149}$$

and $D(r) = \kappa^2 - \omega_m^2$ is P's 'distance' from resonance (in frequency^2 units),
ψ_m = forcing function,
$\phi_m = m^{th}$ Fourier component of disturbing potential Φ_1,
and $\omega_m = m(\Omega - \Omega_{ps})$ = forcing frequency of the m^{th} disturbance.

A resonance = a site $r = r_0$ where a star's forced response r_f or θ_f is large.

A *Lindblad* resonance is the radius $r = r_L$ where $D(r_L) = 0$:

$$\omega_m = m(\Omega - \Omega_{ps}) = \varepsilon\kappa(r_{LR}) \quad \text{where } \varepsilon = \pm 1 \tag{3.150}$$

$$\text{so} \quad \Omega(r_{LR}) - \Omega_{ps} = \varepsilon\kappa(r_{LR})/m \tag{3.151}$$

is the condition for a Lindblad resonance (LR).

In a planetary environment, we usually call this a *mean motion* or a *secular* resonance; I'll distinguish the two later.

First, define the corotation radius $r_0 = r_{CR}$ as the site where the angular velocity of P's guiding center, $\Omega(r_{CR})$, matches the pattern speed Ω_{ps}; P would appear to corotate with the disturbance.

Note that most systems (planetary & galactic) have $\kappa(r) > 0$ and an $\Omega(r)$ that usually decreases with distance r.

Thus a $\varepsilon = -1$ LR is located where $\Omega(r_L) < \Omega_{ps}$.
The resonance condition, Eqn' (3.151), would thus be satisfied at some site $r_L > r_{CR}$, also known as the m^{th} 'outer' LR or OLR.

At an OLR, the Doppler–shifted forcing freq' $\omega_m = m(\Omega - \Omega_{ps}) < 0$, so the crests of the perturbing potential appear to overtake P.

Likewise, the $\varepsilon = +1$ LR requires $\Omega(r_L) > \Omega_{ps}$;
this is the m^{th} inner LR (or ILR) since $r_{ILR} < r_{CR}$;
At an ILR, $\omega_m > 0$ and P orbits faster than the disturbing potential.

Note also that P's forced response $r_f \propto \psi_m$ has a term $\propto 1/\omega_m$, so the site where $\omega_m = 0$ is known as a *corotation* resonance, since that is where $r_0 = r_{CR}$ and $\Omega(r_{CR}) = \Omega_{ps}$, P corotates the disturbance.

So for each m there are a pair of LRs that straddle the CR:

simple example:

LRs in a barred galaxy having a flat rotation curve

Suppose a barred disk galaxy has a flat rotation curve and a central bar that rotates with angular velocity Ω_{ps}. Find this system's LRs in terms of the corotation radius r_{CR}, which is the radius where a star would corotates with the bar.

Since the circular speed $v_c = r\Omega =$ constant, a star's angular velocity $\Omega \propto 1/r$ can be written as $\Omega(r) = (r_{CR}/r)\Omega_{ps}$. Then

$$\kappa^2 = 4\Omega^2 + r\frac{\partial\Omega^2}{\partial r} = 2\Omega_{ps}^2(r_{CR}/r)^2 \tag{3.152}$$

$$\text{so} \quad \kappa = \sqrt{2}\Omega \tag{3.153}$$

The LRs are located where $\Omega - \Omega_{ps} = \epsilon\kappa/m$, so

$$r_{LR} = \left(1 - \frac{\sqrt{2}\varepsilon}{m}\right) r_{CR} \tag{3.154}$$

so the m^{th} OLR is at $r_{OLR} = (1 + \sqrt{2}/m)r_{CR}$,
and the ILR is at $r_{ILR} = (1 - \sqrt{2}/m)r_{CR}$.

Obviously, the CR lies at $r = r_{CR}$.

Where is the $m = 1$ ILR?

The central potential for a planetary system is nearly keplerian, so $\kappa \simeq \Omega$, and the Lindblad resonance condition $\Omega - \Omega_{ps} = \epsilon\kappa/m$ can be written as

$$\frac{\Omega_{ps}}{\Omega} \simeq \frac{T_{orb}}{T_{CR}} = \frac{m - \varepsilon}{m} \tag{3.155}$$

where T_{orb} is the particle's orbital period,
and $T_{CR} = 2\pi/\Omega_{ps}$ is the orbital period at corotation.

The next section will show that if the perturber is an orbiting secondary (planet, satellite, etc) of semimajor axis a_s,
then the corotation radius is $r_{CR} = a_s$,
and the pattern speed is $\Omega_{ps} = \Omega_s =$ the secondary's angular velocity.

Such resonances are called *mean motion* (MMR) or *commensurability* resonances, since the mean motion are ratios of similar whole numbers.

Since $\Omega(r) = n(a) \propto a^{-3/2}$, MMRs are located at

$$a_{MMR} \simeq \left(\frac{m - \varepsilon}{m}\right)^{2/3} a_s \tag{3.156}$$

where $\varepsilon = \pm 1$.

In homework, you will show that if the secondary has an eccentricity $e_s > 0$, then its radial motion gives rise to additional MMRs having
$\varepsilon \rightarrow \varepsilon\ell$ where $\varepsilon = \pm 1$ and $\ell = 1, 2, 3...$

For instance, the $m = 3, \varepsilon = +1, \ell = +2$ resonance with Jupiter is called the 1:3 resonance, since T_{orb}:T_s =1:3.

Since $a_s = 5.2$ AU, this resonance lies at $a_{1:3} = (1/3)^{2/3} a_s = 2.50$ AU, and is responsible for clearing a prominent gap in the asteroid belt (see Fig. 3.10); this is one of the famous *Kirkwood gaps*.

The 2:5 resonance has $m = 5$, $\varepsilon = +1$, $\ell = +3$,
and is at $a_{2:5} = (2/5)^{2/3} a_s = 2.82$ AU,
which is the site of another Kirkwood gap in Fig. 3.10.

Evidently the Solar System is *dense* with an infinite sea of MMRs,
but we shall see that those higher–order resonances
tend to be weaker by higher powers of e_s.

Note also that the region nearest a planet gets very dense
with high–m MMRs, since a_{MMR} accumulates at a_s as $m \rightarrow \infty$.

Assignment #5
due ?
at the start of class

3. Eqn' (3.156) gives an unreliable result for the $m = 1$ ILR. Show that this resonance at $r = r_{ILR}$ is actually located where $\dot{\tilde{\omega}}(r_{ILR}) = \Omega_{ps}$.

4. A secondary orbits a star with a semimajor axis a_s. Show that high–m LRs having $m \gg 1$ lie $\Delta a_m \simeq -2\varepsilon a_s/3m$ away from a secondary's orbit, and that the distance between adjacent resonances is $\simeq 2a_s/3m^2$.

Problem 3 locates a *secular* resonance:

If the secondary is precessing at the rate $\dot{\tilde{\omega}}_s = \Omega_{ps}$ (perhaps due to perturbations from other planets, or planetary oblateness, etc), then this secular resonance is the site where a particle's longitude of periapse precesses at the same rate as the secondary's, ie $\dot{\tilde{\omega}} = \dot{\tilde{\omega}}_s$, which can pump of the particle's eccentricity.

Lastly, I note that if the perturber's orbit is inclined wrt' the system midplane, then its vertical forcing can excite inclinations at vertical resonances analogous to the MMRs.

Similarly, a vertical secular resonance exists at the site where there is a match in the precession rates for the ascending nodes: $\dot{\Omega}_{node} = \dot{\Omega}_s$.

