Lecture Notes for AST 5622
Astrophysical Dynamics

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The Three–Body Problem

This lecture is drawn from M&D, Chapter 3.

The simplest 3–body problem is the *restricted* 3–body problem (R3BP):

this is the study of the motion of
a massless particle P (also called a test particle)
that is being perturbed by a secondary mass m_2 (say, a planet)
that is in a circular orbit about a primary mass m_1 (say, the Sun).

This problem is the easiest of all N–body problem where $N > 2$,
since the motion of the primary & secondary are known exactly.

However it is still challenging,
since there is no general analytic solution for particle P's motion.

This problem is most relevant to the study of the motion of small bodies
(ie, comets, asteroids, dust grains, etc) when they are perturbed by Jupiter
(which, by the way, has a small but non–zero eccentricity $e = 0.05$),

or by a ring particle that is perturbed by a satellite.

Equations of Motion (EOM)

Lets derive the EOM for particle P:

This time place the origin at the system's center of mass (COM), and put m_1 and m_2 in the inertial $\hat{\mathbf{X}}$–$\hat{\mathbf{Y}}$ plane.
Note that P need not reside in this plane.

P's acceleration is

$$\mathbf{A}_{stat} = -\frac{Gm_1}{R_1^3}\mathbf{R_1} - \frac{Gm_2}{R_2^3}\mathbf{R_2} \tag{2.1}$$

relative to the *stationary* origin.

Next, switch to a reference frame that rotates with the angular velocity $n = \sqrt{G(m_1 + m_2)/a^3} = m_2$'s mean–motion with a its semimajor axis.
The $\hat{\mathbf{x}}$–$\hat{\mathbf{y}}$ axes describe the orientation of this rotating reference frame.

Note that P's position vector is

$$\mathbf{R} = -X_1\hat{\mathbf{x}} + \mathbf{R_1} = X_2\hat{\mathbf{x}} + \mathbf{R_2} \tag{2.2}$$

$$\text{so} \quad \mathbf{R_1} = \mathbf{R} + X_1\hat{\mathbf{x}} \quad \text{and} \quad \mathbf{R_2} = \mathbf{R} - X_2\hat{\mathbf{x}} \tag{2.3}$$

are P's position relative to the m_i.

We want P's acceleration measured in the rotating reference frame, $\mathbf{A}_{rot}$.

Recall from your classical mechanics class:

$$\mathbf{A}_{rot} = \mathbf{A}_{stat} - \vec{\omega} \times (\vec{\omega} \times \mathbf{R}) - 2\vec{\omega} \times \mathbf{V} \tag{2.4}$$

where the middle & right terms are the centrifugal & Coriolis acceleration that occur in this rotating reference frame,
$\vec{\omega} = n\hat{\mathbf{z}}$ is the rotation axis,
and $\mathbf{V} = d\mathbf{R}/dt$ is P's velocity measured in the *rotating* coordinate system.
(to refresh your memory, see page 8 of my PHY 3405 notes, posted at http://apwww.stmarys.ca/~jhahn/phy3405/2005fall/chap10.pdf)

The centrifugal acceleration is

$$\mathbf{A}_{cent} = -\vec{\omega} \times (\vec{\omega} \times \mathbf{R}) \tag{2.5}$$

$$\text{where} \quad \vec{\omega} \times \mathbf{R} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & n \\ X & Y & Z \end{vmatrix} \tag{2.6}$$

$$= -nY\hat{\mathbf{x}} + nX\hat{\mathbf{y}} \tag{2.7}$$

$$\text{so} \quad \mathbf{A}_{cent} = -\vec{\omega} \times (\vec{\omega} \times \mathbf{R}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & -n \\ -nY & nX & 0 \end{vmatrix} \tag{2.8}$$

$$= n^2 X\hat{\mathbf{x}} + n^2 Y\hat{\mathbf{y}} \tag{2.9}$$

Similarly, the Coriolis acceleration is

$$\mathbf{A}_{cor} = -2\vec{\omega} \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & -2n \\ \dot{X} & \dot{Y} & \dot{Z} \end{vmatrix} \tag{2.10}$$

$$= 2n\dot{Y}\hat{\mathbf{x}} - 2n\dot{X}\hat{\mathbf{y}} \tag{2.11}$$

so P's acceleration in this rotating coordinate system is

$$\ddot{\mathbf{R}} = \mathbf{A}_{rot} \tag{2.12}$$

$$= -\frac{Gm_1}{R_1^3}\mathbf{R_1} - \frac{Gm_2}{R_2^3}\mathbf{R_2} + n^2(X\hat{\mathbf{x}} + Y\hat{\mathbf{y}}) + 2n\dot{Y}\hat{\mathbf{x}} - 2n\dot{X}\hat{\mathbf{y}} \tag{2.13}$$

Since the origin is at the system's COM,

$$
\begin{aligned}
m_1 X_1 &= m_2 X_2 \qquad \text{and} \qquad X_1 + X_2 = a && (2.14)\\
\text{so} \quad m_1(a - X_2) &= m_2 X_2 && (2.15)\\
(m_1 + m_2)X_2 &= m_1 a && (2.16)\\
\Rightarrow X_2 &= \frac{m_1}{m_1 + m_2} a \equiv \mu_1 a \qquad \text{and} \qquad X_1 = a - X_2 \equiv \mu_2 a && (2.17)\\
\text{where} \quad \mu_i &\equiv \frac{m_i}{m_1 + m_2} = m_i\text{'s mass in fractional units} && (2.18)\\
\text{and} \quad \mu_1 + \mu_2 &= 1 && (2.19)
\end{aligned}
$$

Lets also switch to dimensionless lengths via

$$
\begin{aligned}
\mathbf{r} &= \mathbf{R}/a = \text{P's dimensionless position vector}, && (2.20)\\
\text{so} \quad x_i &= X_i/a, \qquad \dot{x}_i = \dot{X}_i/a \quad \mathbf{r_i} = \mathbf{R_i}/a, \quad \text{etc} && (2.21)
\end{aligned}
$$

ie, lowercase symbols are dimensionless lengths while
uppercase symbols have units of length.
Thus $x_1 = \mu_2$ and $x_2 = \mu_1$.

