Lecture Notes for ASTR 5622
Astrophysical Dynamics

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Fluid Dynamics & Astrophysical Disks

Thus far we have employed a *Lagrangian* approach to study the dynamics of gravitating systems; that approach is most convenient when interested in the evolution of a handful of discrete particles.

But when your system is crowded with many bodies (stars in a galaxy, molecules in a circumstellar disk, etc), it will be convenient to treat your system as a continuous fluid or particle gas, since all fluids are just swarms of discrete particles. Note that our use of continuum fluid mechanics will only apply to spatial scales $\gg$ particle spacings.

These lectures will address the following:
1. Distinctions between Eulerian & Lagrangian dynamics.

2. The fluid EOM (continuity & Euler's eqn),
gravity, pressure, viscosity, enthalpy, the equation of state, sound waves, etc.

3. Fluid disks: galactic, circumstellar, and circumplanetary disks,
their equilibrium state, and viscous evolution.

4. Stability analysis for a gravitating fluid.

5. Spiral wave theory.

6. Planet migration, time permitting.

Eulerian vs Lagrangian dynamics

The *Lagrangian* approach to dynamics usually begins with NII: $\ddot{\mathbf{r}} = -\nabla\Phi(\mathbf{r})$ where $\mathbf{r}$ is the position vector of particle P, and Φ its potential. In Lagrangian dynamics, the goal is usually to solve the EOM for P's trajectory, $\mathbf{r}(t)$:

The particle's velocity is $\mathbf{v} = \dot{\mathbf{r}}$,
and $\mathbf{r}$ and $\mathbf{v}$ are to be regarded as functions of time t only.

This approach is most useful when studying the motion of a discrete particle. Note that you effectively "keep your eye on the particle" when you calculate its velocity $\mathbf{v}(t)$ at some time t in its trajectory.

This is distinct from the Eulerian (or fluid) approach to dynamics,
where one regards $\mathbf{r}$ as pointing to some fixed spot,
where you wish to monitor the fluid velocity $\mathbf{v}(\mathbf{r}, t)$ as that fluid rushes by:

This discussion is excerpted from B&T, Appendix 1.E,
and Chapter 2 of Faber's *Fluid Dynamics for Physicists*

Suppose this fluid has some property $f = f(\mathbf{r}, t)$;
f could, for example, represent the fluid's density ρ.

Now lets calculate df/dt in the Lagrangian sense, which is the time rate–of–change in f that occurs while following the motion of some fluid parcel.

At time t_0, the fluid parcel is at $\mathbf{r}_0$ and has velocity $\mathbf{v}$,
and it has a quantity $f(\mathbf{r}_0, t_0) = f_0$.

A little while later, $t = t_0 + \Delta t$, and the fluid parcel is at $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}\Delta t$, where its $f = f(\mathbf{r}_0 + \mathbf{v}\Delta t, t_0 + \Delta t)$.

In Cartesian coordinates, $f = f(x_0 + v_x\Delta t, y_0 + v_y\Delta t, \ldots, t_0 + \Delta t)$,
which is Taylor expanded as

$$f(\mathbf{r}, t) = f(\mathbf{r}_0, t_0) + \left.\frac{\partial f}{\partial x}\right|_{\mathbf{r}_0} v_x\Delta t + \ldots + \left.\frac{\partial f}{\partial t}\right|_{\mathbf{r}_0} \tag{4.1}$$

$$\text{so} \quad \frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(\mathbf{r}, t) - f(\mathbf{r}_0, t_0)}{\Delta t} = \frac{\partial f}{\partial x}v_x + \frac{\partial f}{\partial y}v_y + \frac{\partial f}{\partial z}v_z + \frac{\partial f}{\partial t} \tag{4.2}$$

$$\text{or} \quad \frac{df}{dt} = (\mathbf{v} \cdot \nabla)f + \frac{\partial f}{\partial t} \tag{4.3}$$

df/dt is sometimes called the *convective*, or Lagrangian derivative of f, since it is the time rate–of–change in f that you sense while following the motion of the fluid parcel at $\mathbf{r}(t)$.

Noting that the fluid velocity is $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$,
it then follows that the fluid parcel's acceleration can be written as

$$\frac{d\mathbf{v}}{dt} = (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \tag{4.4}$$

Note, however, that $(\mathbf{v} \cdot \nabla)\mathbf{v}$ is a *vector operator*. Its particular form (eqns 1B53–1B56 of B&T) depends on your choice of coordinate system (cartesian, cylindrical, etc.)

The Fluid EOM

Now lets derive the principle eqn's for fluid dynamics:
the *continuity* eqn, and *Euler's* eqn.

the continuity eqn'

Consider a distribution of matter of density $\rho(\mathbf{r}, t)$
which has a velocity *field* $\mathbf{v}(\mathbf{r}, t)$.

Now consider a volume V that is enclosed by a arbitrary surface S:

The total mass inside V is

$$M = \int \rho(\mathbf{r}', t)dV' = \int dx \int dy \int dz \rho(x', y'z'), \tag{4.5}$$

Also let $d\mathbf{a} = \hat{\mathbf{n}}da$ = a small differential area element on the surface S, with $\hat{\mathbf{n}}$ = unit vector normal to S.

Suppose there is some matter flowing into or out of the volume V. This flow has a flux $= \rho\mathbf{v}$ (mass/area/time).

After time Δt, mass $\Delta m = \rho\mathbf{v}\cdot\mathbf{da}\Delta t$ will flow across area $\mathbf{da}$, so $dm/dt = -\rho\mathbf{v}\cdot d\mathbf{a}$ = rate at which mass flows across surface element $d\mathbf{a}$; with the sign indicating that an outward flow causes the mass in V to decrease:

Thus the total mass M inside V changes at the rate

$$\frac{dM}{dt} = \int_V \frac{\partial\rho}{\partial t}dV' = \int_S \frac{dm}{dt} = -\int_S \rho\mathbf{v}\cdot d\mathbf{a} \tag{4.6}$$

Now apply the divergence theorem, Eqn' (3.55), to the RHS:

$$\int_V \nabla\cdot\mathbf{A}dV = \int_S \mathbf{A}\cdot d\mathbf{a} \tag{4.7}$$

$$\int_V \nabla f dV = \int_S f d\mathbf{a} \tag{4.8}$$

Consequently

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right] dV' = 0 \tag{4.9}$$

This must be true for any arbitrarily–shaped volume V.
What does this then say about the integrand?

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{4.10}$$

which is the continuity equation. It simply says that mass is conserved.

What if mass were not conserved?
What if mass were being converted to, so, energy, or vise-versa?

Euler's eqn'

Let $p(\mathbf{r}, t)$ = pressure at point $\mathbf{r}$ at time t.
The total pressure–force that volume V exerts on the surrounding material *external* to V is

$$\int_S p d\mathbf{a}' \tag{4.11}$$

$$\text{Consequently} \qquad \mathbf{F_p} = -\int_S p d\mathbf{a}' \tag{4.12}$$

is the force on volume V due to the pressure of the surrounding environment.

Let $\mathbf{F_e} = -M\nabla\Phi(\mathbf{r}, t)$ = sum of all other conservative forces that are *external* to V, where $M = \int_V \rho dV'$ = total mass of V.

Newton's 2^{nd} law of motion is then

$$
\begin{aligned}
M\frac{d\mathbf{v}}{dt} &= \mathbf{F_p} + \mathbf{F_e} = -\int_S p\,d\mathbf{a}' - M\nabla\Phi && (4.13)\\
\text{so}\quad \int_V \rho\frac{d\mathbf{v}}{dt}dV' &= -\int_S p\,d\mathbf{a}' - \int_V \rho\nabla\Phi dV' && (4.14)\\
\text{But}\quad \int_S p\,d\mathbf{a}' &= \int_V \nabla p dV' \quad \text{by div' formula, Eqn' (4.8)} && (4.15)\\
\text{so}\quad & \int_V \left(\rho\frac{d\mathbf{v}}{dt} + \nabla p + \rho\nabla\Phi\right) dV' = 0 && (4.16)
\end{aligned}
$$

which must hold for any arbitrary volume V, hence the integrand is zero:

$$
\begin{aligned}
\frac{d\mathbf{v}}{dt} &= -\frac{\nabla p}{\rho} - \nabla\Phi && (4.17)\\
&= (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\partial\mathbf{v}}{\partial t} \quad \text{by Eqn' (4.4)} && (4.18)\\
\text{so}\quad \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} &= -\frac{\nabla p}{\rho} - \nabla\Phi && (4.19)
\end{aligned}
$$

which is Euler's eqn' for an inviscid (frictionless) fluid,
also known as the momentum equation.

What if this fluid were subject to an additional nonconservative acceleration $\mathbf{a}$? For instance, $\mathbf{a}$ could represent a drag force.

If the fluid has some viscosity ν, then additional terms appear in the RHS, and Euler's eqn' becomes the *Navier–Stokes* eqn';
we will see viscosity again later when we consider a viscous accretion disk.

the equation of state

The use of Euler's eqn' also requires choosing an equation of state (EOS), which relates the pressure p to density ρ or temperature T (or perhaps entropy S) .

We will employ a *barotropic* EOS, which assumes that pressure depends only on the fluid's density:

$$p(\mathbf{r}, t) = p(\rho) \quad \text{where} \quad \rho = \rho(\mathbf{r}, t) \tag{4.20}$$

This EOS is common to many astrophysical fluids. Some examples are:

(*i.*) isothermal (T =constant in time and/or space) ideal gas which obeys $p = \rho k_B T/m$. You can regard an isothermal system as being attached to an external heat bath, which allows heat (energy) to flow into/out of your system to preserve constant T. Example: a fluid that cools via thermal radiation at the same rate that it is heated by a nearby star. Note that the isothermal EOS does not necessarily conserved energy.

(*i.*) isentropic (constant entropy) gas which obeys $p = K\rho^\gamma$, which is a *polytropic* EOS. An isentropic system is also adiabatic, which means that no heat (energy) is lost or gained. An isentropic system conserves energy. Use this EOS when you system is isolated from the rest of the universe, and when there is no cooling due to thermal radiation.

For a barotropic fluid, it will be convenient to replace the pressure $p(\rho)$ with the *enthalpy* $h(\rho)$ which appears in thermodynamics; enthalpy = specific energy due to the fluid's pressure.

Appendix 1.E.c of B&T uses the fundamental laws of thermodynamics to show that the fluid's enthalpy can be written as

$$
\begin{aligned}
h(\rho) &\equiv \int_0^\rho \frac{1}{\rho'} dp(\rho') = \int_0^\rho \frac{1}{\rho'}\frac{dp}{d\rho'} d\rho' && (4.21)\\
\text{so} \quad \frac{dh}{d\rho} &= \frac{1}{\rho}\frac{dp}{d\rho} && (4.22)\\
\text{and} \quad \nabla h(\rho(\mathbf{r},t)) &= \frac{dh}{d\rho}\frac{\partial\rho}{\partial x}\hat{\mathbf{x}} + \ldots = \frac{dh}{d\rho}\sum_{i=1}^{3}\frac{\partial\rho}{\partial x_i}\hat{\mathbf{x}}_i && (4.23)\\
&= \frac{1}{\rho}\sum_i \frac{dp}{d\rho}\frac{\partial\rho}{\partial x_i}\hat{\mathbf{x}}_i = \frac{1}{\rho}\sum_i \frac{dp}{dx_i}\hat{x}_i && (4.24)\\
&= \frac{1}{\rho}\nabla p && (4.25)
\end{aligned}
$$

which relates the fluid's pressure gradient to its enthalpy gradient. Inserting this result into Euler's eqn then yields

$$
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla(h + \Phi) \qquad (4.26)
$$

for a barotropic fluid.

Evidently, replacing the system's pressure p with the enthalpy h allows us to combine pressure with the potential Φ to form a 'fluid potential' $h + \Phi$.

linearize the dynamical equations

Lets consider an inviscid, gravitating, barotropic fluid.
Its motions must satisfy three dynamical equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{continuity eqn'} \tag{4.27}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla(h + \Phi) \quad \text{Euler's eqn'} \tag{4.28}$$

$$\nabla^2 \Phi = 4\pi G \rho \tag{4.29}$$

where the last is Poisson's eqn (3.60),
with Φ being the system's gravitational potential.

Lets linearize these equations of motion, which means we are going to assume that some small perturbation is going to push the system a small distance from some undisturbed state that is time independent, ie,

$$\rho(\mathbf{r}, t) = \rho_0(\mathbf{r}) + \epsilon \rho_1(\mathbf{r}, t) \tag{4.30}$$

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{v_0}(\mathbf{r}) + \epsilon \mathbf{v_1}(\mathbf{r}, t) \tag{4.31}$$

$$p(\mathbf{r}, t) = p_0(\mathbf{r}) + \epsilon p_1(\mathbf{r}, t) \tag{4.32}$$

$$h(\mathbf{r}, t) = h_0(\mathbf{r}) + \epsilon h_1(\mathbf{r}, t) \tag{4.33}$$

$$\Phi(\mathbf{r}, t) = \Phi_0(\mathbf{r}) + \epsilon \Phi_1(\mathbf{r}, t) \tag{4.34}$$

where quantities having the 0 subscript are the system's undisturbed equilibrium state, and the 1 subscript is the system's response to the perturbation, which is also assumed small ie, $\epsilon \ll 1$.