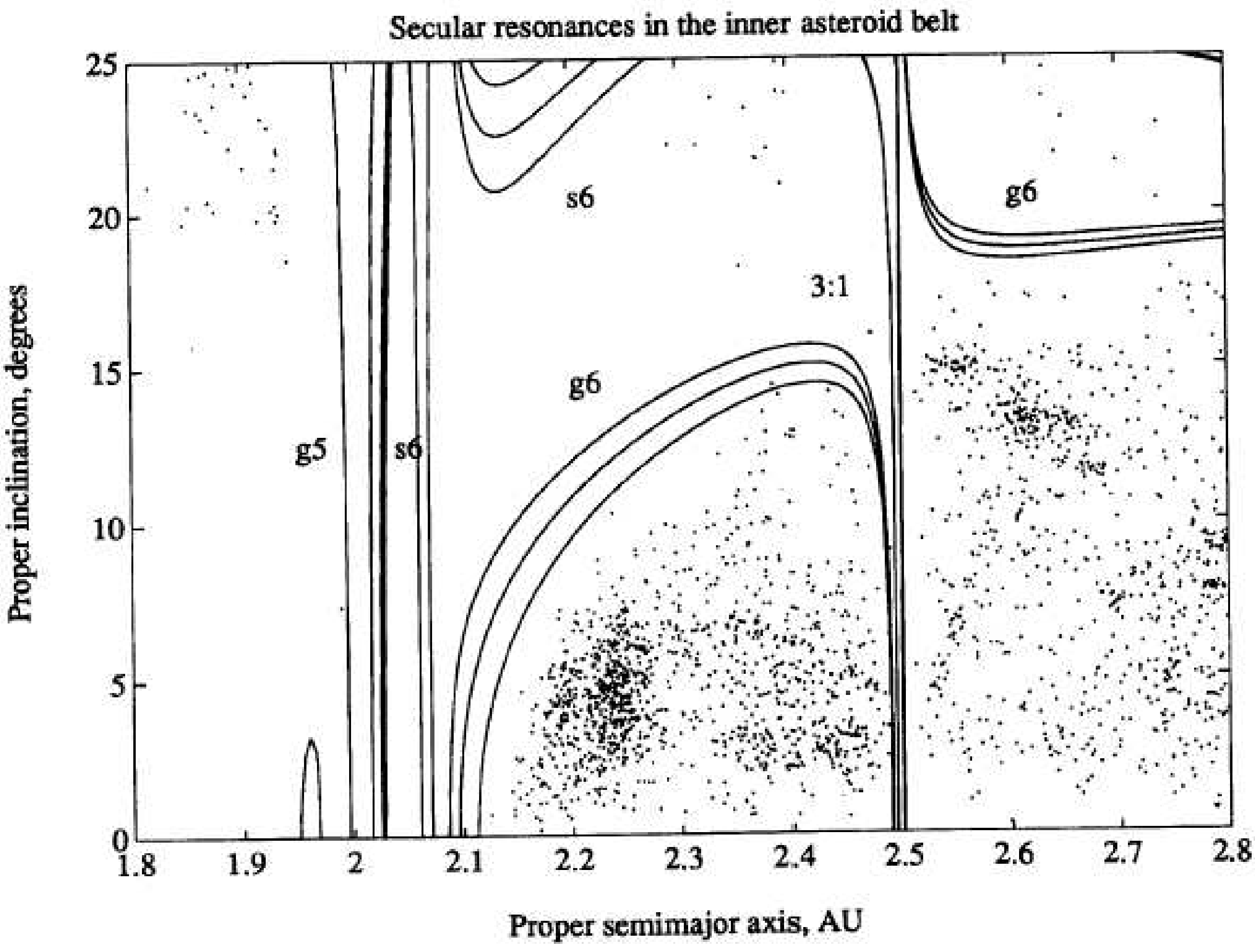


Figure 3.12: from Knezevic et al 1991.

Forcing function ψ_m for a point–mass perturber

These results apply to any orbiting perturber:
a planet, satellite, or companion star.

Recall that particle P's forced motion is

$$r_f(t) = -\frac{\psi_m(r_0)}{D(r_0)}\cos(m\theta_0 + \omega_m t) \tag{3.157}$$

$$\text{where} \quad \psi_m(r) \equiv \frac{\partial \phi_m}{\partial r} + \frac{2m\Omega}{\omega_m}\frac{\phi_m}{r} \tag{3.158}$$

A quantitative understanding of P's motion will require knowing ϕ_m and the disturbing frequency ω_m, which we get from a Fourier expansion of the disturbing potential Φ_1:

$$\Phi_1(r, \theta, t) = \frac{1}{2}\phi_0(r) + \sum_{m=1}^{\infty}\phi_m(r)\cos[m(\theta - \Omega_{ps}t)] \tag{3.159}$$

Calculate ϕ_m for a system having a primary of mass m_p orbited by secondary m_s having a perturbing potential $\Phi_1 = -Gm_s/\Delta$.

Assume the system is coplanar (for now), so
$\Delta^2 = r^2 + r_s^2 - 2rr_s\cos(\theta - \theta_s)$ = distance^2 between m_s and particle P.

Assume the secondary is on a circular orbit: $r_s = a_s$ and $\theta_s = \Omega_s t$, so μ_s is crossing the x–axis at time $t = 0$.

Also recall that we are to evaluate the perturbations at P's *guiding center* at $r(t) = a$ and $\theta(t) = \theta_0 + \Omega_0 t$.

To calculate ϕ_m, multiply Φ_1 by $\cos(m'\theta)$, where m' = arbitrary integer, and integrate over all θ:

$$-\int_{-\pi}^{\pi} \frac{Gm_s}{\Delta}\cos(m'\theta)d\theta = \sum_m \phi_m(r)\int_{-\pi}^{\pi}\cos(m\theta - m\Omega_{ps}t)\cos(m'\theta)d\theta \quad (3.1$$

$$= \pi\sum_m \phi_m \cos(m\Omega_{ps}t)\delta_{mm'} = \pi\phi_{m'}\cos(m'\Omega_{ps}t) \quad (3.1$$

so

$$\phi_m = -\frac{Gm_s}{\pi a_s\cos(m\Omega_{ps}t)}\int_{-\pi}^{\pi}\frac{\cos(m\theta)d\theta}{[1+\beta^2-2\beta\cos(\theta-\theta_s)]^{1/2}} \quad \text{where } \beta \equiv \frac{r}{a_s} \quad (3.162)$$

$$= -\frac{2Gm_s}{a_s}\frac{\cos(m\Omega_s t)}{\cos(m\Omega_{ps}t)}\int_0^{\pi}\frac{\cos(m\phi)d\phi}{(1+\beta^2-2\beta\cos\phi)^{1/2}} \quad \text{where } \phi = \theta - \theta_s \quad (3.163)$$

The time–dependence must disappear, so identify the pattern speed with the secondary's mean motion: $\Omega_{ps} = \Omega_s$.

Next, define the *Laplace coefficient* as

$$b_s^{(m)}(\beta) \equiv \frac{2}{\pi}\int_0^{\pi}\frac{\cos(m\phi)d\phi}{(1+\beta^2-2\beta\cos\phi)^s} \quad (3.164)$$

so

$$\phi_m(r) = -\frac{Gm_s}{a_s}b_{1/2}^{(m)}(\beta) \quad (3.165)$$

where $\beta = r/a_s$. Actually, this result is correct only for $m \geq 2$ terms.

The $m = 0$ term needs to be multiplied by 1/2—see Eqn' (3.159).

The $m = 1$ term also needs to be corrected for the fact that our origin is attached to the primary, which itself is accelerated by the secondary.

Let $\mathbf{r}_{COM}$ be P's position vector relative to the system's COM.

NII says $\ddot{\mathbf{r}}_{COM} = -\nabla\Phi$ where $\Phi = \Phi_0 + \Phi_1$ is the system's potential.