P's acceleration is now

$$
\begin{aligned}
\ddot{\mathbf{r}} &= -\frac{Gm_1}{a^3}\frac{\mathbf{r_1}}{r_1^3} - \frac{Gm_2}{a^3}\frac{\mathbf{r_2}}{r_2^3} + n^2(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) + 2n\dot{y}\hat{\mathbf{x}} - 2n\dot{x}\hat{\mathbf{y}} && (2.22)\\
\text{since} \quad \frac{Gm_i}{a^3} &= \frac{G(m_1 + m_2)}{a^3}\frac{m_i}{m_1 + m_2} = n^2 \mu_i, && (2.23)\\
\ddot{\mathbf{r}} &= 2n\dot{y}\hat{\mathbf{x}} - 2n\dot{x}\hat{\mathbf{y}} + n^2(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) - n^2\left[\frac{\mu_1 \mathbf{r_1}}{r_1^3} + \frac{\mu_2 \mathbf{r_2}}{r_2^3}\right] && (2.24)
\end{aligned}
$$

$$
\begin{aligned}
\text{Now note} \quad \mathbf{r_1} &= \mathbf{R_1}/a = (x + \mu_2)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} && (2.25)\\
\text{and} \quad \mathbf{r_2} &= \mathbf{R_2}/a = (x - \mu_1)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} && (2.26)
\end{aligned}
$$

so $\ddot{\mathbf{r}}$ has cartesian coordinates

$$\begin{aligned}
\ddot{x} &= 2n\dot{y} - n^2\left[\frac{\mu_1}{r_1^3}\mathbf{r_1}\cdot\hat{\mathbf{x}} + \frac{\mu_2}{r_2^3}\mathbf{r_2}\cdot\hat{\mathbf{x}} - x\right] && (2.27)\\
&= 2n\dot{y} - n^2\left[\frac{\mu_1(x+\mu_2)}{r_1^3} + \frac{\mu_2(x-\mu_1)}{r_2^3} - x\right] && (2.28)\\
\ddot{y} &= -2n\dot{x} - n^2\left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3} - 1\right]y && (2.29)\\
\ddot{z} &= -n^2\left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right]z && (2.30)
\end{aligned}$$

Note that the RHS differs from the text due to the definition of μ_i: $\mu_i(lecture) = \mu \times \mu_i(text)$.
The text also chooses units such that a, n, and μ are unity,
but the text is not always consistent about setting these to 1...

Now if we let

$$\begin{aligned}
U(x,y,z) &= n^2\left[\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} + \frac{1}{2}(x^2+y^2)\right] && (2.31)\\
&= \text{'effective' potential} && (2.32)\\
&= -1\times\ \text{(gravitational + centrifugal) potential energies} && (2.33)
\end{aligned}$$

Then

$$\begin{aligned}
\ddot{x} &= 2n\dot{y} + \frac{\partial U}{\partial x} && (2.34)\\
\ddot{y} &= -2n\dot{x} + \frac{\partial U}{\partial y} && (2.35)\\
\ddot{z} &= \frac{\partial U}{\partial z} && (2.36)
\end{aligned}$$

are P's equations of motion in this rotating reference frame;
this can be written more compactly as

$$\ddot{\mathbf{r}} = \nabla U - 2\vec{\omega}\times\dot{\mathbf{r}} \qquad (2.37)$$

Keep in mind that all lengths are dimensionless;
multiply the eqn's by a to get real units of length.

First, confirm Eqn's (2.28) & (2.34):

$$\frac{\partial U}{\partial x} = n^2\left[-\frac{\mu_1}{r_1^2}\frac{\partial r_1}{\partial x} - \frac{\mu_2}{r_2^2}\frac{\partial r_2}{\partial x} + x\right] \tag{2.38}$$

$$\text{where} \quad r_1 = \sqrt{(x+\mu_2)^2 + y^2 + z^2}, \tag{2.39}$$

$$r_2 = \sqrt{(x-\mu_1)^2 + y^2 + z^2} \tag{2.40}$$

$$\text{so} \quad \frac{\partial U}{\partial x} = n^2\left[-\frac{\mu_1(x+\mu_2)}{r_1^3} - \frac{\mu_2(x-\mu_1)}{r_2^3} + x\right] \tag{2.41}$$

which confirms Eqn. (2.28). Ditto for $\partial U/\partial y$ and $\partial U/\partial z$.

The Jacobi Integral

Recall that the 2–body problem has two important integrals, E and $\mathbf{h}$. Since they depend upon the secondary's a and e, these two integrals completely specify the shape of the orbit.

The restricted 3–body problem has only one integral, the Jacobi integral J, and it is a combination of the particle's energy & ang' mom'.

Because this integral does not disentangle the particle's E and h, it does not specify the shape of P's orbit.

Nonetheless, J can be used to determine the range of P's orbital behavior.

Start by forming

$$\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} = 2n\dot{x}\dot{y} + \frac{\partial U}{\partial x}\dot{x} - 2n\dot{x}\dot{y} + \frac{\partial U}{\partial y}\dot{y} + \frac{\partial U}{\partial z}\dot{z} \quad (2.42)$$

$$= \frac{dU}{dt} \quad (2.43)$$

and integrate: $$\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = U + J \quad (2.44)$$

where the integration constant J is the Jacobi integral:

$$J = \frac{1}{2}v^2 - n^2\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right) - \frac{1}{2}n^2(x^2 + y^2) \quad (2.45)$$

Keep in mind that this expression uses dimensionless lengths, so J has units of time^{-2}.

Note that this J differs from the text's Jacobi integral via $C_J = -2J$.

Also note that this is particle P's Jacobi integral evaluated
in the *rotating* reference frame.

It will also be handy to have J written in terms of the inertial coordinates; in particular, we need to replace v with P's velocity V measured in the inertial reference frame.