If the perturber were, say, a galactic bar (or a planet), then ε would likely be the bar/galaxy (or planet/primary) mass ratio.

Insert these trial solutions into the dynamical eqn's & Taylor expand to $\mathcal{O}(\epsilon)$:
For instance, the continuity eqn is

$$\frac{\partial(\rho_0 + \epsilon\rho_1)}{\partial t} + \nabla \cdot [(\rho_0 + \epsilon\rho_1)(\mathbf{v_0} + \epsilon\mathbf{v_1})] \;=\; 0 \tag{4.35}$$

Since ϵ is an arbitrary parameter,
the $\mathcal{O}(\epsilon^0)$ and $\mathcal{O}(\epsilon^1)$ terms must separately sum to zero, so

$$\mathcal{O}(\epsilon^0): \quad \frac{\partial\rho_0}{\partial t} + \nabla \cdot (\rho_0\mathbf{v_0}) \;=\; 0 \tag{4.36}$$

which is the usual continuity eqn'; you likewise recover the usual
Euler & Poisson eqn's for $\rho_0, \mathbf{v_0}, \Phi_0$, etc,

while the dynamical equations for the $\mathcal{O}(\epsilon^1)$ terms are

$$\frac{\partial\rho_1}{\partial t} + \nabla \cdot (\rho_0\mathbf{v_1}) + \nabla \cdot (\rho_1\mathbf{v_0}) = 0 \quad \text{(cont' eqn')} \tag{4.37}$$

$$\frac{\partial\mathbf{v_1}}{\partial t} + (\mathbf{v_1} \cdot \nabla)\mathbf{v_0} + (\mathbf{v_0} \cdot \nabla)\mathbf{v_1} = -\nabla(h_1 + \Phi_1) \quad \text{(Eulers eqn')} \tag{4.38}$$

$$\nabla^2\Phi_1 = 4\pi G\rho_1 \quad \text{(Poisson eqn')} \tag{4.39}$$

We can get rid of the h_1 in the above by noting that

$$h(\rho) = h(\rho_0 + \epsilon\rho_1) \simeq h(\rho_0) + \epsilon \left.\frac{dh}{d\rho}\right|_{\rho_0} \rho_1 = h_0 + \epsilon h_1 \tag{4.40}$$

$$\Rightarrow h_1 = \left.\frac{dh}{d\rho}\right|_{\rho_0} \rho_1 = \left.\frac{dp}{d\rho}\right|_{\rho_0} \frac{\rho_1}{\rho_0} \quad \text{with Eqn' (4.22)} \tag{4.41}$$

where the constant $dp/d\rho|_{\rho_0}$ is obtained from your EOS.

The above constitute the linearized dynamical eqn's for the
disturbances ρ_1 and $\mathbf{v}_1$ that occur in an inviscid, gravitating, barotropic fluid.

simple example: sound waves in a barotropic fluid

Lets perturb a non–gravitating fluid whose unperturbed state is *stationary*.

What is $\nabla\Phi$? What is $\mathbf{v_0}$?
Our linearized fluid eqn's thus become

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0 \quad \text{(cont' eqn')} \tag{4.42}$$

$$\frac{\partial \mathbf{v}_1}{\partial t} = -\nabla h_1 = -\left.\frac{1}{\rho_0}\frac{dp}{d\rho}\right|_{\rho_0} \nabla \rho_1 \quad \text{(Euler eqn')} \tag{4.43}$$

These eqn's tell us how the fluid velocity $\mathbf{v}_1$ changes
if we squeeze the fluid somewhere such that $\rho_1 \neq 0$.

To solve these coupled eqn's, differentiate the cont' eqn' wrt t,
and then insert the Euler eqn:

$$\frac{\partial^2 \rho_1}{\partial t^2} - v_s^2 \nabla^2 \rho_1 = 0 \tag{4.44}$$

$$\text{where} \quad v_s^2 \equiv \left.\frac{dp}{d\rho}\right|_{\rho_0} \tag{4.45}$$

This of course is the wave eqn'. For a 1D sound waves, the solution for the fluid's motion has the form $\rho_1(x, t) = A\cos(kx - \omega t)$. Inserting this into the wave eqn' yields the familiar dispersion relation for traveling waves:

$$\omega^2 = v_s^2 k^2 \tag{4.46}$$

where ω = waves' angular frequency, k = wavenumber, and $v_s = \omega/k = \lambda/T$ is the soundspeed, where wavelength $\lambda = 2\pi/k$, period $T = 2\pi/\omega$.

If the fluid is an ideal gas, then $p = \rho k_B T/m$,
and the soundspeed^2 is $v_s^2 = dp/d\rho|_{\rho_0} = k_B T/m$ for our 1D system.

Gravitational instabilities in astrophysical fluids

We will perform a linearized stability analysis of the fluid equations for three types of systems:

- an infinite, uniform fluid, which can suffer a *Jean's* instability,
- gravitational instabilities in a differentially rotating disk (planetary or galactic), which can suffer a *ring* instability,
- gravitational instabilities in fluid cylinders, which will lead to the *sausage* instability that can occur in astrophysical jets, and (strangely enough) comet Shoemaker–Levy 9.

Jean's instability

Apply the fluid EOM of an infinite fluid that, in its undisturbed state, has a uniform density $\rho_0(\mathbf{r}) =$ constant.

The fluid is also assumed static, so $\mathbf{v_0}(\mathbf{r_0}) = 0$, $p(\mathbf{r}) =$ constant, and $\Phi_0(\mathbf{r}) =$ constant.

Although this seems like plausible initial conditions, we immediately run into problems since Euler's eqn' tells us that $\nabla\Phi_0 = 0$ (since $\mathbf{v_0} = 0$).

However this contradicts Poisson's eqn', $\nabla^2\Phi_0 = 4\pi G\rho_0$, except for the uninteresting case of where $\rho_0 = 0$.

But this inconsistency is an artifact of the unphysical assumption that the fluid has an infinite extent.

If we instead gave our fluid real boundaries, then $\nabla\Phi_0 \neq 0$. However we would rather not have to deal with these boundaries, and their boundary conditions...

B&T (Section 5.1) suggests sidestepping these details by invoking the 'Jean's swindle', that is, to assume that Poisson's eqn' only applies to the perturbed quantities Φ_1 and ρ_1.

It turns out that this 'Jean's swindle' is valid provided you apply it to regions far from the fluid's boundaries, and that you consider spatial scales λ that are small compared to that over which $\nabla\Phi$ varies.

In that case, the linearized fluid eqn's for a barotropic fluid are

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v_1} = 0 \qquad \text{(CE)} \tag{4.47}$$

$$\frac{\partial \mathbf{v_1}}{\partial t} = -\nabla\left(h_1 + \Phi_1\right) \tag{4.48}$$

$$= -\frac{v_s^2}{\rho_0}\nabla\rho_1 - \nabla\Phi_1 \qquad \text{(EE)} \tag{4.49}$$

$$\nabla^2\Phi_1 = 4\pi G\rho_1 \qquad \text{(PE)} \tag{4.50}$$

To derive the Jean's instability, calculate $\partial(CE)/\partial t$,
and then insert the EE and PE, which yields

$$\frac{\partial^2 \rho_1}{\partial t^2} - v_s^2\nabla^2\rho_1 - 4\pi G\rho_0\rho_1 = 0 \tag{4.51}$$

Note that when the fluid is non–gravitating, ie $G = 0$,
and we get the classical wave eqn', which admits sound waves.

And for a gravitating fluid with $G \neq 0$,
the particular solution to the EOM can also be oscillatory:

$$\rho_1(\mathbf{r}, t) = \Re\left[Ce^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] \tag{4.52}$$

As long as the arguments of the exponential are real,
then Eqn' (4.52) describes spherical wave.

Regard this as a gravitationally–modified soundwave
that has a complex amplitude $C(k)$= fluid's response to an
imposed disturbance having a wavenumber $\mathbf{k}$, frequency ω.

Choose a spherical coordinate system, and examine these 'waves' at some site downstream of the disturbance, so $\mathbf{k} \cdot \mathbf{r} = kr$ and $\rho_1 = C(k)\cos(kr - \omega t)$.

Inserting this trial solution into the EOM yields the system's *dispersion relation*:

$$\omega(k)^2 \;=\; v_s^2 k^2 - 4\pi G\rho_0 \tag{4.53}$$

You should regard the frequency ω as a function of the perturbation's wavenumber k or its wavelength $\lambda = 2\pi/k$.

So when the perturbation has a wavelength/wavenumber such that $\omega^2 > 0$, the fluid's response is oscillatory, indicating that the fluid is gravitationally stable.

However if the perturbation has a wavenumber such that $\omega(k)^2 < 0$,
ie, $\omega = i|\omega|$, then

$$\rho_1 = \Re\left[Ce^{i(kr - i|\omega|t)}\right] \propto e^{|\omega|t} \tag{4.54}$$

which indicates that the fluid is gravitationally unstable since the disturbance causes ρ_1 to blow up exponentially over timescale $\tau_c \sim 2\pi/|\omega|$.

The dispersion relation (4.53) tells us that a higher sound speed, and hence a higher temperature (since $v_s^2 = dp/d\rho \propto T$) tends to stabilize the fluid *against* gravitational collapse, while a higher density ρ_0 tends to encourage instability.

In other words, pressure gradients are stabilizing,
while gravity is destabilizing.

If the fluid is indeed unstable, then the gravity term in eqn' (4.53) dominates, $|\omega| \sim \sqrt{4\pi G\rho_0}$ and the fluid density grows as $\rho_1 \propto e^{|\omega|t}$, which corresponds to a gravitational collapses over an e–fold timescale of $\tau_c \sim 1/\sqrt{4\pi G\rho_0}$.

This is roughly the time needed for a particle to sink to the center of a uniform cloud of density ρ_0:

Consider the motion of a particle at the outer edge of a sphere of radius r and mass M:

$$\begin{aligned} \ddot{r} &= -\frac{GM}{r^2} = -\frac{4\pi}{3}G\rho_0 r = -\omega_0^2 r \quad &(4.55) \\ \text{which has solution} \quad r(t) &= A\cos(\omega_0 t) \quad &(4.56) \\ \text{so} \quad \tau_{ff} &= \frac{2\pi}{\omega_0} = \sqrt{\frac{3\pi}{G\rho_0}} \quad &(4.57) \end{aligned}$$

is the particle's free–fall timescale.

In should be noted that any arbitrary perturbing acceleration a_p will not be characterized by a single wavelength λ or wavenumber k.

Nonetheless, any arbitrary disturbance can always be Fourier decomposed as

$$a_p(r) = \int c(k)e^{ikr}dk, \quad (4.58)$$

and the fluid's response to this more general perturbation will have the form

$$\rho_1(\mathbf{r}, t) = \int C(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}d^3\mathbf{k}. \quad (4.59)$$

where the response amplitudes $C(k)$ are proportional to the perturbing amplitudes $c(k)$.

Note that this arbitrary perturbation still satisfies the same dispersion relation:

We call the threshold wavenumber k_J where $\omega(k_J)^2 = 0$
the *Jean's* wavenumber:

$$k_J \;=\; \sqrt{\frac{4\pi G\rho_0}{v_s^2}} \quad \text{and} \quad \lambda_J = \frac{2\pi}{k_J} = \sqrt{\frac{\pi v_s^2}{G\rho_0}} \tag{4.60}$$

where λ_J is the Jean's wavelength.

The dispersion relation shows that an *infinite* fluid is susceptible to gravitational collapse due to all perturbations, including those of but a microscopic amplitude, that have wavenumbers $k < k_J$ or wavelengths $\lambda > \lambda_J$.

However, a real fluid blob will have a finite extent ℓ,
so it can only be perturbed over wavelengths of $\lambda \leq \ell$.

Thus if the blob suffers an infinitesimal perturbation have a spatial scale of $\ell > \lambda > \lambda_J$, those perturbations will grow exponentially, causing the fluid to break up into contracting blobs of size $\sim \lambda_J$ and masses

$$M_J \sim \frac{4\pi}{3}\rho_0 \left(\frac{\lambda_J}{2}\right)^3 \sim \frac{\pi}{6}\rho_0 \left(\frac{\pi v_s^2}{G\rho_0}\right)^{3/2} \tag{4.61}$$

If the fluid is an ideal gas, $v_s^2 = k_B T/m$ and

$$M_J \sim \frac{\pi}{6}\rho_0 \left(\frac{\pi k_B T}{G\rho_0 m}\right)^{3/2} \tag{4.62}$$

where $m =$ molecule mass.

Example: the dense core of a giant molecular cloud has a hydrogen number density $n_H \sim 10^8$ atoms/cm^3, $\rho_0 = m_H n_H \sim 2 \times 10^{-16}$, and $T \sim 150$ K, so if it is more massive than $M_J \sim 10$ M$_\odot$ it will contract and ultimately form clusters of stars, each containing perhaps ~ 10 solar–mass protostars. This collapse will occur quite quickly in only $\tau_c \sim \sqrt{3\pi/G\rho_0} \sim 10^4$ years.