Since $\mathbf{r} = \mathbf{r}_{COM} - \mathbf{r}_p$, our EOM is $\ddot{\mathbf{r}} = -\nabla(\Phi_0 + \Phi_1) - \ddot{\mathbf{r}}_p$, also written as

$$\ddot{\mathbf{r}} = -\nabla(\Phi_0 + \Phi_1 + \phi_{id}) \tag{3.166}$$

$$\text{where} \quad \nabla\phi_{id} \equiv \ddot{\mathbf{r}}_p = \frac{Gm_s}{r_s^2}\hat{\mathbf{r}}_s \tag{3.167}$$

is the called the *indirect potential* ϕ_{id}; this additional perturbation accounts for the fact that origin is co–moving with the primary.

The indirect potential only appears in systems where a moving perturber is offset from the origin, as in a planetary system or a binary star system.

If the perturber is coincident with the origin (like a galactic bar), then $\phi_{id} = 0$.

Assignment #5
due ?
at the start of class

5. The indirect potential can be written

$$\phi_{id} = \frac{Gm_s}{r_s^3}\mathbf{r}\cdot\mathbf{r}_s. \tag{3.168}$$

Confirm this by showing that this ϕ_{id} does indeed satisfy Eqn' (3.167).

6. Show that

$$\frac{db_s^{(m)}}{d\beta} = s\left[b_{s+1}^{(m-1)}(\beta) - 2\beta b_{s+1}^{(m)}(\beta) + b_{s+1}^{(m+1)}(\beta)\right] \tag{3.169}$$

7. Set $\beta = 1 + x$, and show that when $|x| \ll 1$,

$$b_{1/2}^{(m)}(\beta) \simeq \frac{2}{\pi}K_0(m|x|) \tag{3.170}$$

$$\frac{db_s^{(m)}}{d\beta} \simeq -\frac{2m}{\pi}\mathrm{sgn}(x)K_1(m|x|), \tag{3.171}$$

where the K_ν are modified Bessel functions of order ν.
See Goldreich & Tremaine (1980), *ApJ*, **241**, p. 428 for a helpful hint.

Thus

$$\phi_{id} = \frac{Gm_s}{a_s}\beta\cos(\theta-\theta_s) \tag{3.172}$$

$$\text{and} \quad \phi_1 \rightarrow -\frac{Gm_s}{a_s}[b_{1/2}^{(m)}(\beta)-\beta] \tag{3.173}$$

So

$$\phi_m(r) = -\frac{Gm_s}{(1+\delta_{m0})a_s}[b_{1/2}^{(m)}(\beta)-\delta_{m1}\beta] \tag{3.174}$$

accounts for the corrections to the $m=0$ & $m=1$ terms.

The forcing function for an $m \geq 2$ disturbance,
evaluated at resonance ($\omega_m = \varepsilon\kappa$) is

$$\psi_{m\geq 2}(r) \equiv \frac{\partial\phi_m}{\partial r} + \frac{2m\Omega}{\omega_m}\frac{\phi_m}{r} \tag{3.175}$$

$$= -\frac{Gm_s}{a_s^2}\left[\frac{db_{1/2}^{(m)}}{d\beta} + 2m\varepsilon\frac{\Omega}{\kappa}\frac{b_{1/2}^{(m)}}{\beta}\right] \tag{3.176}$$

For a particle orbiting at high–m resonance that lies near the secondary,
we can use the results of problem 4
(which says the resonance is $|x| = |\Delta a_m/a_s| \simeq 2/3m$ away from a_s),
and problem 7:

$$\psi_{m\gg 1}(r) \simeq -\frac{2\varepsilon m}{\pi}\frac{Gm_s}{a_s^2}\left[2K_0(2/3)+K_1(2/3)\right] \tag{3.177}$$

$$= -\frac{2\varepsilon fm}{\pi}\frac{Gm_s}{a_s^2} \tag{3.178}$$

where $f \simeq 2.52$ is the coefficient introduced in Eqn. (2.123).

Although the above result is formally valid only for $m \gg 1$ resonances,
comparison with an exact calculation of ψ_m shows that the above provides
a reliable estimate of ψ_m down to $m=2$, with errors$\lesssim$ 25%

Now we can calculate the particle's forced motion. Recall Eqn' (3.143):

$$r_f(t) \;=\; -\frac{\psi_m(r_0)}{D(r_0)}\cos(m\theta_0+\omega_m t) \tag{3.179}$$

$$\text{where} \quad D(r) \;\equiv\; \kappa^2-\omega_m^2 \quad \text{and } \omega_m=\varepsilon\kappa \tag{3.180}$$

At a LR, $D=0$, and linearized theory breaks down since $r_f\to\infty$. In reality, a more sophisticated nonlinear theory will show that r_f stays finite at exact resonance.

However, we can calculate r_f for particle P orbiting just off exact resonance; do this by Taylor expanding $D(r)$ about resonance: $r=r_L+\Delta r$:

$$D(r) \;\simeq\; D(r_L)+\Delta r\frac{dD}{dr}=3(\varepsilon m-1)\Omega^2\frac{\Delta r}{r} \tag{3.181}$$

Plugging this into r_f in the $m\gg 1$ approximation yields

$$r_f(t) \;\simeq\; -\frac{2f\mu_s}{3\pi x}a_s\cos(m\theta_0+\varepsilon\kappa t) \tag{3.182}$$

where $\mu_s\equiv m_s/m_p$ is the perturber's mass in units of the primary's, and $x=\Delta r/a_s=$ P's fractional distance from the m^{th} LR.

The complete solution for the particle's motion radial motion is

$$r_f(t) \;=\; r_0+r_e(t)+r_f(t) \tag{3.183}$$

$$\;=\; r_0[1-e_e\cos\kappa t-e_f\cos(m\theta_0+\varepsilon\kappa t)] \tag{3.184}$$

where e_e is the particle's free (or epicyclic) eccentricity, and

$$e_f(x) \;=\; \frac{r_f}{r_0}\simeq\frac{2f\mu_s}{3\pi|x|} \tag{3.185}$$

is P's forced e that is excited by the secondary.

First, note that e_f is singular at resonance;
this is an unphysical consequence of using linearized EOM.

Had we preserved small terms to second–order in the (more complicated) EOM, we would find that e_f remains finite as $x \rightarrow 0$.

I suspect that our linearized sol'n, Eqn' (3.185), is reliable where $e_f \lesssim 0.2$, ie, at distances $|x| \gtrsim 3\mu_s$ beyond exact resonance.

It is straightforward to derive similar eqn's for P's azimuthal coordinate: $\theta(t) = \theta_0 + \Omega t + \theta_e(t) + \theta_f(t)$, where θ_e and θ_f are P's free (epicyclic) and forced motions.

Likewise, if P and m_s have a mutual inclination, one can solve the linearized EOM for $\ddot{z}$, which will yield $z(t) = z_0 + z_e(t) + z_f(t)$ where z_e represents P's free (epicyclic) vertical displacement, and z_f would represent the forced motions excited by a nearby *vertical resonance*.

Lastly, I note that the preceding results were derived for motion in a nearly keplerian potential having $\kappa \simeq \Omega$.

However, one could easily generalize the preceding results for a particle orbiting in a non–keplerian potential, and that would revise ϕ_m, ψ_m, and e_f by additional factors of $\sim \kappa/\Omega$, which is $\sim \sqrt{2}$ in a galaxy having a flat rotation curve. Thus the results just obtained for a planetary system would also provide a rough estimate for, say, a star perturbed by another galactic point–mass perturber, such as a GMC, or a satellite galaxy.