Recall from your classical mechanics text (page 6 of my PHYS 3405 notes):

$$\mathbf{V}_{stat} = \mathbf{v}_{rot} + \vec{\omega} \times \mathbf{r} \tag{2.46}$$

which relates P's velocity in the stationary frame, $\mathbf{V}_{stat} = \mathbf{V}$,
to its velocity $\mathbf{v}_{rot} = \mathbf{v}$ measured in the rotating reference frame. Thus

$$v^2 = \mathbf{v}_{rot}^2 = \mathbf{V}^2 - 2\mathbf{V} \cdot (\vec{\omega} \times \mathbf{r}) + |\vec{\omega} \times \mathbf{r}|^2 \tag{2.47}$$

$$\text{Since} \quad \vec{\omega} \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & n \\ x & y & z \end{vmatrix} = -ny\hat{\mathbf{x}} + nx\hat{\mathbf{y}}, \tag{2.48}$$

$$\text{so} \quad |\vec{\omega} \times \mathbf{r}|^2 = n^2(x^2 + y^2) \tag{2.49}$$

We can also use the vector identity

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) \tag{2.50}$$

$$\text{to write} \quad \mathbf{V} \cdot (\vec{\omega} \times \mathbf{r}) = \vec{\omega} \cdot (\mathbf{r} \times \mathbf{V}) \tag{2.51}$$

$$= \vec{\omega} \cdot \left[\mathbf{r} \times \left(\frac{d\mathbf{r}}{dt}\right)_{stat}\right] \tag{2.52}$$

$$= \vec{\omega} \cdot \mathbf{h} \tag{2.53}$$

where $\mathbf{h} = \mathbf{r} \times \mathbf{V}$
is P's ang' mom' measured in the *stationary* reference frame.

Thus

$$
\begin{aligned}
v^2 &= V^2 - 2\vec{\omega}\cdot\mathbf{h} + n^2(x^2+y^2) && (2.54)\\
\text{and}\quad J &= \frac{1}{2}V^2 - n^2\left(\frac{\mu_1}{r_1}+\frac{\mu_2}{r_2}\right) - \vec{\omega}\cdot\mathbf{h} && (2.55)
\end{aligned}
$$

is the Jacobi integral for coordinates in the stationary reference frame. Again, this formula uses dimensionless lengths.

We can also express J in terms of its energy E and ang' mom' $\mathbf{h}$:

$$
a^2n^2\mu_i = \frac{a^2G(m_1+m_2)}{a^3}\frac{m_i}{m_1+m_2} = \frac{Gm_i}{a} \qquad (2.56)
$$

and then multiply J by a^2 so that our lengths are no longer dimensionless:

$$
\begin{aligned}
V &= \frac{V'}{a}, \qquad \mathbf{h} = \frac{\mathbf{h}'}{a^2} \quad \text{ie, primed quantities are nondimensional} && (2.57)\\
\vec{\omega}\cdot\mathbf{h}' &= nh'_z && (2.58)\\
\text{so}\quad J' &= Ja^2 = \frac{1}{2}V'^2 - \left(\frac{Gm_1}{R_1}+\frac{Gm_2}{R_2}\right) - nh'_z && (2.59)\\
&= E' - nh'_z && (2.60)\\
\text{where}\quad E' &= \frac{1}{2}V'^2 - \left(\frac{Gm_1}{R_1}+\frac{Gm_2}{R_2}\right) && (2.61)
\end{aligned}
$$

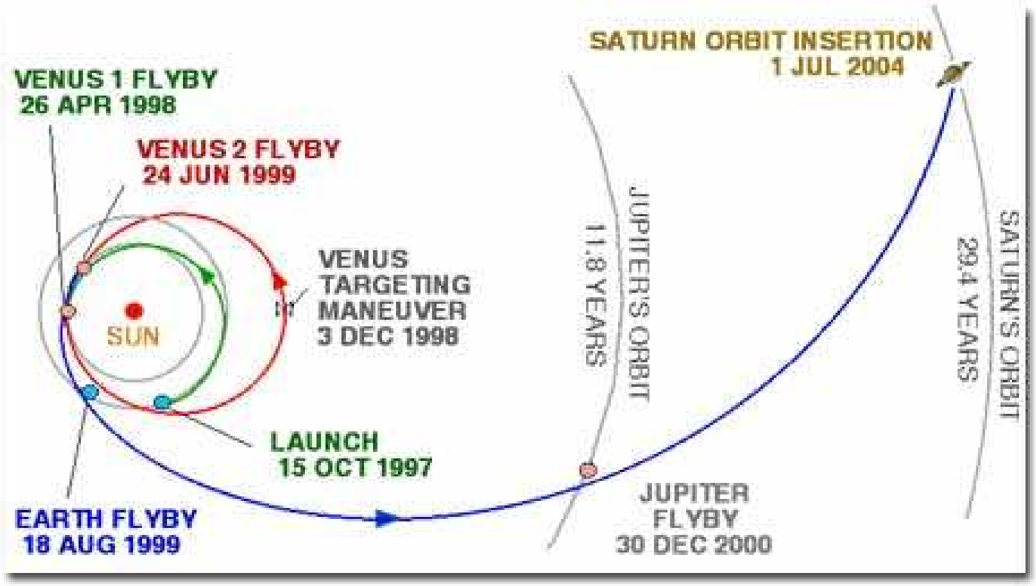
is P's specific energy in the COM coordinate system.

Note that in the R3BP, the particle's energy E' and ang' mom' h'_z are *not* individually conserved.

Instead, it is the combination $J = E' - nh'_z$ that is conserved.

On other words, the particle can alter its E' and h'_z by interacting with the secondary planet m_2.

This allows for the 'gravity assist', whereby a close approach to a planet slingshots a spacecraft into a more (or less) energetic orbit.



Osculating Orbital Elements

In the 2–body problem, m_2's orbit elements are constant.

But in the R3BP, particle's P's orbit elements $a, e, i, \Omega, \omega, M$ are not constants—they should be regarded as functions of time t that are known as *osculating* orbit elements.

To calculate the osculating orbit elements at time t, you need to know P's position $\mathbf{r}(t)$ and velocity $\dot{\mathbf{r}}(t)$ relative to the primary in the stationary reference frame.

Then calculate P's orbit elements assuming that P is the secondary, this provides a, e, i, etc, *for that instant of time* t.

You can then use Kepler's equation to calculate P's $\mathbf{r}(t')$ and $\dot{\mathbf{r}}(t')$ at other times t'.

However that of course will only yield an approximate solution for P's motion since that calculation ignores perturbations from any other planets, stars, other forces, etc. The quality of this approximate solution will steadily degrade over time.