However if the cloud has a size $R \ll \lambda_J$ and mass $M \ll M_J$, that cloud will be immune to the destabilizing long–wavelength perturbations $\lambda_J > R$. Thus λ_J and M_J should be regarded as upper limits on the size & mass of an interstellar cloud.

This limit can also be regarded as a lower limit on a stable cloud's temperature T_{min}:

$$T_{min} = \left(\frac{6M}{\pi\rho_0}\right)^{3/2} \frac{G\rho_0 m}{\pi K_B T} \tag{4.63}$$

a cloud with $T < T_{min}$ is gravitationally unstable.

Gravitational stability of a rotating disk

Lets consider the gravitational stability of a thin disk in orbit about some center; our results will apply to a nearly keplerian circumstellar disk orbiting a young star, but also to a (non–keplerian) disk, such as stars orbiting a galaxy's center.

Use the linearized fluid EOM to do the stability analysis.

Consider the disk's unperturbed state—steady, circular, coplanar motion:

$$\mathbf{v}_0 = r_0\Omega_0\hat{\theta} \quad \text{where} \quad \Omega^2 = \frac{1}{r}\frac{\partial\Phi_0}{\partial r} \quad \text{fluid angular velocity}^2 \tag{4.64}$$

and $\rho_0 = \rho(r_0)$ is the density of the unperturbed fluid,
at the guiding center at $r = r_0$,
and Φ_0 is its unperturbed gravitational potential.

Now reach in and perturb this system slightly:
$\rho \rightarrow \rho_0 + \varepsilon\rho_1(r, t)$
$\mathbf{v} \rightarrow \mathbf{v}_0 + \varepsilon\mathbf{v}_1(r, t)$, etc., where $\varepsilon \ll 1$.

Lets disturb this system so that the perturbations are sinusoidal:
$\rho_1(r, t) = \rho_2 e^{i(kr-\omega t)}$, where the amplitude ρ_2 can be infinitesimally small;
the other perturbations $\mathbf{v}_1$, Φ_1 will have similar forms.

This is equivalent to 'seeding' your disk with some small sinusoidal density perturbation of magnitude ρ_2, wavenumber k, wavelength $\lambda = 2\pi/|k|$:

Note also that we have assumed an *axisymmetric* perturbation.
Note that we could have considered a more general *non–axisymmetric* disturbance having the form

$$\rho_1(r, \theta, t) = \rho_2 e^{i(kr+m\theta-\omega t)} \tag{4.65}$$

where m is the familiar azimuthal wave number,
equivalent to seeding the disk with an m–armed spiral density pattern.

Had we tackled this more general problem, we would find that most disks are stable against non–axisymmetric perturbations having $m \geq 1$,
so we will concentrate only on the disk's response to $m = 0$ perturbations.

We will then insert this perturbation into the EOM, and derive the system's dispersion relation $\omega(k) =$ system's angular frequency.

First, note that our perturbation is assumed to be sinusoidal.
Is this appropriate? Why?

Suppose we find that $\omega(k)$ is real for *all* perturbations having *any* k.
Is this stable or unstable?

But what if $\omega(k)$ has an imaginary part? Does the sign matter?

We begin with the fluid EOM: the CE, EE, and PE.
They form a coupled set of rather complicated–looking PDEs.

However the EOM simplify considerably when the disk is infinitesimally thin:

$$\rho(r, z, t) = \sigma(r, t)\delta(z) \tag{4.66}$$

where $\delta(z)$ is the Dirac delta function (B&T, Appendix 1.C),
and the disk *surface* density of matter is obtained by integrating vertically through the disk along the z axis:

$$\sigma(r, t) = \int_{-\infty}^{\infty} \rho(r, z', t)dz' \tag{4.67}$$

We can also make our 3D linearized CE, Eqn (4.37),
2D by vertically integrating through the disk:

$$\frac{\partial \sigma_1}{\partial t} + \nabla \cdot (\sigma_0 \mathbf{v_1}) + \nabla \cdot (\sigma_1 \mathbf{v_0}) = 0 \tag{4.68}$$

However our linearized PE, Eqn' (4.39) requires special handling:

$$\nabla^2 \Phi_1 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi_1}{\partial \theta^2} + \frac{\partial^2 \Phi_1}{\partial z^2} = 4\pi G \rho_1 \tag{4.69}$$

in cylindrical coord's; see B&T Eqn' (1B–50).

Note that the RHS is zero when $z \neq 0$,
yet the individual terms on the left are probably nonzero.

Note that the potential Φ_1 associated with the density perturbation ρ_1 must satisfy two constraints:
(i.) $\nabla^2 \Phi_1 = 0$ in regions where $z \neq 0$, and
(ii.) Φ_1 has the form $\Phi_1(r, 0, t) = \Phi_2 e^{i(kr - \omega t)}$ in the $z = 0$ plane.

Lets consider the trial solution $\Phi_1(r, z, t) = \Phi_2 e^{i(kr - \omega t) - |kz|}$,
which clearly satisfies point *(ii.)* .

But can it satisfy *(i.)*?

Check by inserting Φ_1 into the PE, and vertically integrate from $z = -a$ to $+a$, where $|a| \ll \lambda$ is some tiny distance just above/below the disk plane:

$$\int_{-a}^{a} \nabla^2 \Phi_1 dz' \simeq \frac{2a}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi_1}{\partial r}\right) + \left.\frac{\partial \Phi_1}{\partial z}\right|_{z=+a} - \left.\frac{\partial \Phi_1}{\partial z}\right|_{z=-a} = 4\pi G \sigma_1 \tag{4.70}$$

Since a is arbitrary, take the limit where $a \to 0$.
Since $e^{-|kz|} = e^{-s_z|k|z}$ where $s_z = \text{sgn}(z)$,

$$\frac{\partial \Phi_1}{\partial z} = -s_z|k|\Phi_1 \tag{4.71}$$

$$\text{so} \quad \Phi_1 = -2\pi G\sigma_1/|k| \tag{4.72}$$

Evidently, a thin disk has a rather simple relationship between its perturbed density σ_1 and its gravitational potential Φ_1.

Lets examine the linearized CE:

$$\frac{\partial \sigma_1}{\partial t} + \nabla \cdot (\sigma_0 \mathbf{v_1}) + \nabla \cdot (\sigma_1 \mathbf{v_0}) = 0 \tag{4.73}$$

$$\text{where} \quad \mathbf{v_0} = r\Omega\hat{\theta} = \text{ unperturbed fluid velocity} \tag{4.74}$$

Our trial solutions for the perturbed quantities will have the form

$$\sigma_1 = Se^{i(kr-\omega t)} \tag{4.75}$$

$$\text{and} \quad \mathbf{v_1} = v_r\hat{\mathbf{r}} + v_\theta\hat{\theta} \tag{4.76}$$

$$= V_r e^{i(kr-\omega t)}\hat{\mathbf{r}} + V_\theta e^{i(kr-\omega t)}\hat{\theta} \tag{4.77}$$

To use the CE, we will need the divergence of the generic vector $\mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\theta} + A_z\hat{\mathbf{z}}$ in cylindrical coordinates. According your studies of vector calculus (and Eqn' 1B–45 of B&T),

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \tag{4.78}$$

So what is $\nabla \cdot (\sigma_0 \mathbf{v_1})$? and $\nabla \cdot (\sigma_1 \mathbf{v_0})$?
With this in mind, the CE then becomes

$$-i\omega\sigma_1 + \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_0 v_r) = 0 \tag{4.79}$$

$$\text{so} \quad \sigma_1 = \frac{1}{ir\omega}\frac{\partial}{\partial r}(r\sigma_0 v_r) \tag{4.80}$$

Next, lets anticipate that any unstable disturbances in the disk will have a radial wavelength is small compared to the disk radius r,
so $\lambda/r = 2\pi/|kr| \ll 1$.
You will confirm this assumption in Assignment #6.

Lets also assume that the unperturbed disk surface density varies slowly with distance r, perhaps like a power–law: $\sigma_0(r) \propto r^{-\alpha}$ where $\alpha \sim \mathcal{O}(1)$.

Since $|kr| \gg 2\pi$, the CE becomes

$$\sigma_1 = \frac{[-(\alpha - 1) + ikr]\sigma_0 v_r}{ir\omega} \simeq \frac{k\sigma_0 v_r}{\omega} \tag{4.81}$$

Now recall the linearized EE, Eqn' (4.38):

$$\begin{aligned}
\frac{\partial \mathbf{v_1}}{\partial t} + (\mathbf{v_1} \cdot \nabla)\mathbf{v_0} + (\mathbf{v_0} \cdot \nabla)\mathbf{v_1} &= -\nabla(h_1 + \Phi_1) && (4.82)\\
-i\omega\mathbf{v_1} + (\mathbf{v_1} \cdot \nabla)\mathbf{v_0} + (\mathbf{v_0} \cdot \nabla)\mathbf{v_1} &= -\nabla\left(\frac{v_s^2\sigma_1}{\sigma_0} + \Phi_1\right) && (4.83)\\
&= -\nabla\left(v_s^2 - \frac{2\pi G\sigma_0}{|k|}\right)\sigma_1/\sigma_0 && (4.84)\\
&= -i(v_s^2 k - 2\pi G\sigma_0 s_k)(\sigma_1/\sigma_0)\hat{\mathbf{r}} && (4.85)\\
&= -i(v_s^2 k^2 - 2\pi G\sigma_0|k|)(v_r/\omega)\hat{\mathbf{r}} && (4.86)
\end{aligned}$$

since $h_1 = v_s^2\rho_1/\rho_0 = v_s^2\sigma_1/\sigma_0$ in the $z = 0$ plane,
where $\Phi_1 = -2\pi G\sigma_1/|k|$, and $s_k = \mathrm{sgn}(k)$.

So now we are down to a single PDE containing a single unknown, $\mathbf{v_1}$.

But we still have to deal with terms like $(\mathbf{v_1} \cdot \nabla)\mathbf{v_0}$,
which requires the use of Eqn' (1B–54) of B&T:

$$\begin{aligned}
(\mathbf{v_1} \cdot \nabla)\mathbf{v_0} &= -v_\theta\Omega\hat{\mathbf{r}} + v_r\frac{\partial(r\Omega)}{\partial r}\hat{\theta} && (4.87)\\
\text{and}\quad (\mathbf{v_0} \cdot \nabla)\mathbf{v_1} &= -\Omega v_\theta\hat{\mathbf{r}} + \Omega v_r\hat{\theta} && (4.88)
\end{aligned}$$

Please confirm this on your own.

Insert these results into the EE, and consider the $\hat{\theta}$ part of that eqn':

$$v_\theta = \frac{2\Omega + r\frac{\partial\Omega}{\partial r}}{i\omega}v_r = \frac{2(\Omega - A)}{i\omega}v_r = \frac{2iB}{\omega}v_r = -\frac{i\kappa^2 v_r}{2\omega\Omega} \tag{4.89}$$

where the Oort $A\&B$ constants of Eqn's (3.120–3.121) are invoked:
$A = -(r/2)(\partial\Omega/\partial r)$, $B = A - \Omega$, and $\kappa^2 = -4B\Omega$.

Plug this result into the $\hat{\mathbf{r}}$ part of EE:

$$-i\omega v_r - 2\Omega v_\theta = -i\left(\omega - \frac{\kappa^2}{\omega}\right)v_r = -i(v_s^2k^2 - 2\pi G\sigma_0|k|)(v_r/\omega) \tag{4.90}$$

$$\text{so} \quad \omega^2 = v_s^2k^2 - 2\pi G\sigma_0|k| + \kappa^2. \tag{4.91}$$

This is the dispersion relation (DR)
for a gravitating, rotating, pressure–supported disk.

These results apply to a nearly keplerian system (like a circumstellar disk), and a non–keplerian one (like a galactic disk.

Recall that $\omega(k)$ is the frequency of the disk's oscillations due to perturbations having a wavenumber k and wavelength $\lambda = 2\pi/|k|$.

What conditions must be satisfied to be assured of gravitational stability?

Inspect this dispersion relation: is pressure stabilizing or destabilizing?
What about gravity?
And rotation?

Consider two extreme cases:
the non–gravitating disk: is it stable or unstable?
the pressureless (or dynamically cold) disk—is it stable?