Assignment #5
due ?
at the start of class

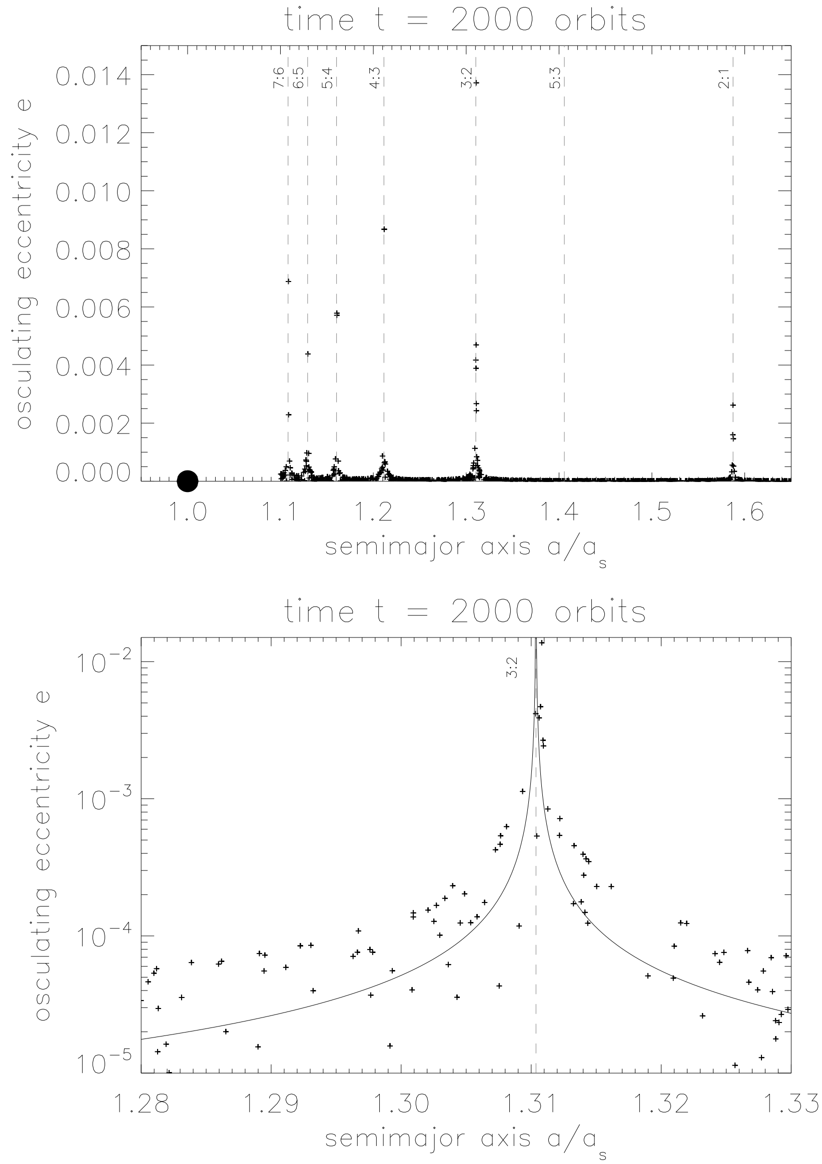
8. a) Show that $\phi_0(r)$ is equivalent to the gravitational potential of a uniform ring of mass m_s and radius a_s evaluated in the ring plane a distance r from its center.

b). Use Gauss' planetary eqns' to show that ϕ_0 term in a planet's gravitational potential will cause a particle's longitude of periapse to precess at the time–averaged rate

$$\dot{\tilde{\omega}} \simeq \frac{1}{4}\mu_s\beta b_{3/2}^{(1)}(\beta)n \tag{3.186}$$

where μ_s = planet/primary mass ratio, $\beta = a/a_s$ is the particle/planet semimajor axes, and n is the particle's mean motion. Assume the particle is in a low–e orbit. Do the $\phi_{m\geq1}$ terms drive any long–term precession? Explain.

Additional problems pending...



Results of an Nbody simulation (MERCURY, available at http://star.arm.ac.uk/~jec/mercury) of 1000 particles perturbed by a $\mu_s = 10^{-6}$ secondary (third of an Earth–mass). The particles were initially on circular orbits, and the figure shows their osculating orbit elements, $e(a)$, at time $t = 2000$ orbits later. Various mean motion resonances are indicated, and the lower figure zooms in on the 3:2 resonance.

Also plotted is Eqn' (3.185), the particle's expected forced e's.

Why do the osculating e's differ from the expected e_f by factors of ~ 2?

Why is there no disturbance at the 5:3?

Resonance Trapping

So far we have considered orbital evolution due to drag forces (like orbit decay due to PR drag), and resonant perturbations.

Lets put these phenomena together to consider the orbital drift of a particle due to a drag force, which can deliver it to a Lindblad resonance. This will lead to the phenomenon known as *resonance trapping*.

Lets consider the motion of a dust grain that is perturbed by an orbiting secondary of mass μ_2. The grain's orbit will decay PR drag.

[Alternately, we could have considered the motion of, say, a recently–consumed satellite galaxy whose orbit is decaying via dynamical friction, which could deliver the satellite to a LR with a bar (assuming galactic tide didn't already disrupt the satellite...). Although the force law might differ some from PR drag, this problem and its solution will be quite similar to our dust grain's motion.]

The PR drag acceleration is Eqn. (3.69)

$$\mathbf{a}_{PR} = a_r\hat{\mathbf{r}} + a_\theta\hat{\theta} \tag{3.187}$$

$$\text{where} \quad a_r = -\frac{2a_{rad}\dot{r}}{c} \equiv -2\alpha n\dot{r} \tag{3.188}$$

$$\text{and} \quad a_\theta = -\frac{a_{rad}r\dot{\theta}}{c} = -\alpha an^2 \tag{3.189}$$

$$\text{where} \quad a_{rad} = \beta g_\star = \frac{\beta GM_\star}{r^2} \simeq \beta an^2 \tag{3.190}$$

where the constant β is the radiation pressure/stellar gravity ratio, and $\alpha \equiv \beta an/c \ll 1$ is a small dimensionless coefficient for PR drag.

The grain is also disturbed by a secondary's m^{th} LR,
so its EOM is $\ddot{\mathbf{r}} = -\nabla\Phi + \mathbf{a}_{PR}$ where

$$\begin{aligned} \Phi(r,\theta,t) &= \Phi_0(r) + \Phi_1(r,\theta,t) \simeq \Phi_0 + \phi_m(r)\cos[m(\theta - \Omega_{ps}t)] && (3.191) \\ &= \Phi_0 + \mathrm{Re}[\phi_m e^{im(\theta-\Omega_{ps}t)}] && (3.192) \end{aligned}$$

where we will use the complex notation $e^{i\phi} = \cos\phi + i\sin\phi$ for convenience.

Assume nearly circular orbits:

$$\begin{aligned} r(t) &= r_0 + r_1(t) && (3.193) \\ \theta(t) &= \theta_0 + \Omega_0 t + \theta_1(t) && (3.194) \end{aligned}$$

where r_1 and θ_1 are assumed small, and

$$\begin{aligned} \Omega^2 &= \frac{1}{r}\frac{\partial\Phi_0}{\partial r} && (3.195) \\ \text{and} \quad \kappa^2 &= 3\Omega^2 + \frac{\partial^2\Phi_0}{\partial r^2} = 4\Omega^2 + r\frac{\partial\Omega^2}{\partial r} && (3.196) \end{aligned}$$

are the grain's angular and epicyclic frequencies^2.

In component form, this EOM is

$$\begin{aligned} \hat{\mathbf{r}} &: \ddot{r} - r\dot{\theta}^2 = -\frac{\partial\Phi}{\partial r} - 2\alpha\Omega_0\dot{r} && (3.197) \\ \hat{\theta} &: \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} - \alpha r_0\Omega_0^2 && (3.198) \end{aligned}$$

where I have replaced semimajor axis $a \to r_0$ and mean motion $n \to \Omega_0$.