An exact solution for P's motion over time requires solving Newton's laws (usually numerically) in a manner that accounts for all other perturbing forces.

some more Classical Mechanics

In my earlier review of classical mechanics,
I neglected to note a few other fundamental formulas:

1. We can usually write the force as $\mathbf{F} = -\nabla U$,
where $U(\mathbf{r})$ is the system's*potential energy* (or PE), and

$$\nabla U = \frac{\partial U}{\partial x}\hat{\mathbf{x}} + \frac{\partial U}{\partial y}\hat{\mathbf{y}} + \frac{\partial U}{\partial z}\hat{\mathbf{z}} \quad \text{in Cartesian coordinates} \qquad (2.62)$$

$$= \frac{\partial U}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\hat{\phi} + \frac{\partial U}{\partial z}\hat{\mathbf{z}} \quad \text{in cylindrical coord's} \qquad (2.63)$$

2. For a particle of mass m, we call $\Phi(\mathbf{r}) = U/m = m$'s *potential*,
ie, its potential energy per unit mass.
Then by Newton's 2nd law, m's EOM is $\ddot{\mathbf{r}} = -\nabla\Phi(\mathbf{r})$.

3. suppose the scalar function $f = f(\mathbf{r}, t) = f(x, y, z, t)$ is a function of the spatial coordinate $\mathbf{r}$ and time t. Then by the Chain Rule,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t} = (\nabla f)\cdot\dot{\mathbf{r}} + \frac{\partial f}{\partial t} \qquad (2.64)$$

We will use this in the next homework.

A More General Derivation of J

The preceding derivation of the Jacobi integral J, which we was obtained by considering the R3BP, is *not* the most general derivation.

Consider a particle that is subject to the potential Φ that is *stationary* (ie, time-independent) in a reference frame that rotates with constant angular velocity ω about some axis.

Does the potential experienced by the particle in the R3BP satisfy this constraint?

In the following homework assignment, you will show that the particle also has a conserved Jacobi integral

$$J = E - \vec{\omega} \cdot \mathbf{L} \tag{2.65}$$

where E and $\mathbf{L}$ are the particle's *total* energy and angular momenta measured in the inertial reference frame.

We will find this Jacobi integral very handy when we start studying the motion of perturbed bodies, such as:

- a star in a galaxy that is disturbed by rotating central bar,
- or a particle disturbed by a planet in a circular orbit.

Assignment #2
due Thursday February 2
at the start of class

1. Derive Eqn. (2.65) for a particle that is subject to a potential Φ that is stationary in a reference frame that rotates at a constant angular velocity ω about the rotation axis $\vec{\omega} = \omega\hat{\mathbf{z}}$.

a.) Begin by writing the EOM for the particle's position vector $\mathbf{r}$ in this rotating reference frame. Then consider the product $\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}}$ to show that

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\mathbf{r}}^2 + \Phi\right) \;=\; -[\vec{\omega} \times (\vec{\omega} \times \mathbf{r})] \cdot \dot{\mathbf{r}} \qquad (2.66)$$

b.) identify the RHS with Eqn. (2.64) so that $\nabla f = -\vec{\omega} \times (\vec{\omega} \times \mathbf{r})$;
then verify that $f = (\vec{\omega} \times \mathbf{r})^2/2$ satisfies the preceding result.
Tip: use cylindrical coordinates.

c.) use the above result to show that

$$J \;=\; \frac{1}{2}\dot{\mathbf{r}}^2 + \Phi \;-\; \frac{1}{2}(\vec{\omega} \times \mathbf{r})^2 \qquad (2.67)$$

is a constant of the motion. Keep in mind that $\dot{\mathbf{r}}$ is the particle's velocity in the *rotating* reference frame.

d.) replace $\dot{\mathbf{r}}$ with the particle's velocity as measured in the *inertial* reference frame, which then yields a jacobi integral

$$J \;=\; E - \vec{\omega} \cdot \mathbf{L} \qquad (2.68)$$

for a particle having specific energy and ang' mom' E and $\mathbf{L}$.

3. Show that a dimensionless Jacobi integral,

$$J' = \frac{J}{E_2} = \frac{a}{a_P} + 2\sqrt{\left(1+\frac{m_2}{m_1}\right)\frac{a_P}{a}(1-e_P^2)}\cos i_P + \frac{m_2}{m_1}\frac{a}{R_2}, \quad (2.69)$$

when you write J in term's of particle P's osculating orbit elements a_P, e_P, i_P, where E_2 is the secondary's specific energy in the COM reference frame.

4. The Tisserand parameter T is Eqn. (2.69) with the rightmost term neglected, since that term is usually quite small. The Tisserand parameter is useful in studies of cometary orbits, since it is approximately conserved in a simple 1–planet system.

An active comet is one that gets close enough to the Sun for its icy surface to sublimate; active comets usually have perihelia $q \lesssim 2.5$ AU. Such bodies are usually crossing the orbit of Jupiter, so their dynamics tend to be dominated by that planet.

An important class of comets are the Oort Cloud comets; they reside in nearly parabolic orbits with semimajor axes $\mathcal{O}(10^4)$ AU. Show that an active comet from the Oort Cloud will have a Tisserand parameter of $T \lesssim 2$ when measured with respect to Jupiter.

(Another important class of comets are the ecliptic comets; these are comets that likely diffused inwards from the Kuiper Belt. Ecliptic comets have $2 \lesssim T \lesssim 3$, so one can easily distinguish these two types of comets by inspecting their Tisserand parameters.)

Zero velocity curves

One useful property of the Jacobi integral J is that it can be used to provide constraints on particle P's allowed range of motion.

Recall that the rotating coordinate system,

$$J = \frac{1}{2}v^2 - U(x,y,z) \tag{2.70}$$

$$\text{so} \quad U = \frac{1}{2}v^2 - J \tag{2.71}$$

Since $v^2 \geq 0$, this implies that

$$U(x,y,z) = n^2\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right) + \frac{1}{2}n^2(x^2+y^2) \geq -J \tag{2.72}$$

Note that for a particle which has some value for J,
there may be some regions (x,y,z) that do not satisfy the above relation.

Consequently, the particle is excluded from that region.