Assignment #6
due Thursday April 6
at the start of class

1. a.) Consider a disk that is gravitationally unstable. Show that the fastest–growing unstable mode in this disk has a wavenumber

$$|k_Q| \equiv \frac{\pi G \sigma_0}{v_s^2} \tag{4.92}$$

b.) The stability of a disk can be quantified by its *Toomre stability parameter*,

$$Q \equiv \frac{\kappa v_s}{\pi G \sigma_0} \tag{4.93}$$

(adapted from Toomre, 1964). Show that a disk is stable when $Q \geq 1$, and is gravitationally unstable otherwise.

c.) Show that the fastest–growing disturbance in an unstable disk grows as $S \propto e^{t/\tau_Q}$, where

$$\tau_Q = \frac{1}{\kappa\sqrt{Q^{-2} - 1}} \tag{4.94}$$

is the e–fold timescale for growth. Thus the instabilities in a disk having, say $Q \sim 0.5$ and $\kappa \sim \Omega$, will manifest themselves in an orbital–period timescale.

d.) Show that the disk scale height (ie, the vertical half–thickness of the disk) is $h \sim v_s/\Omega$. Next, recall that we assumed $|kr| \gg 2\pi$ in our derivation of instabilities, and that we still need to confirm the validity of that assumption. Do this by showing that unstable modes do indeed have $|kr| \gg 2\pi$ when $h \ll r$. In other words, our system really needs to be disklike (which requires $h \ll r$), rather than spherical or ellipsoidal (which would have $h \sim r$).

2. a.) Obtain a rough estimate of Q for the solar nebula, which is the circumsolar gas disk from which the Solar System formed. Assume the solar nebula had a mass $M_{disk} \sim 0.01\mathrm{M}_{\odot}$ (which is typical of the disk that form around young stars), and that the bulk of this mass resides interior to Saturn's orbit (where most of the Solar System's mass resides). Treat the nebula as a blackbody whose temperature is determined by solar heating.

b.) Saturn's main A& B rings have a surface density of $\sigma \sim 100$ gm/cm^2. What is the minimum vertical thickness for Saturn's rings, h/r, in fractional units? How does that compare to the fractional thickness of a sheet of paper? Note that the thickness of Saturn's rings probably is close to the limit you obtained here.

3. A resonance trapping problem is also pending...

Spiral density waves

We will examine the physics of a self–interacting system in greater detail. In fact, we have already examined several self–interacting systems:

- sound waves that propagate in a pressure–supported astrophysical fluid;
- a self–gravitating fluids that are susceptible to gravitational instabilities, when they are too cool.

We will now examine the acoustic (ie, pressure) and/or gravity waves that a perturber can launch in a rotating disk—spiral density waves.

We anticipate that these waves might get launched at a Lindblad resonance, since that is a site where disk particles have their eccentricities pumped up by a perturber.

We will see that if those particles at resonance can communicate their disturbed motions to adjacent particles (via pressure and/or gravity), then that disturbance ca propagate away as a wave.

We will tackle the general problem of waves in a variety of disks:

- pressure–dominated gas disks having $Q \gg 1$,
  such as a circumstellar gas disk
- gravity–dominated $Q \sim 1$ particle disks:
  - waves in Saturn's rings that are launched by a satellite;
    (an example of waves in a nearly keplerian system)
  - waves in a disk galaxy that are launched by a central bar;
    (a non–keplerian example)

the linearized fluid EOM

Since the disk is self–gravitating, the system's total gravitational potential is $\Phi = \Phi_0 + \Phi_1 = \Phi_0 + \Phi^d + \Phi^p$,
where Φ^d is the gravitational potential due to the disturbance in the disk,
and Φ^p is the potential due to the perturber
(which could be a planet in a circumstellar disk, or a bar in a galactic disk).

As usual, we will assume weak disturbances: $|\nabla(\Phi^d + \Phi^p)| \ll |\nabla\Phi_0|$.

The fluid velocities are again $\mathbf{v} = \mathbf{v_0} + \mathbf{v_1}$,
where $\mathbf{v_0} = r\Omega\hat{\theta}$ is the fluid's undisturbed circular velocity,
and $\mathbf{v_1}$ is the velocity of the fluid's perturbed motions.

We will also assume the disk is thin, so that $\rho(r,\theta,z,t) = \sigma(r,\theta,t)\delta(z)$, where $\sigma(\mathbf{r},t) = \sigma_0(\mathbf{r}) + \sigma_1(\mathbf{r},t) =$ unperturbed + perturbed surface densities.

The perturber's gravitational potential can always be Fourier expanded as

$$\Phi^p(r,\theta,t) = \sum_{m=0}^{\infty} \phi_m^p(r) e^{im(\theta - \Omega_{ps}t)} \tag{4.95}$$

(see Eqn' 3.122, and note our switch to complex notation)
where Ω_{ps} is the pattern speed.

From our earlier discussion of Lindblad resonances,
we know that if the perturber is an orbiting secondary (ie, planet),
then Ω_{ps} = secondary's angular velocity Ω_s.

And if the perturber is a bar, then Ω_{ps} = bar's rotational angular velocity.

We also know that Lindblad resonances tend to be segregated spatially,
so we can assume that a fluid parcel orbiting near the m^{th} resonance
senses only the m^{th} term in the sum:

$$\Phi^p(r,\theta,t) \simeq \phi^p_m(r)e^{im(\theta-\Omega_{ps}t)}. \tag{4.96}$$

These perturbations are sinusoidal in time and azimuth,
and we anticipate the disk will to respond similarly,
with perturbed quantities having the form:

$$\sigma_1(r,\theta,t) = S(r)e^{im(\theta-\Omega_{ps}t)} = \text{perturbed disk surface density} \tag{4.97}$$

$$\mathbf{v_1} = v_r\hat{\mathbf{r}} + v_\theta\hat{\theta} = \text{p'ed fluid velocities} \tag{4.98}$$

$$= V_r(r)e^{im(\theta-\Omega_{ps}t)}\hat{\mathbf{r}} + V_\theta(r)e^{im(\theta-\Omega_{ps}t)}\hat{\theta} \tag{4.99}$$

$$\Phi^d(r,\theta,t) \simeq \phi^d_m(r)e^{im(\theta-\Omega_{ps}t)} = \text{p'ed gravitational potential.} \tag{4.100}$$

The linearized CE for a 2D disk is Eqn' (4.73):

$$\frac{\partial\sigma_1}{\partial t} + \nabla\cdot(\sigma_0\mathbf{v_1}) + \nabla\cdot(\sigma_1\mathbf{v_0}) = 0 \tag{4.101}$$

where again we need the divergence of $\sigma_0\mathbf{v_1}$ and $\sigma_1\mathbf{v_0}$ using Eqn' (4.78).

What is $\nabla\cdot(\sigma_0\mathbf{v_1})$?

$\nabla\cdot(\sigma_1\mathbf{v_0})$?

Since $\partial/\partial t \to -im\Omega_{ps}$, the CE becomes

$$\text{CE:}\quad i\omega_m\sigma_1 + \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_0 v_r) + \frac{im\sigma_0}{r}v_\theta = 0 \tag{4.102}$$

where $\omega_m = m(\Omega - \Omega_{ps})$ is the familiar doppler–shifted forcing frequency, Eqn' (3.132).

The linearized EE for a disk is Eqn' (4.82)

$$\frac{\partial \mathbf{v_1}}{\partial t} + (\mathbf{v_1}\cdot\nabla)\mathbf{v_0} + (\mathbf{v_0}\cdot\nabla)\mathbf{v_1} = -\nabla\left(\frac{v_s^2}{\sigma_0}\sigma_1 + \Phi^d + \Phi^p\right) \tag{4.103}$$

Use Eqn' (1B–54) of B&T to evaluate the *convective operators*:

$$(\mathbf{v_1}\cdot\nabla)\mathbf{v_0} = -\Omega v_\theta \hat{\mathbf{r}} + v_r\frac{\partial(r\Omega)}{\partial r}\hat{\theta} \tag{4.104}$$

$$\text{and}\quad (\mathbf{v_0}\cdot\nabla)\mathbf{v_1} = \Omega(imv_r - v_\theta)\hat{\mathbf{r}} + \Omega(imv_\theta + v_r)\hat{\theta}. \tag{4.105}$$

Be sure that you are able to confirm this step on your own.

The radial and angular parts of the EE thus becomes

$$\hat{\mathbf{r}}\cdot\text{EE:}\quad i\omega_m v_r - 2\Omega v_\theta = -\frac{\partial}{\partial r}\left(\frac{v_s^2}{\sigma_0}\sigma_1 + \Phi^d + \Phi^p\right) \tag{4.106}$$

$$\hat{\theta}\cdot\text{EE:}\qquad \left[\frac{\partial(r\Omega)}{\partial r} + \Omega\right]v_r + i\omega_m v_\theta = \frac{\kappa^2}{2\Omega}v_r + i\omega_m v_\theta = -\frac{im}{r}\left(\frac{v_s^2}{\sigma_0}\sigma_1 + \Phi^d + \Phi^p\right) \tag{4.107}$$

where the $[\,] = 2(\Omega - A) = -2B = \kappa^2/2\Omega$ according to page 24.

Lastly, the linearized PE is

$$\nabla^2\Phi^d = 4\pi G\rho_1 \tag{4.108}$$

So we have three nasty–looking PDEs (CE, $\hat{\mathbf{r}}\cdot$EE, $\hat{\theta}\cdot$EE) to describe our four unknowns: $v_r, v_\theta, \Phi^d, \sigma_1$.

These eqn's simplify considerably in the *tight–winding* limit, which assumes that the waves' radial wavelength λ is smaller that the disk scale–length r.

Equivalently, we will assume that $|kr| \gg 2\pi$, where wavenumber $|k| = 2\pi/\lambda$.

The tight–winding approximation approximation is an excellent one for studies of spiral waves in planetary rings.
For instance, waves in Saturn's rings have $\lambda \sim 10^{-4}r$.

However this assumption is only marginally satisfied in a circumstellar gas disk, which has density wavelengths of $\lambda \sim 0.2r$.

For galactic spirals, the tight–winding approximation is clearly questionable, since $\lambda \sim r$ in these systems. Despite this, we will still make this assumption when we apply our findings to a spiral galaxy.

It turns out that if you compare the linearized theory for spiral waves, obtained in the tight–winding limit, and then compare those results to a more exact theory (obtained numerically, perhaps), our results will still be a good indicator of wave phenomena in spiral galaxies.

Because of this, it has been said that the tight–winding approximation approximation, $|kr| \gg 2\pi$, works better than we deserve...

The PE for tightly–wound spiral waves is

$$\nabla^2\Phi^d = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi^d}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi^d}{\partial\theta^2} + \frac{\partial^2\Phi^d}{\partial z^2} \tag{4.109}$$

$$\simeq \frac{\partial^2\Phi^d}{\partial r^2} + \frac{\partial^2\Phi^d}{\partial z^2} = 4\pi G\sigma_1\delta(z) \tag{4.110}$$

Recall that we obtained the same eqn' when we considered gravitational instabilities in a disk, so we anticipate that
$\Phi^d \propto e^{-|kz|=-s_z|k|z}$ where $s_z =$ sgn(z), and $k(r)$ is the wavenumber.

Thus when $z \neq 0$, the PE tells us

$$\frac{\partial^2\Phi^d}{\partial z^2} = |k|^2\Phi^d \tag{4.111}$$

$$\text{so} \quad \frac{\partial^2\Phi^d}{\partial r^2} = -|k|^2\Phi^d \tag{4.112}$$

$$\text{which can satisfied by} \quad \frac{\partial\Phi^d}{\partial r} = ik\Phi^d \tag{4.113}$$

In fact, if k were a constant, then Φ^d would be sinusoidal in r, ie, wavelike. But keep in mind that k might vary with r...

Since $k(r)$ can vary with r, the above can be solved via a *WKB* approximation; WKB is shorthand for the solution to Schrödinger eqn' given by Wentzel, Kramers, & Brillioun:

$$\Phi^d(r,\theta,t) = A(r,\theta,t)e^{i\int_{r_0}^{r}k(r')dr'} \tag{4.114}$$

where $k(r)$ is the wavenumber,
and $A(r,\theta,t)$ is the amplitude of the wave of wavelength $\lambda = 2\pi/|k|$,
and $r_0 =$ resonance radius.

In the WKB approximation, you assume that the wave amplitude $A(r)$ varies slowly over spatial scales much larger than the wavelength λ,
so $\partial\Phi^d/\partial r \simeq ik\Phi^d$, as is required above.

We will also assume that *all* the perturbed quantities, σ_1, v_r, v_θ have a WKB form.

With these thoughts in hand, we can again use our trick of vertically integrating the PE over some tiny distance $a \ll \lambda$,
and then taking the limit as $a \to 0$:

$$\left.\frac{\partial \Phi^d}{\partial z}\right|_{z=a} + \left.\frac{\partial \Phi^d}{\partial z}\right|_{z=-a} = -2|k|\Phi^d = 4\pi G\sigma_1 \tag{4.115}$$

$$\text{so} \quad \sigma_1 = -\frac{|k|\Phi^d}{2\pi G} \tag{4.116}$$

$$\text{but} \quad k\Phi^d = -i\frac{\partial \Phi^d}{\partial r} \tag{4.117}$$

$$\text{so} \quad \sigma_1 = \frac{is_k}{2\pi G}\frac{\partial \Phi^d}{\partial r} \tag{4.118}$$

where $s_k = \mathrm{sgn}(k)$. This is our linearized PE in the tight–winding limit, upon making the WKB approximation.