Next, linearize this EOM, ie, drop small^2 terms, and evaluate the perturbations at the grain's guiding circle, $\mathbf{r}_0 = (r_0, \theta_0 + \Omega_0 t, 0)$:

$$\begin{aligned} \ddot{r}_1 + (\kappa_0^2 - 4\Omega_0^2)r_1 - 2r_0\Omega_0\dot{\theta}_1 &\simeq -\left.\frac{\partial\phi_m}{\partial r}\right|_{r_0} e^{i(m\theta_0+\omega_m t)} - 2\alpha\Omega_0\dot{r}_1 && (3.199) \\ r_0\ddot{\theta}_1 + 2\Omega_0\dot{r}_1 &\simeq -\frac{im\phi_m(r_0)}{r_0}e^{i(m\theta_0+\omega_m t)} - \alpha r_0\Omega_0^2 && (3.200) \end{aligned}$$

where it is understood that once these eqn's are solved,
we preserve only the real parts.

Note that these are the same EOM for a particle orbiting near an m^{th} LR, but with a drag force added to the RHS.

The RHS has oscillatory ($e^{i\omega_m t}$) and secular (steady) driving terms ($\dot{r}_d$, etc). Thus we anticipate a solution having oscillatory & secular parts:

$$r_1(t) = r_e(t) + r_f(t) + r_d(t) \quad (3.201)$$

$$\theta_1(t) = \theta_e(t) + \theta_f(t) + \theta_d(t) \quad (3.202)$$

where (r_e, θ_e) describes the grain's unforced epicyclic motion that satisfies the unforced EOM (with $\phi_m = 0$); these solutions resemble Eqns. (3.105) but with their amplitudes (or free eccentricities) damped by factor $e^{-\alpha\Omega_0 t} = e^{-t/T_d}$ where $T_d = 1/\alpha\Omega_0 = T_{orb}/2\pi\alpha$ the e–fold damping timescale. We will ignore these damped transient motions.

The (r_f, θ_f) are the forced oscillatory motions excited by the sinusoidal potential; they will again have the form

$$r_f(t) = \mathrm{Re}[R_m e^{i(m\theta_0 + \omega_m t)}] \quad (3.203)$$

$$\text{and} \quad \theta_f(t) = \mathrm{Re}[\Theta_m e^{i(m\theta_0 + \omega_m t)}] \quad (3.204)$$

where the complex constants R_m, Θ_m are the amplitudes of the grain's forced motions.

The grain's secular drift due to the drag is described by (r_d, θ_d).

Plug these anticipated solutions into the linearized EOM,
and collect oscillatory terms on the RHS, and secular terms on the left:

$$e^{i(m\theta_0+\omega_m t)}\left[(\kappa^2 - \omega_m^2 - 4\Omega^2 + 2i\alpha\Omega\omega_m)R_m - 2ir_0\Omega_0\omega_m\Theta_m + \frac{\partial\phi_m}{\partial r}\right]_{r_0}$$
$$= -\ddot{r}_d + 3\Omega^2 r_d + 2r_0\Omega_0\dot{\theta}_d - 2\alpha\Omega_0\dot{r}_d$$
$$\text{and} \quad e^{i(m\theta_0+\omega_m t)}\left[-r\omega_m^2\Theta_m + 2i\Omega\omega_m R_M + \frac{im\phi_m}{r}\right]_{r_0}$$
$$= -r_0\ddot{\theta}_d - 2\Omega_0\dot{r}_d - \alpha r_0\Omega_0^2 \quad (3.205)$$

Note the the RHS is oscillatory, and the LHS varies secularly with time t.

What does this tell us about the RHS? the LHS?

What about the stuff in the []?

Differentiate the upper RHS and solve for $\ddot{\theta}_d$:

$$\ddot{\theta}_d = \frac{\dddot{r}_d}{2r_0\Omega_0} - \frac{3\Omega_0\dot{r}_d}{2r_0} + \frac{\alpha\ddot{r}_d}{r_0} \tag{3.206}$$

Plug this into lower RHS:

$$-\frac{\dddot{r}_d}{2\Omega_0} - \alpha\ddot{r}_d - \frac{1}{2}\Omega_0\dot{r}_d = \alpha r_0\Omega_0^2 \tag{3.207}$$

what is the solution to this eqn'?

Earlier we found that PR drag has $\dot{r}_d$ = constant, so try $\ddot{r}_d = 0 = \dddot{r}_d$, which yields $\dot{r}_d = -2\alpha r_0\Omega_0$ = orbit decay rate due to PR drag.

Comparing this to Eqn (3.62) of Assignment #4 shows that our results indeed equivalent to our earlier findings when $e \ll 1$.

The grain's secular tangential motion is

$$\ddot{\theta}_d = 3\alpha\Omega_0^2 \tag{3.208}$$

$$\text{so} \quad \dot{\theta}_d(t) = 3\alpha\Omega_0^2 t \tag{3.209}$$

$$\text{and} \quad \theta_d(t) = \frac{3}{2}\alpha\Omega_0^2 t^2 \tag{3.210}$$

$$\text{while} \quad r_d(t) = r_0 - 2\alpha r_0\Omega_0 t \tag{3.211}$$

Evidently, the drag delivers a grain into smaller, *faster* orbits, causing its longitude to lead ahead of an unperturbed grain.

The grain's forced motions R_m and Θ_m are obtained by setting the $[] = 0$:

$$\Theta_m = \frac{2i\Omega_0 R_m}{r_0\omega_m} + \frac{im\phi_m}{(r_0\omega_m)^2} \tag{3.212}$$

$$\text{and thus} \quad R_m = \frac{-\psi_m}{D + 2i\alpha\Omega\omega_m} = \frac{-\psi_m D + 2i\psi_m\alpha\Omega\omega_m}{D^2 + (2\alpha\Omega\omega_m)^2} \tag{3.213}$$

$$\text{where} \quad D(r) = \kappa^2 - \omega_m^2 \tag{3.214}$$

$$\text{and} \quad \psi_m(r) = \frac{\partial\phi_m}{\partial r} + \frac{2m\Omega}{r\omega_m}\phi_m(r) \tag{3.215}$$

is the familiar forcing function ψ_m,
with all quantities are evaluated at the guiding center, $r = r_0$.

This of course is the familiar solution to a damped, driven SHO,
where α is the dimensionless damping coefficient.

When $\alpha = 0$, we recover our earlier results for a particle at an m^{th} LR.

The drag force alters the particle's motion in 2 ways:

(1.) P's forced motion is no longer singular at a $D = 0$ LR:

$$Re(r_f) = \frac{-\psi_m[D\cos(m\theta_0 + \omega_m t) + 2\alpha\Omega\omega_m \sin(m\theta_0 + \omega_m t)]}{D^2 + (2\alpha\Omega\omega_m)^2} \tag{3.216}$$

(2.) near exact resonance when $D \simeq 0$, P's response is $\sim 90°$ out of phase with the perturber's forcing, which varies as $\cos \omega_m t$

In the absence of damping ($\alpha = 0$), P's response is either in phase, or out of phase by 180°.

This is equivalent to saying that the particle's forced longitude of periapse is aligned or anti–aligned with the perturber at conjunction:

Recall from your studies of the damped SHO: if the particle's motion gets out of phase from the driver, then the driver can do *work* on the particle, possibly trapping it at resonance.

If resonance trapping does occur, this is possible only if the particle achieves a balance of torques: the torque due to the drag force is counterbalanced by the gravitational torque exerted by the secondary.

resonance trapping via a torque balance

Calculate the specific torque that the secondary m_2 exerts on a particle at the m^{th} LR:

$$\mathbf{T}_2 = \mathbf{r}\times(-\nabla\Phi_1) = -\mathrm{Re}\left[\frac{\partial\Phi_1}{\partial\theta}\hat{\mathbf{z}}\right], \tag{3.217}$$

$$\text{so} \quad T_2 = m\phi_m(r)\sin[m(\theta - \Omega_{ps}t)] \tag{3.218}$$

is the torque that m_2 exerts on particle P at $\mathbf{r} = (r, \theta)$.