The boundary of that forbidden region is the particle's
zero–velocity surface which satisfies

$$U(x,y,z) = n^2\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right) + \frac{1}{2}n^2(x^2+y^2) = -J \tag{2.73}$$

It will be convenient to choose units so that $\mu = G(m_1 + m_2) = 1$;
this is equivalent to choosing a unit of time such that the planet's orbital frequency $n = \sqrt{\mu/a^3} = 1$ and period $T = 2\pi/n = 1$;
note that our lengths are already dimensionless lengths, ie $a = 1$.

You may regard the zero–velocity surface as a minimum–energy orbit,
one that minimized P's kinetic energy in the rotating coordinate system.

Fig. 3.2 of M&D shows zero–velocity curves for a binary star system with masses $\mu_1 = 0.8$ and $\mu_2 = 0.2$.

The figure shows that a particle with $J = -1.95$ (ie, text's $C_J = 3.9$) can reside in a tight orbit about μ_1 (region I, a circumprimary orbit),
a tight orbit about μ_2 (region II, a satellite orbit),
or in a wide orbit about the binary (region III, a circumbinary orbit).

Note, however, that mass transfer from $\mu_1 \leftrightarrow \mu_2$ is forbidden,
and that transitions from tight$\leftrightarrow$wide orbits is also impossible.

However mass transfer is possible for a particle with $J = -1.85$ ($C_J = 3.7$):

This graphical analysis of a particle's zero–velocity curves tells you the possible range of that particle's motions.

For instance, you can use these curves to determine whether a satellite's orbit is always confined to the vicinity of m_2,
or if it can also wander into an orbit about m_1.

In particular, you can use these curves to determine if mass transfer is possible in a binary star system.

Suppose a particle at some instant of time is sitting on a zero velocity curve. Will that particle remain motionless at that site for all times?

Why?
If a particle is stationary at all times, then it is at a site where all the forces on that particle (gravity + centrifugal) sum to zero; those sites are known as the Lagrange equilibrium points.

Lagrange Equilibrium Points

An equilibrium point = a site where the force on a motionless particle P in the reference frame that co–rotates with m_1–m_2.

P's EOM in the rotating reference frame is from Eqn. (2.24):

$$\ddot{\mathbf{r}} = \nabla U - 2\vec{\omega} \times \dot{\mathbf{r}} \tag{2.74}$$

where U = P's effective potential due to gravity + centrifugal acceleration and the right term is the Coriolis acceleration.

A particle at an equilibrium point has $\ddot{\mathbf{r}} = 0$ and $\dot{\mathbf{r}} = 0$,
so these equilibria satisfy $\nabla U(x, y, z) = 0$,
ie, $\partial U/\partial x = \partial U/\partial y = \partial U/\partial z = 0$.

From Eqn. (2.30),

$$\frac{\partial U}{\partial z} = -n^2 \left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right] z = 0 \tag{2.75}$$

so the equilibrium points lie in the $z = 0$ plane.

Section 3.6 of M&D show there are two *triangular* Lagrange points (LP);
these sites are equidistant from the m_i at separations $r_1 = 1 = r_2$,
(see text for derivation) so P, m_1, and m_2 form an equilateral triangle:

where $x = 1/2 - \mu_2$.

Confirm that $\partial U/\partial x = 0 = \partial U/\partial y$ at the triangular LPs:

Inspection of Eqn's (2.28–2.29) shows that

$$\frac{\partial U}{\partial x} = -n^2\left[\frac{\mu_1(x+\mu_2)}{r_1^3} + \frac{\mu_2(x-\mu_1)}{r_2^3} - x\right] \tag{2.76}$$

$$\frac{\partial U}{\partial y} = -n^2\left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3} - 1\right] y \tag{2.77}$$

Since $\mu_1 + \mu_2 = 1$, $\partial U/\partial y = 0$ $\surd$.

And since $x = 1/2 - \mu_2$,

$$\frac{\partial U}{\partial x} = -n^2\left[\frac{1}{2}\mu_1 - \frac{1}{2}\mu_2 - \frac{1}{2} + \mu_2\right] = 0 \quad \surd \tag{2.78}$$

Eqn. (2.77) indicates there may be additional equilibrium sites along $y = 0$:

Since

$$r_1 = |x+\mu_2| \equiv s_1(x+\mu_2) \quad \text{where } s_1 = \text{sgn}(x+\mu_2), \tag{2.79}$$

$$r_2 = s_2(x-\mu_1) \quad \text{where } s_2 = \text{sgn}(x-\mu_1) \tag{2.80}$$

$$\text{then} \quad \frac{\partial U}{\partial x} = -n^2\left[\frac{\mu_1}{s_1^3(x+\mu_2)^2} + \frac{\mu_2}{s_2^3(x-\mu_1)^2} - x\right] = 0 \tag{2.81}$$

$$\text{so} \quad \frac{s_1\mu_1}{(x+\mu_2)^2} + \frac{s_2\mu_2}{(x-\mu_1)^2} = x \tag{2.82}$$

This equation has three real solutions, x_{L1}, x_{L2}, x_{L3},
which are the positions of the *colinear* Lagrange points L1, L2, L3.
Their location must be solved for numerically as a function of the mass ratio $\mu_2/\mu_1 = m_2/m_1$.

Fig. 3.9 of M&D shows zero–velocity curves for a secondary of mass $\mu_2 = 0.01$.

Always keep in mind that zero–velocity curves are *not* orbits; rather, they are boundaries that cannot be crossed by a low–energy particle having a particular J.

Recall that an equilibrium point is a site where all the forces (here, gravity + centrifugal) sum to zero.

A *stable* equilibrium point is a site where, if a particle at equilibrium is displaced slightly, then the forces cause that particle to oscillate about equilibrium. Friction would also drive the particle back into the equilibrium.

But at an *unstable* equilibrium point, the forces on a displaced particle drive it further away from equilibrium.

According to Fig. 3.9, which LP's are stable, and which are unstable? Why

Section 3.7 examines rigorously the stability of particles at the LP's; and shows that L4, L5 are *stable* when $\mu_2 \lesssim 0.04$ (about 40 Jupiter masses).

If you displace a particle slightly from L4 or L5, it will oscillate about that site in a *tadpole* orbit (green). Such objects are sometimes called Trojans, since Jupiter's Trojan asteroids are the most famous example of bodies in tadpole orbits; see Fig. 3.23.