Now write the CE and EE in the tight–winding limit:

$$\text{CE:} \quad i\omega_m\sigma_1 + \sigma_0\frac{\partial v_r}{\partial r} + \frac{im\sigma_0}{r}v_\theta = 0 \tag{4.119}$$

$$\hat{\mathbf{r}}\cdot\text{EE:} \quad \frac{v_s^2}{\sigma_0}\frac{\partial \sigma_1}{\partial r} + i\omega_m v_r - 2\Omega v_\theta = -\frac{\partial}{\partial r}\left(\Phi^d + \Phi^p\right) \tag{4.120}$$

$$\hat{\theta}\cdot\text{EE:} \quad \frac{imv_s^2}{r\sigma_0}\sigma_1 + \frac{\kappa^2}{2\Omega}v_r + i\omega_m v_\theta = -\frac{im}{r}\left(\Phi^d + \Phi^p\right) \tag{4.121}$$

This is a system of 3PDEs in three unknowns: the disk's perturbed velocities v_r, v_θ, and the surface density $\sigma_1 \propto \partial\Phi^d/\partial r$.

The general solution to this problem is given in Goldreich & Tremaine (1978). Although this problem is analytic, the solution is a bit complicated, and we won't do it here...

There are, however, two useful limiting cases:

(i.) A pressure–dominated disk where self–gravity is unimportant. Circumstellar gas disks around young stars, which have a stability parameter $Q \gg 1$, are pressure–dominated. This limit is obtained by setting $\Phi^d = 0$ in the above EOM. The solution is given in Ward (1986), which shows that pressure–driven spiral density waves are launched at a LR.

(ii.) A gravity–dominated disk having $Q \sim 1$ can have gravity–driven spiral density waves launched at LRs in the disk. This limit is obtained by setting $v_s = 0$. Gravity–dominated waves occur in planetary rings, and can also occur in a stellar galactic disk. The solution in this limit is given in Shu (1984).

Solving the amplitude of a spiral wave requires solving eqn's 4.119 for σ_1. With that solution, you could then calculate the torque that the perturber exerts on the disk, and then consider how the disk & perturber mutually *shepherd* each other (ie, push each other around) due to the resulting angular momentum exchanged between disk & perturber.

If the perturber were a planet, you could then assess the rate at which the disk might drive *type I* planet migration. But if the perturber is massive enough to open a gap in the disk, then type I motion stalls, and *type II* migration begins. These issues are of great importance to studies of planet and satellite formation, as well as the origin of the orbits of extra–solar planets.

However, will simply solve for the dispersion relation for spiral waves, which will tell us about the properties of these waves, *downstream* of the resonance.

Assume that our perturber has already managed to launch a wave at a LR, and lets focus our attention to the *downstream* part of the wave that is already far from resonance.

Consider the EOM for the disk at site where the perturber's influence is now small compared to disk's internal forces.
Do this by setting $\Phi^p = 0$ in the EOM, eqn' (4.119).

As usual, the DR is obtained by inserting our assumed solutions for the disk's perturbed quantities, S, Φ^d, V_r, V_θ, into the EOM (4.119) all of which are assumed to have the WKB form

$$\sigma_1(r,\theta,t) = S(r)e^{i\int_{r_0}^{r} k(r')dr + im\theta - im\Omega_{ps}t}, \tag{4.122}$$

$$\tag{4.123}$$

But first, lets confirm that the above WKB form can indeed represent a spiral density wave.

Do this by mentally putting your finger at some spot (r, θ) in a spiral arm, where $\sigma_1(r, \theta)$ is maximal. Then move your finger outwards a small radial distance Δr and a small angular distance $\Delta\theta$, such that the spiral arm stays under your finger.

As you trace out the spiral arm,
the surface density under your finger should stay constant, so

$$\sigma_1(r + \Delta r, \theta + \Delta\theta, t) \simeq \sigma_1(r, \theta, t)e^{i(k\Delta r + m\Delta\theta)} \simeq \text{ constant} \tag{4.124}$$

$$\text{so} \quad \frac{\Delta\theta}{\Delta r} \simeq -\frac{k}{m} \tag{4.125}$$

The sketches show that you get an m–armed ‘leading’ spiral pattern when the wavenumber $k < 0$, and a ‘trailing’ spiral pattern when $k > 0$.

Now lets derive the DR for spiral density waves. Begin by noting that derivatives of our perturbed quantities that have the WKB form yield

$$\frac{\partial\sigma_1}{\partial r} \simeq ik\sigma_1 \text{ in the tight–winding limit} \tag{4.126}$$

$$\text{so the CE is} \quad \sigma_1 + (krv_r + mv_\theta)\frac{\sigma_0}{r\omega_m} \simeq 0 \tag{4.127}$$

Which term in the parentheses dominates?

To answer that,
you first have to know how the radial v_r compares to the tangential v_θ.

How do they compare for an isolated particle orbiting at a LR?
See eqn's 3.143–3.145.

Thus in the tight–winding limit ($|kr| \gg 2\pi$),

$$v_r \simeq -\frac{\omega_m}{k}\frac{\sigma_1}{\sigma_0} \tag{4.128}$$

Next, plug this result into the radial EE, first noting that

$$\frac{\partial\Phi^d}{\partial r} = -is_k 2\pi G\sigma_1 \quad \text{(from Eqn' 4.118)}, \tag{4.129}$$

$$\text{which yields} \quad v_\theta = \frac{i}{2\Omega k}\left(v_s^2k^2 - 2\pi G\sigma_0|k| - \omega_m^2\right)\frac{\sigma_1}{\sigma_0} \tag{4.130}$$

Next, eliminate Φ^d from the EOM using eqn' (4.116): $\Phi^d = -2\pi G\sigma_1/|k|$.

Inserting all these results into the $\hat{\theta}$ part of EE then yields

$$\left(1 - \frac{2im\Omega}{kr\omega_m}\right)(2\pi G\sigma_0|k| - v_s^2k^2) = D(r) = \kappa^2 - \omega_m^2 \tag{4.131}$$

after a bit of algebra.

$D(r)$ of course is the familiar frequency distance from exact resonance.

How do the terms in the first parentheses compare in the tight–winding limit?

$$\text{Then} \quad \omega_m^2 \simeq v_s^2 k^2 - 2\pi G\sigma_0|k| + \kappa^2 \tag{4.132}$$

Lastly, note that the time dependence of all perturbed quantities vary as $\sigma_1 \propto e^{-im\Omega_{ps}t} = e^{-i\omega t}$ where $\omega \equiv m\Omega_{ps}$ is the angular rate at which the spiral density pattern rotates.

Since $\omega_m = m(\Omega - \Omega_{ps}) = m\Omega - \omega$,
the DR for spiral density waves can be written

$$(\omega - m\Omega)^2 = v_s^2 k^2 - 2\pi G\sigma_0|k| + \kappa^2 \tag{4.133}$$

What is the azimuthal wavenumber m of an axi–symmetric disturbance (one that is ring–like) in the disk?

Note that when we consider an axi–symmetric waves in our disk, we recover the DR for gravitational instabilities in the disk, Eqn' (4.91).

Now lets correct an earlier error.

In our derivation of instabilities in disk, I claimed on page 20 that we did not need to consider non–axisymmetric disturbances having $m \geq 1$ because disks were stable against such perturbations.

That is incorrect. Eqn' (4.133) shows that non–axisymmetric $m \geq 1$ disturbances are no more destabilizing than axisymmetric disturbances having $m = 0$.

Graphical analysis of the spiral wave DR

The DR just derived contains *lots* of information about the properties of spiral density waves.

We can extract much of that info by simply sketching the DR, Eqn' (4.131):

$$D(r) = \kappa^2 - \omega_m^2 = 2\pi G\sigma_0|k| - v_s^2k^2 \tag{4.134}$$

where $D(r)$ is the distance from a LR where $D(r) = 0$,
and $\omega_m = m\Omega - \omega = \epsilon\kappa$,
where $\epsilon = +1(-1)$ at the inner (outer) LR.

Another useful quantity is the waves' *group velocity*: $v_g = \partial\omega/\partial k$. If you recall your past studies of traveling waves, the group velocity is the rate at which energy (or angular momentum) is transmitted by a traveling wave.

Note that the group velocity is easily confused with the wave's *phase velocity*: $v_p = \omega/k$.

The distinction between the two is explained in Appendix 1.E.4 of B&T.

A spiral wave is launched somewhere in the disk were $D(r) = 0$, and it travels 'downstream' where $D(r) \neq 0$.

A sketch of the DR shows that it admits 1 or 2 possible solutions:

- a small $|k|$ solution—'long' waves, since wavelength $\lambda = 2\pi/|k|$
- a large $|k|$ solution—'short' waves

The waves' group velocity can be obtained from

$$\frac{dD}{dk} = -2\omega_m \frac{d\omega_m}{dk} = 2\omega_m \frac{d\omega}{dk} = 2\omega_m v_g = s_k 2\pi G\sigma_0 - 2v_s^2 k \tag{4.135}$$

$$\text{so} \quad v_g = \frac{s_k \pi G\sigma_0 - v_s^2 k}{\omega_m} \simeq \frac{\epsilon dD/dk}{2\kappa} \tag{4.136}$$

where the RHS is approximately true for waves near the LR where $\omega_m \simeq \epsilon\kappa$.

Note that the *direction* of wave propagations is set by the sign of v_g: $\text{sgn}(v_g) = \epsilon\text{sgn}(dD/dk)$.

Inspection of the DR shows that long waves have $\text{sgn}(dD/dk) = +1$, so long waves have $\text{sgn}(v_g) = \epsilon$,

and that short waves have $\text{sgn}(dD/dk) = -1$ and thus $\text{sgn}(v_g) = -\epsilon$.

We will make use of this later...

Set $v_s = 0$ to consider spiral waves in a gravity dominated disk.

Lets consider spiral waves in the vicinity of a LR in a nearly keplerian disk.

Our results would thus apply to waves in Saturn's rings.

According to Eqn' (3.181),

$$D(x) \simeq 3\epsilon(m - \epsilon)\Omega^2 x = 2\pi G\sigma_0|k| > 0 \tag{4.137}$$

where $x = \Delta r/r =$ fractional distance from resonance.

Evidently, gravity–dominated spiral waves only propagate where
D and $\epsilon x > 0$.

Also note that $\mathrm{sgn}(v_g) = \epsilon$, which indicates a long spiral wave.

The sketch shows that a LR in a gravity–dominated launches long spiral density waves that propagate towards the CR radius
(ie, towards the perturbing satellite, if we are considering Saturn's rings).

The group velocity for a gravity–dominated wave that is near a LR (where $\omega_m \simeq \epsilon\kappa$) is

$$v_g \simeq s_k \epsilon \frac{\pi G \sigma_0}{\kappa} \tag{4.138}$$

so the direction of propagation is $\text{sgn}(v_g) = s_k \epsilon = \epsilon$.

Thus s_k =sgn$(k) = +1 \Rightarrow$ a LR in a gravity–dominate disk launches *long, trailing* spiral density waves ($k > 0$) that propagate towards CR.

This DR indicates a long spiral wave that is launched at the LR (where $D = 0$), and that the wave propagates in the $D > 0$ zone in the disk.

But eventually, the wave encounters a site where dD/dk turns over. This site is known as the *Q–barrier*.

Since $v_g \propto dD/dk$, this site is a turning point, which causes the wave to reflect as a short wave, which now propagates towards the launching LR.

The Q–barrier's location in the disk is determined by v_s.

Increasing v_s does what to the Q–barrier—move it close to the LR? or closer to CR?

Recall that for a gas disk, v_s = sound speed, which is about equal random velocity of a typical gas molecule.

Similarly, in a particle disk (Saturn's rings, a stellar disk),
v_s = particles' typical random velocity.

Evidently, there is a *forbidden zone* that is concentric with the CR; waves do not enter the forbidden zone, but reflect at its edges, at the Q–barrier.

The DR shows that the reflected waves are short leading (k¡0) waves that head back to the launching LR.

In Saturn's rings, collisional viscosity damps out the long density waves long before they reach their Q–barrier.

The DR for pressure–dominated spiral waves is obtained by setting $\sigma_0 = 0$,

$$\text{so} \quad D = -v_s^2 k^2 \tag{4.139}$$

$$\text{and} \quad \text{sgn}(D) = \epsilon\text{sgn}(x) = -1 \tag{4.140}$$

$\Rightarrow$ pressure waves propagate where $\epsilon x < 0$, ie, opposite to gravity waves:

The group velocity for spiral waves in a pressure–dominated disk is

$$v_g \simeq -\epsilon \frac{v_s^2 k}{\kappa} \tag{4.141}$$

near the LR ($\omega_m \simeq \epsilon\kappa$).

Although our linearized theory says that these waves can travel off to infinity, nonlinear shocks probably dissipate these waves in a circumstellar gas disk.

Assignment #6
due Thursday April 6
at the start of class

3. Show that the Q–barrier lies a fractional distance

$$x_Q \simeq \frac{1}{3mQ^2} \tag{4.142}$$

from a $m \gg 1$ Lindblad resonance in a nearly keplerian disk.

This explains why a high–Q disk does not sustain gravity–dominated spiral waves—they have no place to go. Instead, such waves escape the resonance by propagating in the opposite direction, as short wavelength pressure waves.