Next, insert $r(t) = r_0 + r_1(t)$ and $\theta(t) = \theta_0 + \Omega_0 t + \theta_1(t)$ into the above and Taylor expand to first order:

$$T_2 \simeq m\left[\phi_m + r_1\frac{\partial\phi_m}{\partial r}\right]_{r_0} \sin(m\theta_0 + \omega_m t + m\theta_1) \tag{3.219}$$

$$\simeq m\phi_m\sin(m\theta_0 + \omega_m t) + m\frac{\partial\phi_m}{\partial r}r_1\sin(m\theta_0 + \omega_m t) + m^2\phi_m\theta_1\cos(m\theta_0 + \omega_m t) \tag{3.220}$$

where henceforth all quantities are evaluated at $r = r_0$.

In the above,

$$r_1(t) = r_e(t) + \mathrm{Re}[R_m e^{i(m\theta_0+\omega_m t)}] + r_d(t) \tag{3.221}$$

$$= r_e(t) + \mathrm{Re}(R_m)\cos(m\theta_0 + \omega_m t) - \mathrm{Im}(R_m)\sin(m\theta_0 + \omega_m t) + r_d(t) \tag{3.222}$$

and likewise for $\theta_1(t)$.

Now, mentally time–average T_2 over the forcing cycle $\Delta t = |2\pi/\omega_m| = T_{orb}$.

Do we need to account for the grain's epicyclic motions r_e, θ_e, when calculating the torque T_2?

Do the grain's secular motions r_d, θ_d contribute to T_2?

Next, time–average torque T_2 over one forcing cycle, noting that only the grain's forced oscillatory motions make a net contribution since

$$< \sin^2(m\theta_0 + \omega_m t) > = \frac{1}{2} =< \cos^2(m\theta_0 + \omega_m t) > \quad (3.223)$$

$$\text{so} \quad < T_2 > = -\frac{1}{2}m\frac{\partial \phi_m}{\partial r}\text{Im}(R_m) + \frac{1}{2}m^2\phi_m\text{Re}(\Theta_m) \quad (3.224)$$

$$\text{where} \quad \text{Re}(\Theta_m) = -\frac{2\Omega}{r\omega_m}\text{Im}(R_m) \qquad \text{(see Eqn' 3.212)} \quad (3.225)$$

$$\text{so} \quad < T_2 > = -\frac{1}{2}m\psi_m\text{Im}(R_m) \quad (3.226)$$

$$= -\frac{m\psi_m^2\alpha\Omega\omega_m}{D^2 + (2\alpha\Omega\omega_m)^2} \quad (3.227)$$

$$= -\frac{\varepsilon m\psi_m^2\alpha\Omega\kappa}{D^2 + (2\alpha\Omega\kappa)^2} \quad (3.228)$$

since $\omega_m = \varepsilon\kappa$ at a LR; this is the
time–averaged *specific* torque that the secondary exerts on the grain.

Note that the sign of $< T_2 >$ is $-\varepsilon\alpha$.
Which resonances can trap grains—the ILR or the OLR?

What if we changed the sign on the drag force, and replace $\alpha \to -\alpha$.

What happens then?

Now calculate the torque the drag exerts on the grain—how do I do this?

Recall Eqn' (2.138) from Assignment #3:

$$T'_d = \frac{T_d}{m} = \frac{1}{2}r_0\Omega_0\dot{r}_d = -\alpha(r_0\Omega_0)^2 \quad (3.229)$$

where T'_d is the specific torque on the grain due to drag.

What is the condition for trapping the grain at the secondary's LR?

The torques on the grain must balance: $< T_2 > + T'_d = 0$.

That balance also tells you how far from resonance the grain gets trapped

$$-\frac{\varepsilon m\psi_m^2\alpha\Omega\kappa}{D^2 + (2\alpha\Omega\kappa)^2} = \alpha(r\Omega)^2 \tag{3.230}$$

$$\text{so} \quad D^2(x) = \frac{m\psi_m^2\alpha\Omega\kappa}{\alpha(r\Omega)^2} - (2\alpha\Omega\kappa)^2 \tag{3.231}$$

at an $\varepsilon = -1$ OLR.

In the $m \gg 1$ approximation,

$$D(x) \simeq 3\varepsilon m\Omega^2 x \tag{3.232}$$

$$\psi_m \simeq -\frac{2\varepsilon f m\mu_s}{\pi}\Omega^2 a_s \tag{3.233}$$

$$\text{so} \quad x^2 \simeq \left(\frac{2f\mu_s}{3\pi}\right)^2 - \left(\frac{2\alpha}{3m}\right)^2 \tag{3.234}$$

is the fractional distance^2 from resonance where the grain gets trapped.

Note that resonance trapping is possible, ie, $x^2 > 0$,
when the drag force is sufficiently weak, ie, $\alpha < \alpha_c$ where

$$\alpha_c = \frac{f m^{3/2}\mu_s}{\pi} \tag{3.235}$$

is the critical drag parameter necessary for trapping at the m^{th} OLR.

What happens to those grains that have $\alpha > \alpha_c$?

Consider dust generated by colliding Kuiper Belt Objects (KBOs), which spiral inwards due to PR drag.

Lets assumed the grain has already crossed Neptune's $m = 1$ OLR (ie, the 2:1 MMR), which is weakened some by the indirect potential (see Eqn' 3.174).

Neptune has mass $\mu_s \simeq 5 \times 10^{-5}$ in solar units, and $a_s \simeq 30$ AU.
At Neptune's $m = 2$ OLR (the 3:2 MMR),

$$\alpha_c = \frac{fm^{3/2}\mu_s}{\pi} \simeq 1 \times 10^{-4} = \beta_c \left(\frac{r\Omega}{c}\right) \simeq 0.2 \left(\frac{1\mu\text{m}}{R_c}\right) \sqrt{\frac{Gm_2}{a_s c^2}} \quad (3.236)$$

$$\simeq 4 \times 10^{-6} \left(\frac{1\mu\text{m}}{R_c}\right) \quad (3.237)$$

So $R_c \simeq 0.04\mu$m is the threshold radius for trapping dust grains at Neptune's $m = 2$ OLR.

So what happens to grains larger than R_c? smaller than R_c?

Caveat: the above size threshold is probably somewhat unreliable, since our estimate of the parameter β (the ratio of radiation pressure/solar gravity ratio) was acquired in the *geometric optics* limit (ie, the grain sizes are $>$ wavelength of sunlight). Thus we are probably overestimating the PR drag parameter $\alpha \propto \beta$ some for grains smaller than $\sim 1\mu$m. Due to this complication, it would be prudent to merely assume that grain larger than $\sim 1\mu$m get trapped at Neptune's $m = 2$ OLR.

Nonetheless, it is evident that Neptune should be quite effective at trapping dust at its MMRs. However, telescopic searches for a faint IR glow from to this anticipated Kuiper Belt dust have so far not yielded any detections...

We have just quantified the conditions under which a particle whose orbit decays due to do the drag acceleration of the form

$$\mathbf{a}_{PR} = -2\alpha\Omega\dot{r}\hat{\mathbf{r}} - \alpha r\Omega^2\hat{\theta} \quad (3.238)$$

which is the drag law for PR drag.

It turns out that most drag laws relevant to planetary phenomena have the form

$$\mathbf{a}_{PR} = -2\alpha_1\Omega\dot{r}\hat{\mathbf{r}} - \alpha_2 r\Omega^2\hat{\theta} \quad (3.239)$$

If you were to plug this drag force into Gauss' planetary equations, you would find that α_1 controls the rate of eccentricity damping, and α_2 controls the orbital decay rate (assuming the $\alpha_i > 0$).

Examples of such drag forces include:

- atmospheric drag experienced by a low–altitude satellite
- aerodynamic drag that a planetesimal feels while orbiting within the solar nebula gas disk
- the torque that a disk exerted on an embedded companion (ie, satellite orbiting near a planetary ring, a planet embedded in a circumstellar gas disk, etc) mimics a drag force.
- a star suffering dynamical friction with a galactic disk

Although our preceding results were derived for a PR–like drag force ($\alpha_1 = \alpha_2$), you could easily rederive other trapping criteria for any alternate drag force having $\alpha_1 \neq \alpha_2$.