Neptune also has a few Trojans; they resemble icy Kuiper Belt Objects.

Trojans also occur in the Saturnian system:
For example, Tethys ($D \sim 1000$ km) has Telesto @L and Calypso @L5; these small satellites have $D \sim 20$ km.

Section 3.7 shows that the L1, L2, L3 points are always *unstable*.

If a particle is displaced slightly from L3, it enters what resembles a *horseshoe* orbit (blue) in this rotating reference frame.

Note that the horseshoe orbits circumscribe the tadpole orbits.

Objects in horseshoe or tadpole orbits are known as *coorbitals*, since their motion relative to the secondary is usually quite slow.

An interesting coorbital pair are the Saturnian satellites Janus & Epimetheus. Their masses are $m_{epi} \simeq 0.25 m_{janus}$,
so they seem to violate the spirit of the R3BP...

Note the zero velocity curve that passes through the unstable equilibria L1 and L2; recall that a level curve that passes through an unstable site is a *separatrix*, which is a barrier that divides the different kinds of motion.

If a particle is displaced inwards from L1, it will orbit the primary μ_1.
If displaced outwards from L1, it will orbit μ_2.
Similarly, the L2 sites separates satellite orbits from circumbinary orbits.

You can think of L1 and L2 as narrow 'gates' or 'keyholes' through with a particle can transit from circumplanetary↔circumstellar orbits.

The example of Comet Shoemake–Levy 9...

The Hill Sphere and the Roche Limit

The Hill sphere = volume around m_2 interior to the LP's L_1 and L_2.
This the volume where satellite orbits are stable,
and escape to a circumstellar orbit is forbidden.

The radius of the Hill sphere, R_H, is very important dynamical parameter.

For instance, if you are interested in the motion of a satellite orbiting m_2 at distances of $r_2 \ll R_H$, then you can often ignore the Sun's gravity. This is because the satellite is deep within m_2 potential well,
and its motion is essentially 2–body motion.

However if the satellite is in a wide orbit around m_2 such that $r_2 \sim R_H$, then you cannot ignore the Sun, and you must treat this as a 3–body problem.

Similarly, if particle P is in a low–energy orbit about the primary m_1, and it approaches m_2 to within a distance $r_2 \sim R_H$, then large changes in P's orbit are likely. These encounters are known as *gravitational scattering.*

But if the separations stay large, ie, $r_2 \gg R_H$, then the effects of gravitational scattering is weak, and only small orbit changes occur (usually)...

R_H is also applicable to binary star systems.

For example, R_H determines the volume in which planets might orbit m_2.

Or if star m_2 expands beyond its Hill sphere,
then m_2 can shed some mass!

Assignment #2, continued
due Thursday February 2
at the start of class

5. Use Eqn. (2.82) to show that a low mass secondary ($m_2 \ll m_1$) has a Hill radius

$$R_H \simeq \left(\frac{m_2}{3m_1}\right)^{1/3} a, \tag{2.83}$$

where a is the secondary's semimajor axis.

6. The Roche limit is usually thought of as the semimajor axis a_R where the primary's gravitational tide just exceeds the secondary's self–gravity, which causes the secondary to lose mass.

The Roche limit can also be regarded as the semimajor axis a_R of a body whose physical radius just exceeds its Hill radius R_H. Supposing that the primary and secondary are uniform spheres, show that this definition lead to a Roche limit of

$$a_R = \left(\frac{3\rho_1}{\rho_2}\right)^{1/3} R_1 = 1.44 \left(\frac{\rho_1}{\rho_2}\right)^{1/3} R_1 \tag{2.84}$$

where R_1 is the primary's radius.

Actually, assuming m_2 is a uniformly dense sphere is a poor approximation, since m_1's tide will distort m_2.

Also, the densities of m_1 and m_2 (which could be a planet or star) are not uniform.

An improved solution is obtained by allowing m_2 to be tidally distorted, yet in hydrostatic equilibrium; that alters the numerical coefficient $1.44 \rightarrow 2.45$.

Also, if m_2 is a rocky or icy satellite,
then its tensile strength resists tidal disruption,
which effectively pushes a_R inwards and closer to the primary.

Nonetheless, it is worth noting that all of the major planet's satellites reside at $a \gtrsim a_R$, while all the planetary rings reside at $a \lesssim a_R$.

However, small ($R \lesssim 10$ km) satellites can be found inside the giant planets' Roche limits.

An example is Pan, which orbits in Saturn's main A ring,
which is close to or interior to Saturn's Roche limit.

Hill's Equations

Hill's eqn's were derived by G. Hill in 1878,
are are another linearized set of EOM for the R3BP, but with the origin moved away from the system's COM and onto the secondary μ_2.

Moving the origin to μ_2 allows us to study the motion of particles as they are perturbed by a nearby planet/satellite.

These EOM are especially useful in planetary dynamics, and can be used to study:

- the motion of planetary ring particles perturbed by a nearby satellite,
- the accretion of a planet from a swarm of lower–mass bodies,
- the formation of binaries in the Kuiper Belt.

These EOM also have astrophysical applications, since they have been used to study the wake (or disturbance) that a massive body (a star, or a giant molecular cloud) generates as it travels through a galaxy.

Recall that the motion of particle P's guiding center
resembles the zero–velocity curves (ZVC).

The exact EOM for the coplanar problem, Eq. (2.28), is

$$\ddot{x} = 2n\dot{y} - n^2\left[\frac{\mu_1(x+\mu_2)}{r_1^3} + \frac{\mu_2(x-\mu_1)}{r_2^3} - x\right] \quad (2.85)$$

$$\ddot{y} = -2n\dot{x} - n^2\left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3} - 1\right] y \quad (2.86)$$

$$\text{where} \quad r_1 = \sqrt{(x+\mu_2)^2 + y^2} \quad (2.87)$$

$$\text{and} \quad r_2 = \sqrt{(x-\mu_1)^2 + y^2} \quad (2.88)$$

and all lengths are in units of the secondary's semimajor axis a.

Lets move the origin to μ_2, so $x = \mu_1 + x'$ where
x', y' is P's position relative to μ_2.