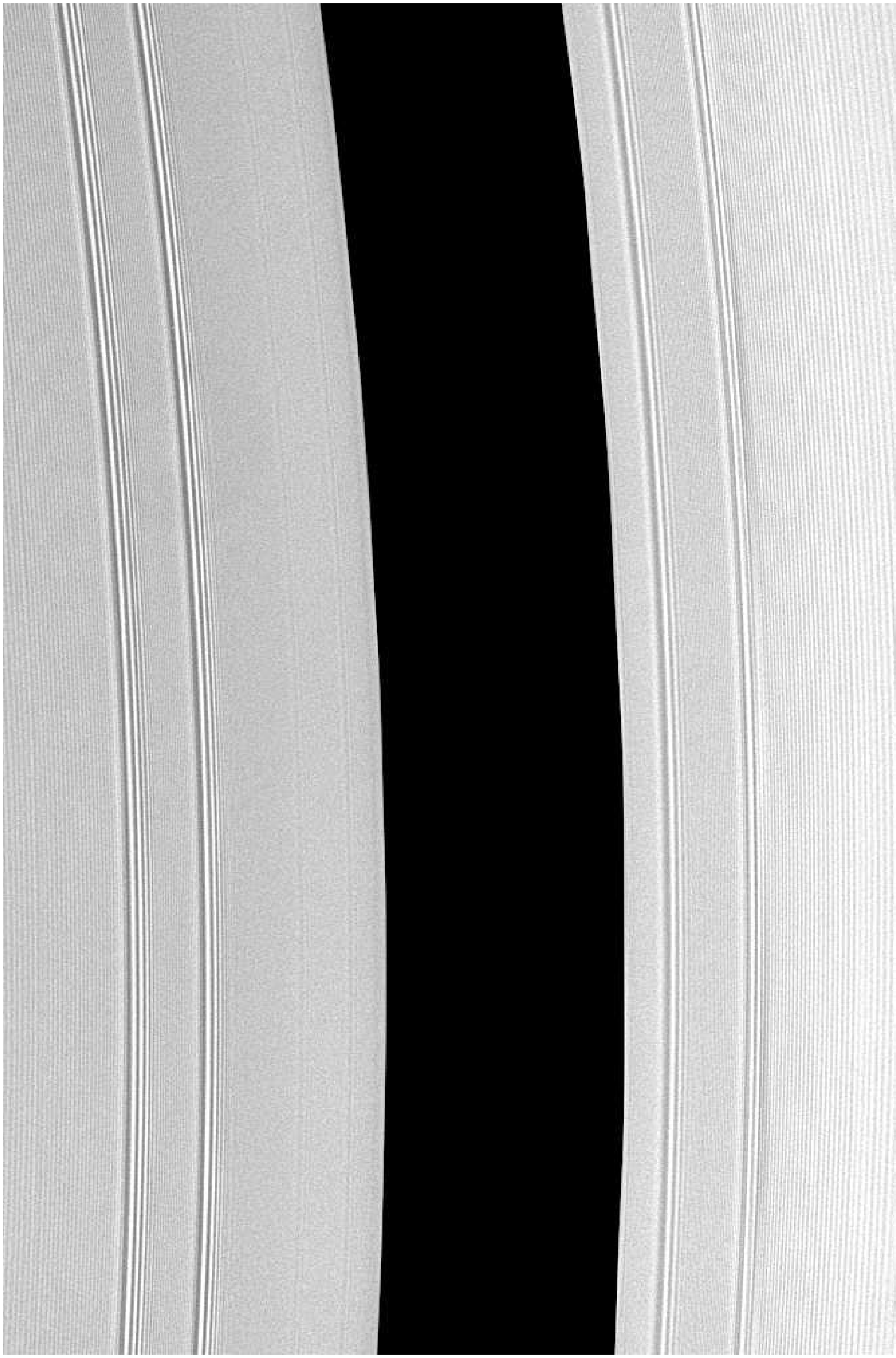

Figure 4.1: The Encke gap as seen by Cassini. Saturn is far to the left. The pairs of bands on either side of the gap are spiral density waves launched by Pandora and Prometheus, small satellites to the right, outside the main rings. Note that these gravity–dominated density waves do indeed propagate towards CR, and that their wavelengths shrink with distance from the LR (a feature you will derive in problem 4). The constant-wavelength features on the right are Pan's wakes, due to Pan's passage through this region about 4 months ago.

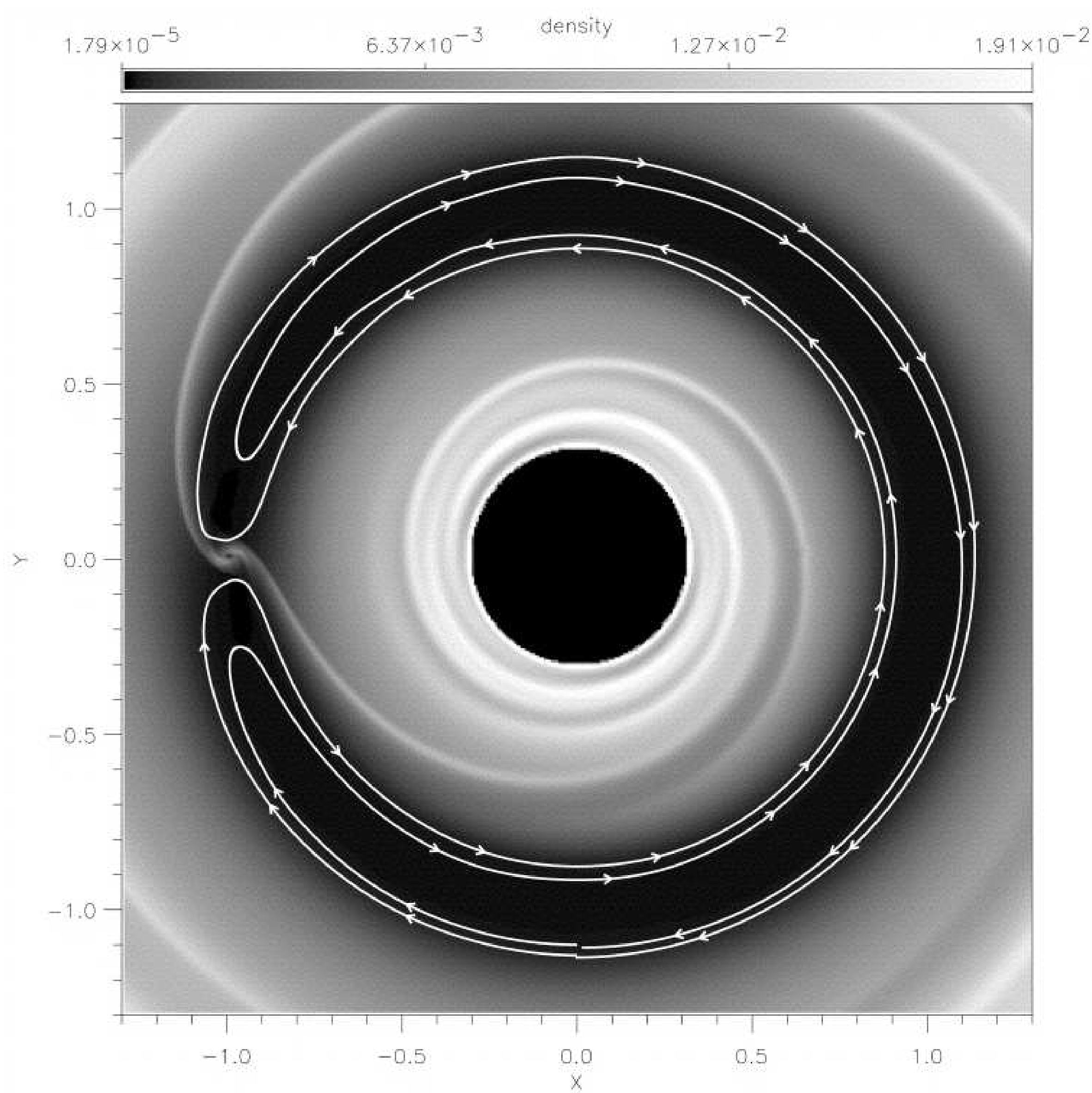

Figure 4.2: A hydrodynamic simulation of a Jupiter–mass planet orbiting within a $0.02\mathrm{M}_{\odot}$ circumstellar gas disk, by Lubow et al (1999). Note that these pressure–dominated waves propagate away from CR. Although we call these 'short' waves, their wavelengths are much longer than the 'long' gravity waves seen in Saturn's rings. Note that the gravitational torque from the planet is enough to shepherd open a gap about its orbits, despite the viscosity in the disk, which wants to close that gap. Nonetheless, some disk matter manages to stream into the gap (indicated by the horseshoe orbits), which also feed a circumplanetary disk, where satellites might presumably form.

Assignment #6
due Thursday April 6
at the start of class

4. The following problems solve for the amplitude and wavelength of spiral waves launched by a perturber in a gravity–dominated, nearly keplerian disk.

a.) Use the CE and PE to show that the fluid's forced radial velocity obeys

$$v_r \simeq \frac{\omega_m \Phi^d}{2\pi G \sigma_0} \tag{4.143}$$

in the tight–winding limit (TWL).

b.) Then use the EE to show that

$$v_r \simeq -\frac{i\omega_m}{D}\left(\frac{\partial}{\partial r} + \frac{2m\Omega}{r\omega_m}\right)(\Phi^p + \Phi^d) \tag{4.144}$$

c.) Combine these results to obtain a single PDE for the disk's potential that describes the wave:

$$\frac{\partial \Phi^d}{\partial r} - \frac{iD}{2\pi G\sigma_0}\Phi^d \simeq -\Psi_m \tag{4.145}$$

where Ψ_m is the usual forcing function.

d.) Show that the above EOM can be written as

$$\frac{\partial \Phi^d}{\partial x} - \frac{ix}{\gamma}\Phi^d \simeq -a_m \Psi_m \tag{4.146}$$

where $\Phi^d(x)$ is now regarded as a function of the fractional distance from resonance x, a_m is the radius of the m^{th} LR, and the constant $\gamma = 2\epsilon\mu_d/3(m-\epsilon)$, where $\mu_d = \pi\sigma_0 r^2/m_1$ is the normalized disk mass of Eqn' (2.143).

e.) Solve the above eqn' using the method of integrating functions. Then show that your solution can be recast in terms of a new radial variable $\xi \equiv \epsilon x/\sqrt{2|\gamma|}$, so that

$$\begin{aligned} \Phi^d(\xi) &= -\epsilon\sqrt{2\pi\gamma}a_m\psi_m H_\epsilon(\xi) \\ \text{where} \quad H_\epsilon(\xi) &\equiv \frac{1}{\sqrt{\pi}}e^{i\epsilon\xi^2}\int_{-\infty}^{\xi} e^{-i\epsilon\tau^2}d\tau. \end{aligned} \tag{4.147}$$

This is a convenient form, because the function $|H_\epsilon(\xi)| \rightarrow 1$ far downstream of the resonance, while $|H_\epsilon(\xi)| \simeq 0$ far away on the 'non–wave' side of the resonance. You are encouraged to convince yourself of that by using MAPLE (or similar) to plot $|H_\epsilon(\xi)|$. Ask me how.

5. Show that the amplitude of a spiral density wave launched at an $m \gg 1$ LR in a gravity–dominated, nearly keplerian disk is

$$\left|\frac{\sigma_1}{\sigma_0}\right| \simeq |x| f \mu_s \sqrt{\frac{3m^3}{\pi\mu_d^3}} \tag{4.148}$$

where μ_s is the perturber's mass in units of the primary's, and $|\sigma_1/\sigma_0|$ is the fractional change in the disk's surface density due to the wave. Note that the wave amplitude is expected to *grow* as the wave propagates away from resonance. Is that happening to the waves seen in Fig. 4.1? Why or why not?

6. The first wavelength λ_1 is the radial separation between the first two adjacent arms in the spiral wave pattern. Use Eqn's (4.147) to show that the the first wavelength in a gravity–dominated, nearly keplerian disk is

$$\lambda_1 = \sqrt{\frac{8\pi\mu_d}{3(m-\epsilon)}}a_m. \tag{4.149}$$

What is λ_1 (in km) for the spiral density waves seen in Fig. 4.1, assuming that the waves were launched by Pandora or Prometheus, and that Saturn's A ring has $\sigma \sim 100$ gm/cm^2.

7. resonance trapping problem...

Viscous Disk Evolution

Astrophysical disks experience differential rotation because their circular velocity $v(r) = r\Omega$ varies with radius r in the disk,
causing fluid parcels in the disk to slide past each other.

There are usually frictional forces in these disks, and such forces usually attempt to 'brake' this differential motion. This friction saps the system of its orbital energy, causing the orbits of the fluid parcels to decay.

In a circumstellar gas disk, that friction is thought to be due to turbulence in the disk—perhaps driven by vertical convection in the disk,
or maybe an MHD instability.

In a planetary ring, friction is a consequence of collisions among ring particles.

Note that collisions are generally too rare to be of consequence in a stellar disk. But stars can gravitationally scatter each other, and its effects can be loosely treated as a viscosity in a fluid.