Assignment #5
due ?
at the start of class

9. Show that particles orbiting near a secondary's LR will cross that resonance when $|x| < x_c$, where x is the particle's fractional distance from resonance, and

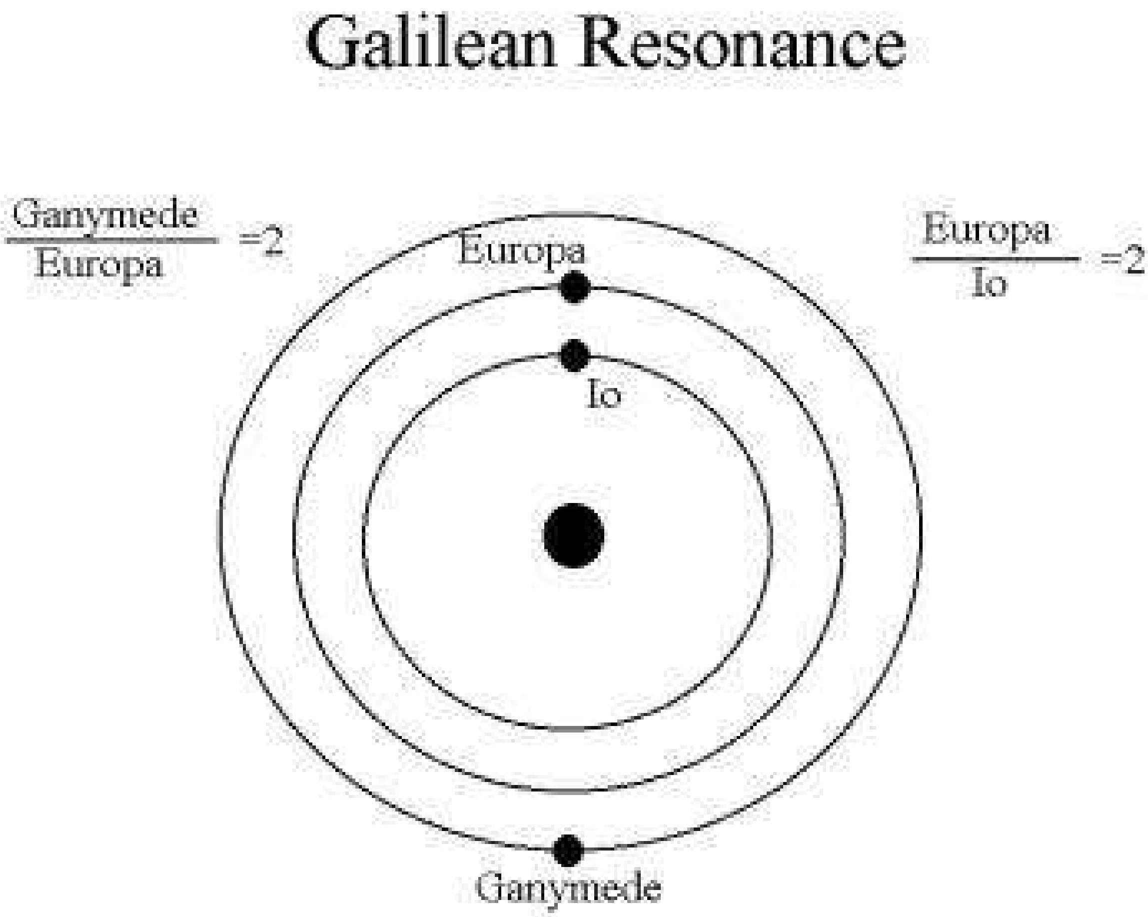
$$\text{where} \quad x_c \leq \sqrt{\frac{2f\mu_s}{3m}}, \tag{3.240}$$

is the threshold distance for resonance–crossing, where μ_s is the secondary's mass.

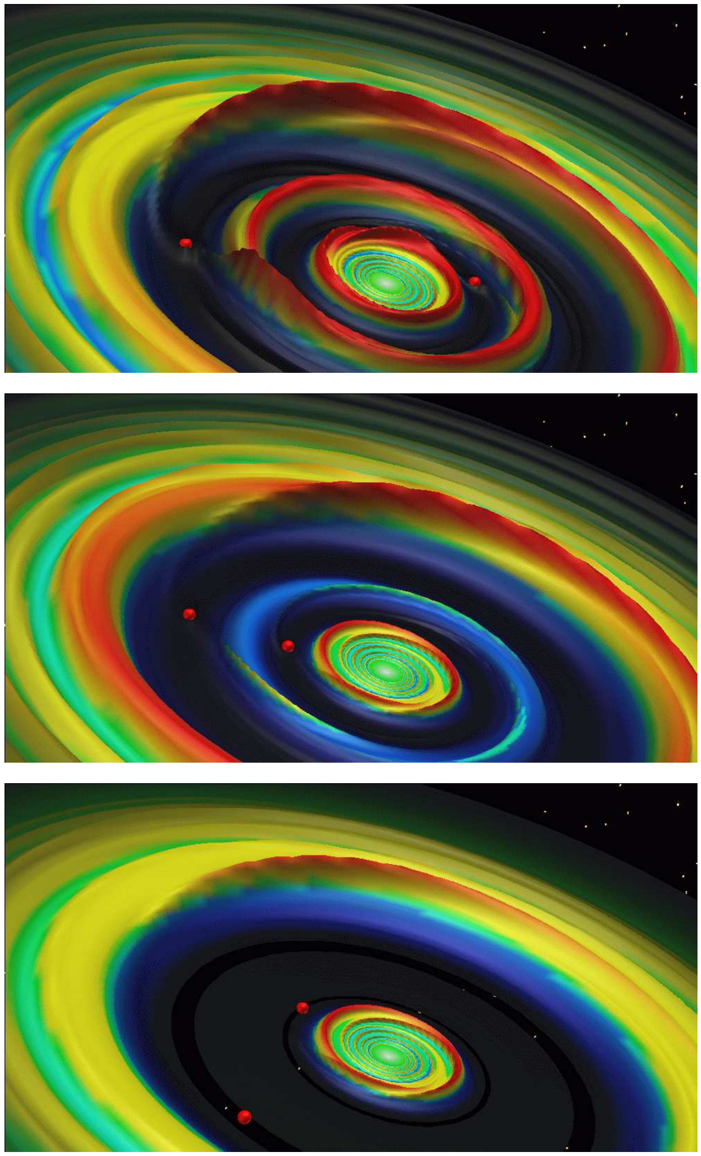
(Thus particles on each side of the resonance can crash into each other when $|x| \leq x_c$. Similarly, stars orbiting sufficiently near a galactic LR can interact with like stars on the other side of the LR.)

10. A dust grain of radius $R = 1\mu$m drifts into Neptune's $m = 2$ OLR. Does it get trapped there? If so, how far from resonance? Evaluate x and its forced eccentricity e_f. Is this orbit stable—will this grain persist in this orbit? Explain.

Suspected cases of resonance trapping



1. Three of the Galilean satellites, Io, Europa, & Ganymede, inhabit the Laplace resonance, ie, in mutual 2:1 resonances. This could be due to outward orbital evolution due to tides with Jupiter, or inwards migration due to interactions with a long–gone circumplanetary disk.



Bryden/Lin 2000
http://www.ucolick.org/~bryden/2planet

Figure 3.13: model of the Gliese 876 exoplanetary system

2. Hydrodynamic simulation of a pair of recently-formed giant planets that are interacting with the circumstellar gas disk from which they formed; this sim' indicates that torques from the dissipating gas disk can drive giant planets into a 2:1 resonance lock.