Our moving origin now co–rotates with the secondary,
such that the $\hat{\mathbf{x}}'$ direction points radially away from μ_1,
and $\hat{\mathbf{y}}'$ points in the direction of μ_2's motion.

In the planetary approximation, $\mu_1 \simeq 1$ and $\mu_2 \ll \mu_1$, and

$$r_1 = \sqrt{(1+x')^2 + y'^2} \simeq \sqrt{1+2x'} \simeq 1 + x' \quad (2.89)$$

$$\text{so} \quad r_1^{-3} \simeq 1 - 3x' \quad (2.90)$$

$$\text{and} \quad \ddot{x}' \simeq 2n\dot{y}' - n^2\left[(1+x')(1-3x') + \frac{\mu_2 x'}{\Delta^3} - 1 - x'\right] \quad (2.91)$$

$$\simeq 2n\dot{y}' + n^2\left[3 - \frac{\mu_2}{\Delta^3}\right] x' \quad (2.92)$$

$$\ddot{y}' \simeq -2n\dot{x}' + n^2\left[3x' - \frac{\mu_2}{\Delta^3}\right] y' \quad (2.93)$$

where $\Delta' = \sqrt{x'^2 + y'^2} = r_2$.

We are specifically interested in the motions of particles in the near–vicinity of μ_2's Hill sphere which has a radius $R_H = (\mu_2/3)^{1/3}$ in dimensionless units. Thus $\Delta, |x|, |y|$ are all of order $\sim \mu_2^{1/3} \ll 1$, while $\mu_2/\Delta^3 \sim \mathcal{O}(1) \gg |x|$ and

$$\ddot{x} \simeq 2n\dot{y} + n^2\left[3 - \frac{\mu_2}{\Delta^3}\right]x \tag{2.94}$$

$$\ddot{y} \simeq -2n\dot{x} - n^2\frac{\mu_2}{\Delta^3}y \tag{2.95}$$

and I've dropped the primes.
These linearized EOM are known as Hill's equations.

Now lets transform to a scale–invariant set of equations that is both dimensionless and contains *no* physical parameters.

1. Express all lengths in terms of R_H, ie, $x_h = x/R_H$, etc.

2. Use the dimensionless time coordinate $\tau = nt = 2\pi t/T$
where T is μ_2's orbit period$\Rightarrow$ τ increments by 2π with each orbit period:

$$\text{Thus} \quad dt = d\tau/n \quad \text{and} \quad dx = R_H dx_h \tag{2.96}$$

$$\text{so} \quad \frac{dx}{dt} = nR_H\frac{dx_h}{d\tau} \tag{2.97}$$

$$\text{and} \quad \frac{d^2x}{dt^2} = n\frac{d}{d\tau}\frac{dx}{dt} = n^2R_H\frac{d^2x_h}{d\tau^2} \tag{2.98}$$

Since $\mu_2 = 3R_H^3$, the EOM become

$$\frac{d^2x_h}{d\tau^2} = 2\frac{dy_h}{d\tau} + 3\left[1 - \frac{1}{\Delta_h^3}\right]x_h \tag{2.99}$$

$$\frac{d^2y_h}{d\tau^2} = -2\frac{dx_h}{d\tau} - \frac{3}{\Delta_h^3}y_h \tag{2.100}$$

$$\text{so} \quad \ddot{x} = 2\dot{y} + 3\left(1 - \frac{1}{\Delta^3}\right)x \tag{2.101}$$

$$\text{and} \quad \ddot{y} = -2\dot{x} - \frac{3}{\Delta^3}y \tag{2.102}$$

where I've dropped the h subscript, and the derivatives are wrt time τ.

Note that μ_2 has disappeared from the problem, which means that these EOM apply to *any* R3BP (provided $\mu_2 \ll 1$ of course),
ie, to binary stars having extreme mass ratios, the Pluto–Charon binary, satellites of the giant planets, etc.

We will use Hill's equations later in a more detailed examination of planetary rings.

Assignment #2, continued
due Thursday February 2
at the start of class

7. Show that the scale–invariant Hill eqn' for a particle's vertical motion is

$$\ddot{z} = -\left(1 + \frac{3}{\Delta^3}\right) z \tag{2.103}$$

where $\Delta =$ the particle's distance from m_2 in Hill units.

8. Start with the scale–invariant Hill eqn's, and *derive* a new Jacobi integral

$$J_h = \frac{1}{2}v^2 - U_h, \tag{2.104}$$

$$U_h(x, y, z) = \frac{3}{2}x^2 - \frac{1}{2}z^2 + \frac{3}{\Delta}, \tag{2.105}$$

where v is the particle's velocity in Hill units,
and U_h is the new effective potential in these units.

Fig. 3.30 shows the results of a numerical integration of Hill's eqns for numerous particles P initially in circular orbits having a wide range of *impact parameters* $x = (a_p - a)/R_H$.

First, note that Hill's approximation essentially removes the curvature of a circular orbit, thus unperturbed particles would move left↔right.

As expected, Ps with impact parameters $|x| \lesssim 1$ are in horseshoe orbits, while more distant Ps get more strongly perturbed, and enter higher e orbits.

But why don't we see the effects of the triangular Lagrange points in this diagram?

Evidently, Hill's eqn's break down as the Ps drift to large longitudes $y \gg 1$.