Collisions and scattering tend to covert ordered orbital energy into disordered energy—they 'heat up' the disk, and that heat can be radiated away. Thus friction in the disk reduces the disk's energy, which implies an inwards flow of matter.

But keep in mind that a disk must also conserve angular momentum $\mathbf{L}$.

Even though the net mass flux is radially inwards,
$\mathbf{L}$ conservation still requires some mass to flow outwards, too
$\Rightarrow$ friction causes a disk to spread.

We usually describe these frictional forces in the disk by its
kinematic shear viscosity ν;

Note that although the *net* mass flux in the disk tends tends to be inwards, the outer parts of the disk must spread outwards, too...
unless the disk hits a barrier, such as a resonance with a perturber...

the EOM for a viscous disk

The usual approach is to add to the RHS of the EE
the acceleration on a fluid element that is due to the disk's kinematic shear viscosity ν:

$$\nu\nabla^2\mathbf{v} + \frac{1}{3}\nu\nabla(\nabla\cdot\mathbf{v}); \tag{4.150}$$

where $\mathbf{v}(\mathbf{r},t)$ is the usual Eulerian fluid velocity.

The resulting EOM is known as the Navier–Stokes eqn'.

Note that there are other sources of viscosity (like the fluid's *bulk viscosity*) which can contribute additional terms; we won't be considering those here.

The derivation of the term in Eqn' (4.150) can be found in any text on fluid dynamics. But it is a bit involved, and we won't pursue it here.

Instead, we will use methods described in a very nice review paper by Pringle (1981), who derives another useful eqn':
a continuity eqn' for the disk's angular momentum.

That eqn' + the usual mass CE can then be used to solve for the disk's surface density $\sigma(r,t)$ and velocity $\mathbf{v}(r,t)$.

ang' mom' continuity

Lets suppose that our disk has a small vertical half–width h,
so the disk's volume density is $\rho = \sigma/2h$.

The angular momentum in some small volume ΔV is $\mathbf{\Delta L} = (\rho\Delta V)\mathbf{r}\times\mathbf{v}$,
so $\vec{\ell}_V \equiv \mathbf{\Delta L}/\Delta V = \rho\mathbf{r}\times\mathbf{v}$ is the *volume density* of ang' mom' in the disk.

For a flat disk that has no vertical motions, $\mathbf{v} = v_r\hat{\mathbf{r}} + r\Omega\hat{\theta}$,
and $\vec{\ell}_V = \rho r^2\Omega\hat{\mathbf{z}} \equiv \ell_V\hat{\mathbf{z}}$.

The ang' mom' *flux density* is $\mathbf{j} = \ell_V\mathbf{v}$,
and the ang' mom' *flux* through surface $\mathbf{A} = A\hat{\mathbf{n}}$ is $\mathbf{j}\cdot\mathbf{A} = \ell_V A\mathbf{v}\cdot\hat{\mathbf{n}}$,
where $\hat{\mathbf{n}}$ is the unit vector normal to area A.

Now lets consider the angular momentum content of some volume V
that is bounded by surface S:

Since $L = \int_V \ell_V dV$ = total ang' mom' in volume V, its rate of change is

$$\frac{dL}{dt} = \int_V \frac{\partial \ell_V}{\partial t} dV = -F + T \tag{4.151}$$

$$\text{where} \quad F = \int_S \mathbf{j} \cdot \mathbf{da} = \int_V \nabla \cdot (\ell_V \mathbf{v}) dV \tag{4.152}$$

is the rate that ang' mom' flows out of V through surface S, making use of the divergence theorem, Eqn' (4.7).

The T term is the rate at which ang' mom' is deposited in V due to external perturbations (ie, the torque on V):

$$T = \int_V \tilde{t} dV \tag{4.153}$$

where $\tilde{t}$ is the *torque volume density* due to external perturbations.
Putting all this together yields

$$\int_V \left(\frac{\partial \ell_V}{\partial t} + \nabla \cdot (\ell_V \mathbf{v}) - \tilde{t} \right) = 0 \tag{4.154}$$

$$\Rightarrow \quad \frac{\partial \ell_V}{\partial t} + \nabla \cdot (\ell_V \mathbf{v}) = \tilde{t} \tag{4.155}$$

since the volume V is arbitrary.
This is the 3D continuity eqn' for the disk's angular momentum.

For many astrophysical disks, σ, ℓ_V ,and the external torque density $\tilde{t}$ are axially symmetric (independent of θ), or nearly so.

Since our disk is also thin, lets simplify by making the above eqn' 2D:
set $\ell_V = \ell/2h$ where ℓ is the disk's ang' mom' *surface density*,
so when we vertically integrate the above eqn' through the disk,

$$\frac{\partial \ell}{\partial t} + \nabla \cdot (\ell \mathbf{v}) = 2h\tilde{t} \tag{4.156}$$

Then let $\partial T/\partial r$ = torque radial density (eg, eqn' 2.141) such that

$$\frac{\partial T}{\partial r}\Delta r = \text{ torque on annulus of radial width } \Delta r \tag{4.157}$$

$$= \tilde{t}2\pi r \cdot 2h\Delta r \quad \rightarrow 2h\tilde{t} = \frac{1}{2\pi r}\frac{\partial T}{\partial r} \tag{4.158}$$

$$\text{and} \quad \frac{\partial \ell}{\partial t} + \nabla \cdot (\ell \mathbf{v}) = \frac{1}{2\pi r}\frac{\partial T}{\partial r} \tag{4.159}$$

is the ang' mom' CE for our axially symmetric 2D disk.

The fluid velocity in an axially symmetric disk is $\mathbf{v} = v_r(r,t)\hat{\mathbf{r}} + r\Omega(r,t)\hat{\theta}$, so the above becomes

$$\frac{\partial \ell}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\ell v_r) = \frac{1}{2\pi r}\frac{\partial T}{\partial r}, \tag{4.160}$$

where $\ell = \sigma r^2 \Omega$.

The other EOM is the mass CE, Eqn' (4.10),
which is similar but with the RHS=0:

$$\frac{\partial \sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\sigma v_r) = 0 \tag{4.161}$$

the viscous torque density

Lets derive $\left.\frac{\partial T}{\partial r}\right|_{\nu}$, which is the radial torque density due to the disk's viscosity.

Begin by dividing up the disk into nested annuli that have a small radial width Δr and vertical half–thickness $2h$:

Let $F/A =$ frictional force per area that one annulus exerts on another one, where $A = 2\pi r 2h =$ area of the border between between annuli.
If $\Delta\Omega =$ difference in angular velocities of adjacent annuli, then we might expect $F \propto r(d\Delta\Omega/\Delta r)$, ie

$$\frac{F}{A} = \eta r \frac{\Delta\Omega}{\Delta r} \rightarrow \eta r \frac{\partial\Omega}{\partial r} \tag{4.162}$$

where the proportionality constant η is the *shear viscosity*.

This should seem reasonable,
since we expect $F \rightarrow 0$ in a rigidly–rotating disk ($\Omega =$ constant).

Since viscosity η is usually proportional to amount of matter in the disk, ie, its density ρ, we usually set $\eta = \rho\nu$, where ν is the *kinematic shear viscosity*.

The total torque on one annulus due to its neighbor is thus

$$T_{\nu} = rF = A\eta r^2 \frac{\partial\Omega}{\partial r} \tag{4.163}$$

The viscous radial torque density is then

$$\left.\frac{\partial T}{\partial r}\right|_\nu = \frac{\partial T_\nu}{\partial r} = \frac{\partial}{\partial r}\left(4\pi r^3 h\rho\nu\frac{\partial\Omega}{\partial r}\right) \tag{4.164}$$

$$= 2\pi\frac{\partial}{\partial r}\left(r^3\sigma\nu\frac{\partial\Omega}{\partial r}\right) \tag{4.165}$$

since $\rho = \sigma/2h$.

Note that if your disk is nearly keplerian, then $\partial\Omega/\partial r = -3\Omega/2r$ and

$$\left.\frac{\partial T}{\partial r}\right|_\nu = -3\pi\frac{\partial(\ell\nu)}{\partial r} \tag{4.166}$$

where $\ell = \sigma r^2\Omega$ is the ang' mom' surface density.

a diffusion eqn for the disk

Lets combine our two CE to obtain a single diffusion eqn for a viscous disk. Eqn' (4.160)/$r^2\Omega$ is

$$\frac{\partial\sigma}{\partial t} + \frac{1}{r^3\Omega}\frac{\partial}{\partial r}(\sigma r^3\Omega v_r) = \frac{\partial\sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\sigma v_r) + \frac{\sigma v_r}{r^2\Omega}\frac{\partial}{\partial r}(r^2\Omega) \tag{4.167}$$

$$= \frac{1}{2\pi r^3\Omega}\left.\frac{\partial T}{\partial r}\right|_\nu = -\frac{3}{2r^3\Omega}\frac{\partial}{\partial r}(\nu\sigma r^2\Omega), \tag{4.168}$$

but the first two right terms are zero by the mass CE,
so we can solve for the disk's radial velocity v_r that is due to viscosity.

Noting that a keplerian disk has $r^2\Omega \propto r^{1/2}$ and $\partial(r^2\Omega)/\partial r = r\Omega/2$, then

$$v_r = -\frac{3}{\sigma r^2\Omega}\frac{\partial}{\partial r}(\nu\sigma r^2\Omega) \tag{4.169}$$

So if you know the disk's surface density σ and viscosity ν, then you can calculate the fluid radial velocity v_r, as well as the mass transport rate $dm/dt = 2\pi\sigma r v_r$ that flows through an annulus in the disk of radius r.

Now use this result to eliminate v_r from the mass CE, Eqn' (4.161):

$$\frac{\partial\sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[\frac{1}{r\Omega}\frac{\partial}{\partial r}(\nu\sigma r^2\Omega)\right] \tag{4.170}$$

we get a single diffusion eq' for the disk surface density $\sigma(r, t)$.

simple example

Lets consider a constant viscosity disk of infinite radial extent, that is also in equilibrium, ie, $\partial\sigma/\partial t = 0$.

How does σ vary with r in a steady–state disk?

What is the disk's radial velocity v_r throughout the disk?

And what is the mass–loss rate dm/dt through annulus r?

Eqn' (4.170) for a steady–state, nearly keplerian disk says that

$$\frac{\partial}{\partial r}(\nu\sigma r^2\Omega) \propto r\Omega \propto r^{-1/2} \tag{4.171}$$

$$\text{so} \quad \nu\sigma r^2\Omega \propto r^{1/2} \tag{4.172}$$

$$\Rightarrow \sigma = \text{ constant} \tag{4.173}$$

According to Eqn' (4.169), the fluid radial velocity is

$$v_r = -\frac{3\nu}{r^2\Omega}\frac{\partial}{\partial r}(r^2\Omega) = -\frac{3\nu}{2r} \tag{4.174}$$

And the mass–loss rate is

$$\frac{dm}{dt} = 2\pi r\sigma v_r = -3\pi\sigma\nu, \tag{4.175}$$

which is constant throughout the disk.

Note that this solution breaks down at small r.

For instance, if there is a star at the center, then pressure effects at the *boundary layer*, which is where the star's surface contacts the inner disk, tends to impede the flow.

the α viscosity law

The viscosity ν in most astrophysical disks usually has a form that is known as the Shakura & Sunyaev viscosity law:

$$\nu = \alpha v_s h \tag{4.176}$$

where v_s is the disk's sound speed, h is the disk scale–height, and α is a dimensionless coefficient. This is sometimes known as the α viscosity law.

Since $v_s = h\Omega$ (see Assignment #6, problem 1d), $\nu = \alpha h^2\Omega$.

All of the physics of viscosity, which is often poorly known,
is buried in the α parameter.

For instance, a circumstellar disk that is viscous due to turbulent vertical convection has an α–type viscosity, as does a slightly ionized disk that suffers the Balbous–Hawley MHD instability.

Note that observations of circumstellar disk usually show that the disk's are slightly *flared*, ie, their angular scale–height h/r increases slowly with r, which would suggests that $\nu = \alpha h^2\Omega$ might vary slowly with r.

However, it is often good enough to treat $\nu(r)$ as a constant.

gap formation in a constant–viscosity disk

Lets consider a constant–viscosity disk which also has a secondary mass μ_s orbiting within. The secondary could be a companion star orbiting in a circumbinary disk, a protoplanet orbiting in a circumstellar disk, or a small satellite orbiting in a planetary ring.

Lets tackle several questions:

How massive must μ_s be in order to shepherd open a gap in the disk?

How wide is the gap?

How fast will μ_s migrate due to *type II* migration?

The relevant EOM is our CE for the disk's $\ell = \sigma r^2\Omega$, Eqn' (4.160)

$$\frac{\partial \ell}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\ell v_r) = \frac{1}{2\pi r}\left(\left.\frac{\partial T}{\partial r}\right|_\nu + \left.\frac{\partial T}{\partial r}\right|_s\right) \tag{4.177}$$

$$\text{where} \quad \left.\frac{\partial T}{\partial r}\right|_\nu = -3\pi\frac{\partial}{\partial r}(\sigma r^2\Omega\nu) = \text{viscous torque density, Eqn' (4.166)} \tag{4.178}$$

$$\text{and} \quad \left.\frac{\partial T}{\partial r}\right|_s = \text{sgn}(x)\frac{32f^2}{81\pi}\left(\frac{a}{x}\right)^4 \mu_d\mu_s^2 m_1 a_s\Omega_s^2 \tag{4.179}$$

is the radial torque density that μ_s exerts on the disk matter that lies a radial distance $x = r - a_s$ away, $\mu_d = \pi\sigma_0 r^2/m_1$; see Eqn' (2.142).