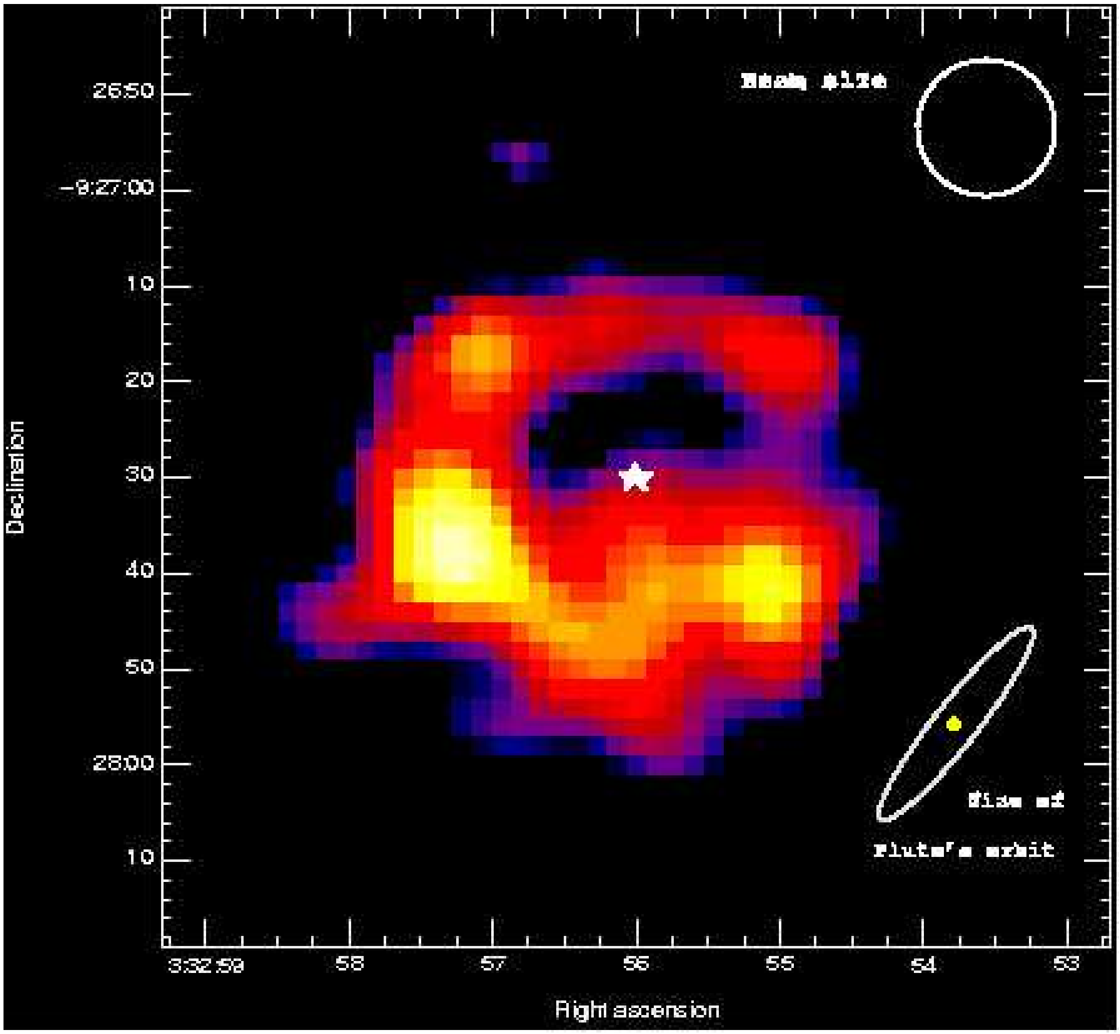


Figure 3.14: ϵ Eridani at submillimeter wavelengths.

3. Numerical simulations by Quillen and Thorndike (2002) show this clumpy circumstellar dust ring could be due to dust delivered (via PR drag) to 5:3 and 3:2 resonances with a Neptune–mass planet at $a = 40$ AU.

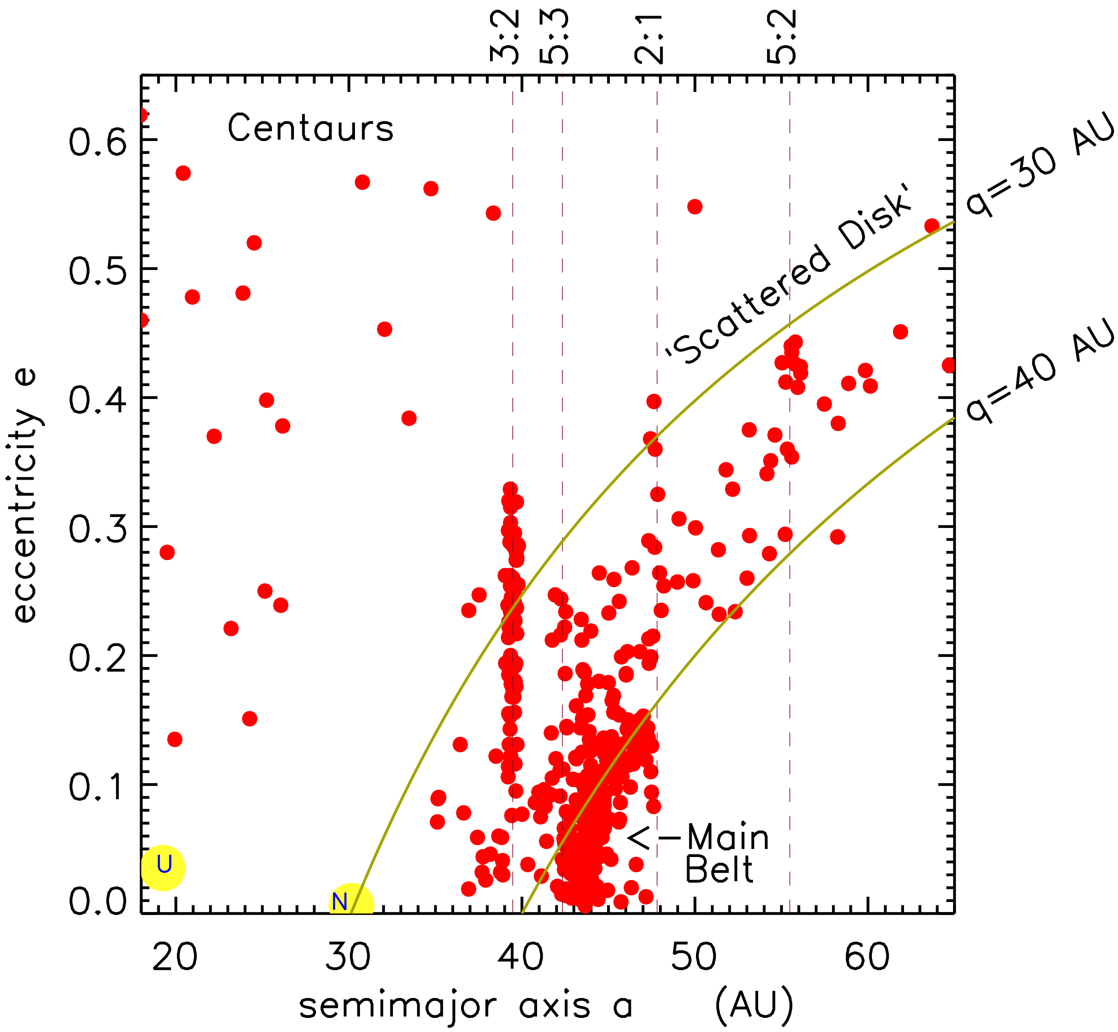


Figure 3.15: KBO orbit elements

4. Many KBOs inhabit 2:1 & 3:2 MMRs with Neptune. This is usually interpreted as evidence for Neptune's *outwards* migration (due to interactions with the natal planet–forming disk). This evolution is a bit different from the PR drag problem, since the planet's migration delivers the resonance to the particle, trapping it at resonance and pumping up its e with further migration.