Although that torque density is formally valid for, say, a satellite that perturbs a particle ring, it is also approximately true for a star/planet that perturbs a circumstellar gas disk, too.

Suppose the system is in quasi–static equilibrium,
which means that there is no motion of the disk relative to the s.

What does that tell us about the torques in this system?

Quasi–static equilibrium means that $\partial\ell/\partial t \simeq 0$ and $v_r \simeq 0$,
which implies a torque balance:

$$3\pi\nu\frac{\partial}{\partial r}(\sigma r^2\Omega) = \mathrm{sgn}(x)\frac{32f^2}{81\pi}\left(\frac{a}{x}\right)^4\mu_d\mu_s^2 m_1 a_s\Omega_s^2 \qquad (4.180)$$

Lets suppose that the gap is a narrow, sharp–edged feature, such that

$$3\pi\nu\frac{\partial}{\partial r}(\sigma r^2\Omega) \simeq 3\pi\nu r^2\Omega\frac{\partial\sigma}{\partial x} \qquad (4.181)$$

$$\text{so}\quad \frac{\partial\sigma}{\partial x} = \mathrm{sgn}(x)\frac{32f^2}{243\pi}\left(\frac{a}{x}\right)^4\frac{\mu_s^2\sigma_0 a_s\Omega_s}{\nu} \qquad (4.182)$$

$$\text{and}\quad \sigma(x) = \int\frac{\partial\sigma}{\partial x}dx = \sigma_0\left[1 - \frac{32f^2}{729\pi}\left|\frac{a}{x}\right|^3\left(\frac{a_s^2\Omega_s}{\nu}\right)\mu_s^2\right] \qquad (4.183)$$

$$\equiv \sigma_0\left(1 - \left|\frac{x_{edge}}{x}\right|^3\right) \qquad (4.184)$$

$$\text{where}\quad x_{edge} = \left[\frac{32f^2}{729\pi}\left(\frac{a_s^2\Omega_s}{\nu}\right)\mu_s^2\right]^{1/3} a_s \qquad (4.185)$$

is the secondary–gap edge separation.

For, say, a Jupiter–mass planet $\mu_s = 10^{-3}$ in a disk having $h \sim 0.1r$ (typical of circumstellar disks) and $\alpha \sim 0.01$ (typical of the MHD instability), $\nu/a_s^2\Omega_s \sim \alpha(h/r)^2 \sim 10^{-4}$.

In this case, the distance to the gap edge is $x_{edge} \sim 0.1a_s$,
so the gap half–width is $\sim 10\%$ of the planet's orbit.

Compare this result to Fig. 4.2.

How massive should the secondary be in order to open a 'clean' gap?

Hint: compare x_{edge} to the disk particles' random epicyclic motions.

$$\Rightarrow \mu_s \gtrsim 3.4\sqrt{\alpha\left(\frac{h}{r}\right)^5} \tag{4.186}$$

which is about 35 $M_\oplus$ for the above example.

If μ_s is less than this limit, then the gap won't be 'clean'.
Rather, it will resemble a surface density 'depression'.

Now lets consider the long–term evolution of this system.

Is our disk really in static equilibrium? ie, is $\partial\sigma/\partial t$ really 0?
And is the fluid radial velocity v_r?

This is why I use the term *quasi–static equilibrium*,
which is meant to imply that there is no fluid motions relative to the secondary.

What is the disk's radial velocity v_r?

And the secondary's?

This is *type II* planet migration. According to Eqn' (4.169), the disk flows inwards with velocity

$$v_r = -\frac{3\nu}{2r} = -\frac{3\alpha}{2}\left(\frac{h}{r}\right)^2 r\Omega. \tag{4.187}$$

What happens to the secondary's orbit?

When the disk is in quasi–static equilibrium, the *secondary's* orbit decays, too, since there is no fluid motion relative to μ_s.

When μ_s opens a gap in the disk, the gap acts like a mass barrier.

But the disk is viscous,
so disk + μ_s spiral into the primary on the disk's viscous timescale

$$\tau_\nu = \left|\frac{r}{v_r}\right| = \frac{P_{orb}}{3\pi\alpha}\left(\frac{r}{h}\right)^2, \tag{4.188}$$

also known as the type–II migration timescale.

For the example above, $\tau_\nu \simeq 10^3$ orbits,
which at $r = 5$ AU is only $\sim 10^4$ years!

Viscous disks are quite perilous to planet–formation!

Actually, observations of disks around young stars suggest that these disks tend to persist for a few million years, which would imply an $\alpha \sim 10^{-4}$.

time–dependent disk evolution

Thus far we have solved for the viscous evolution of quasi–static systems.

This time, lets solve a truly time–dependent problem:
the viscous evolution $\sigma(r,t)$ and $v_r(r,t)$ for an initially narrow ring.

Lets assume the ring has mass m and is initially narrow, with radius r_0:

$$\sigma(r, t = 0) = \frac{m}{2\pi r_0}\delta(r - r_0) \tag{4.189}$$

(check: does ring mass $m = \int \sigma da$?)

The ring's diffusion EOM is Eqn' (4.170)

$$\frac{\partial\sigma}{\partial t} = \frac{3\nu}{r}\frac{\partial}{\partial r}\left[\frac{1}{r\Omega}\frac{\partial}{\partial r}(\sigma r^2\Omega)\right] \tag{4.190}$$

For a nearly keplerian disk, $r\Omega = c/r^{1/2}$ where $c =$ constant, so

$$\frac{\partial \sigma}{\partial t} = \frac{3\nu}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(\sigma r^{1/2})\right] \tag{4.191}$$

Diffusion eqn's like this are usually solved by separation of variables, which means that you assume the solution is the product $\sigma(r,t) = S(r)T(t)$. Insert this into the EOM, and divide by σ:

$$\frac{1}{T}\frac{\partial T}{\partial t} = \frac{3\nu}{Sr}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(Sr^{1/2})\right] \tag{4.192}$$

What is the LHS a function of?
And the RHS?
What does this say about the LHS & the RHS?
The the system's behavior in time must satisfy

$$\Rightarrow \frac{\partial T}{\partial t} = -\omega T \tag{4.193}$$

$$\text{so} \quad T(t) = e^{-\omega t} \tag{4.194}$$

where ω is a constant, and this $T(t)$ is just one possible solution.

But keep in mind that $e^{-\omega' t}$ is another solution, too...

Thus the radial EOM becomes

$$\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(Sr^{1/2})\right] = -\frac{\omega}{3\nu}rS \tag{4.195}$$

$$\text{so} \quad S''r + \frac{3}{2}S' + k^2 rS = 0 \tag{4.196}$$

where $S' = \partial S/\partial r$, etc., and $k^2 \equiv \omega/3\nu$ is a constant.

You could massage the eqn further until it starts to resemble Bessel's eqn... or you can ask MAPLE to solve this, using `dsolve()`:

$$S(r) = \frac{C}{(kr)^{1/4}} J_{1/4}(kr) + \frac{D}{(kr)^{1/4}} Y_{1/4}(kr) \tag{4.197}$$

where the J and Y functions are Bessel fn's of the 1^{st} and 2^{nd} kind, and C & D are constants.

We expect the disk surface density σ to be finite everywhere at times $t > 0$, which implies that $D = 0$, since $Y_{1/4}/r^{1/4}$ diverges as $r \to 0$.

Thus a particular solution to the diffusion EOM is

$$S(r)T(t) = e^{-3\nu k^2 t} \frac{C(k)}{(kr)^{1/4}} J_{1/4}(kr) \tag{4.198}$$

keep in mind that this is merely a single 'mode' having a wavenumber k and amplitude $C(k)$ that satisfies the EOM.
Other modes having other k's also satisfy the EOM, too.

The general solution formed from the superposition of all modes that satisfy initial conditions. This is obtained by replacing $C(k) \to c(k)dk$ and summing over all k:

$$\sigma(r,t) = \int_0^\infty dk e^{-3\nu k^2 t} c(k)(kr)^{-1/4} J_{1/4}(kr) \tag{4.199}$$

The last step is to determine the function $c(k)$ that recover's the disk initial state when $t = 0$ and $\sigma(r,0) = m\delta(r - r_0)/2\pi r_0$.

That last step is rather mathematical...it requires some elaborate orthogonality relations for Bessel fn's...

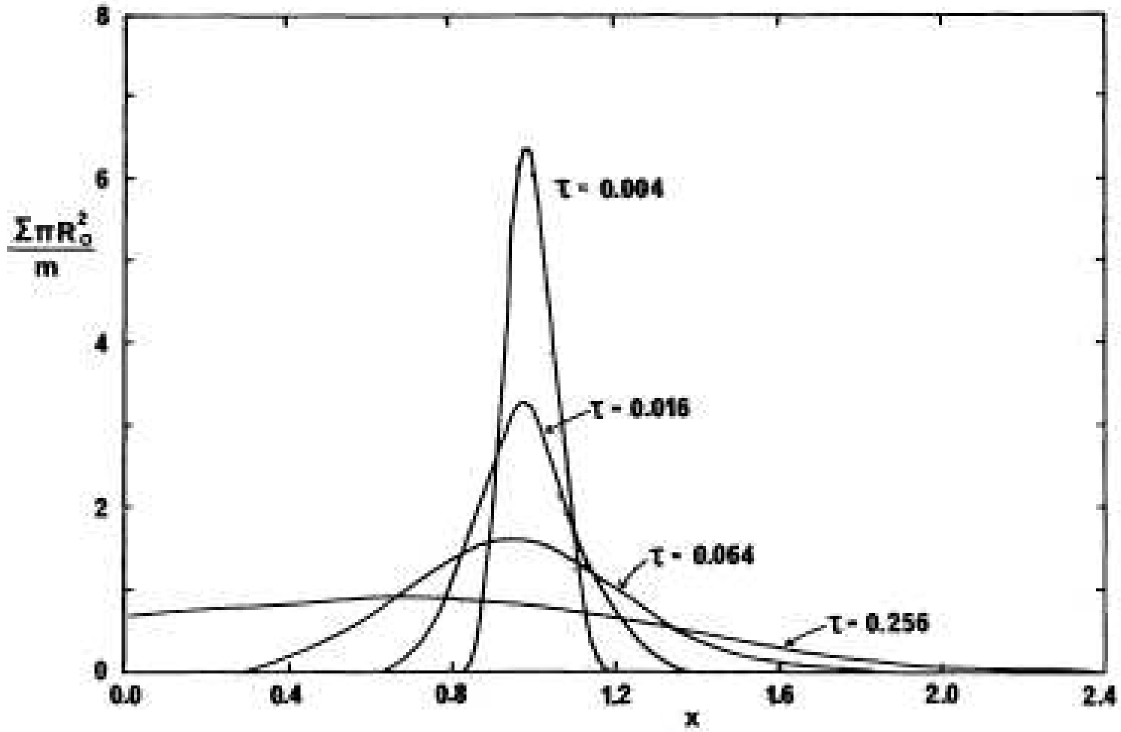

Figure 1 The viscous evolution of a ring of matter of mass m. The surface density Σ is shown as a function of dimensionless radius $x = R/R_0$, where R_0 is the initial radius of the ring, and of dimensionless time $\tau = 12\nu t/R_0^2$ where ν is the viscosity.

We will skip those details [which are given in a classic paper by Lynden–Bell & Pringle (1974)], and merely quote the final result:

$$\sigma(x,\tau) = \left(\frac{m}{2\pi r_0^2}\right)\frac{1}{x^{1/4}\tau}e^{-(x^2+1)/2\tau}I_{1/4}\left(\frac{x}{\tau}\right) \qquad (4.200)$$

where $I_{1/4}$ is a modified Bessel function of the dimensionless distance $x = r/r_0$ and dimensionless time $\tau = 6\nu t/r_0^2$.

The above Figure (from Pringle 1981)
illustrates the main features of any viscous disk:

- the disk spread's radially, both inwards and outwards
- the net mass flux is inwards,
- nonetheless, some mass gets transported outwards, in a manner that preserves the ring's angular momentum.

The viscous timescale t_ν is obtained by setting $\tau = 6\nu t/r_0^2 \equiv t/t_\nu$
$\Rightarrow t_\nu = r_0^2/6\nu$.

This is the time required for the ring to get smeared out into a disk,
losing all memory of its initial state.

For an α–disk, $\nu = \alpha h^2\Omega$, and

$$t_\nu = \frac{r_0^2}{6\nu} = \frac{P_{orb}}{12\pi\alpha}\left(\frac{r}{h}\right)^2 \qquad (4.201)$$

which is comparable to the type–II migration timescale, Eqn' (4.188).