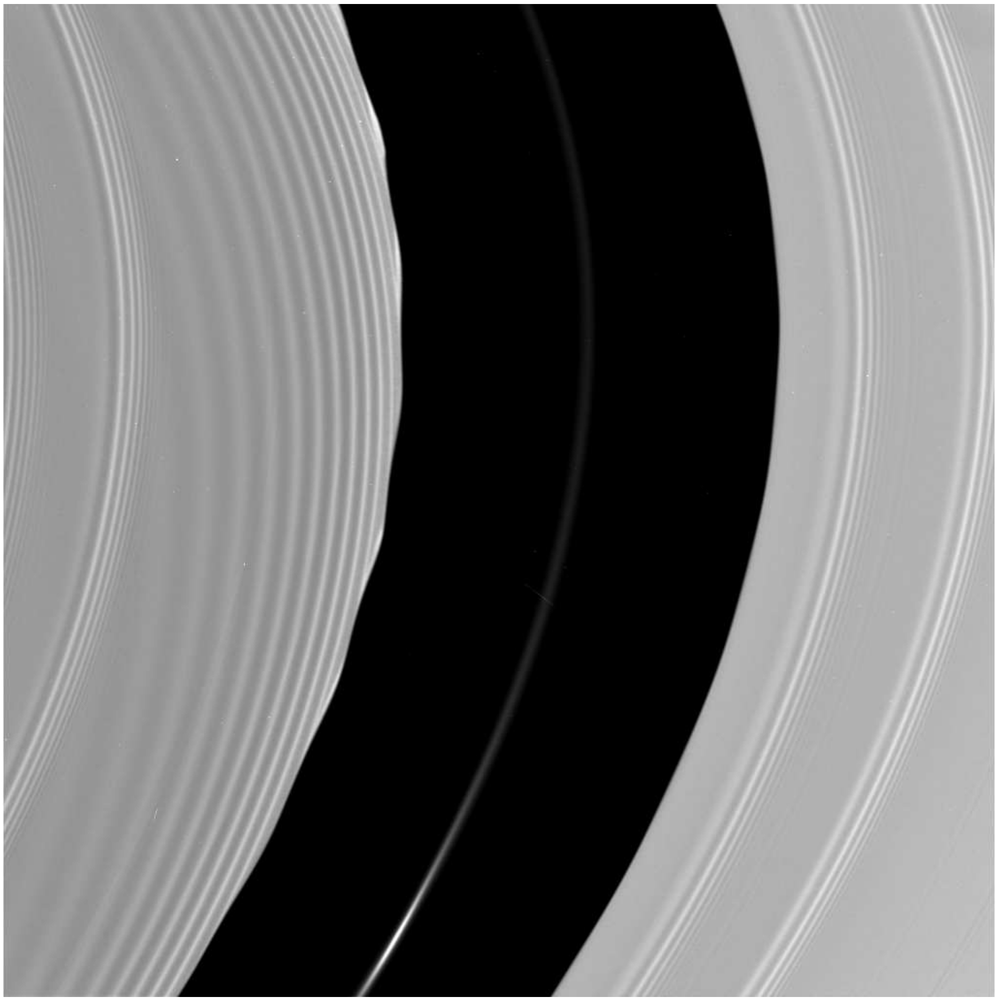
Where is periapse and apoapse in this diagram?

Which trajectories might correspond to ones that might collide with μ_2?

Also note the wavelike streamline in the more distant post–encounter trajectories, which are known as *wakes*.

Voyager saw wakes in Saturn's rings (see Fig 10.21—10.23), which were used to pinpoint the location of the previously unknown satellite Pan, which is $\sim$ 25 km. Pan's wakes are seen quite prominently in the newer Cassini images of the rings.

On the class website I have posted an animation of a numerical solution to Hill's equations for a swarm of ring particles that encounter a satellite. The particles start on circular orbits, and the animation shows horseshoe orbits, wakes, and scattered particles.



This figure shows a Cassini image of the Encke gap, which is a narrow 300 km–wide gap in Saturn's rings that is maintained by the small 25km satellite Pan.

A narrow ringlet also shares Pan's orbits.
What kind of orbits are these ring particles likely in?

Cassini is looking obliquely 'upwards' (northward).
Which way is Saturn?

Pan is not present in this image,
but its wakes run along the left edge of the gap.

In what direction should you look for it?

Other structures are spiral waves launched by satellites...more later.

Examine the motion of a particle that is NOT on a horseshoe orbit; inspection of Fig. 3.30 that the encounter excites the particle's eccentricity e.

Lets calculate that e, since it will be quantify the disturbance that a perturber (like a moon embedded in a planetary ring, or a planet embedded in a planet-forming disk) might excite.

The 2D Hill's EOM are

$$\ddot{x} - 2\dot{y} - 3x = -\frac{3x}{\Delta^3} \tag{2.106}$$

$$\ddot{y} + 2\dot{x} = -\frac{3y}{\Delta^3} \tag{2.107}$$

where distance from perturber m_2 is $\Delta = \sqrt{x^2 + y^2}$,
all lengths are in units of R_H, and the dimensionless time is $\tau = nt$.

First consider unperturbed motion, where $m_2 \to 0$.
Since $R_H \to 0$, $\Delta \to 0$ and RHS$\to 0$.

Unperturbed circular motion thus corresponds to

$$x(t) = b = \text{ constant} \tag{2.108}$$

$$\text{so} \quad \dot{y} = -\frac{3}{2}b \tag{2.109}$$

$$\text{and} \quad y = -\frac{3}{2}bt. \tag{2.110}$$

where the separation is $\Delta = b\sqrt{1 + (3t/2)^2} \equiv \Delta_0(t)$.

We will call this the zeroth order solution.

Note that $\dot{y}$ is the particle's angular velocity relative to m_2,
so the synodic period if $\tau_{syn} = 2\pi/\dot{y}$.
Does this agree with you answer to problem 3 of Assignment #1?

Now assume $m_2 \neq 0$, and that the encounter is distant, $|b| \gg 1$ so that the perturbation is weak. Then

$$x(t) = b + x_1(t) \tag{2.111}$$

$$y(t) = -\frac{3}{2}bt + y_1(t) \tag{2.112}$$

Now turn on the perturbation, and *linearize* the EOM, ie, assume $|x_1|$ & $|y_1|$ are $\ll |b|$.

Also, when calculating the disturbance (terms on RHS of EOM), adopt the particle's undisturbed trajectory:

$$\ddot{x}_1 - 2\dot{y}_1 - 3x_1 = -\frac{3b}{\Delta_0^3} \tag{2.113}$$

$$\ddot{y}_1 + 2\dot{x}_1 = \frac{9bt/2}{\Delta_0^3} \tag{2.114}$$

The solution will be first–order in the perturbation.

An analytic solution for these simple–looking EOM is rather difficult...

Instead, lets obtain a quick 'n dirty solution to this problem:
Assume that P's motion is simple 2-body epicyclic motion when far from m_2, and hyperbolic when near m_2:

The figure shows that P is scattered by m_2 through angle θ_s, which you obtained in problem #3 of Assignment 1:

$$\cos\frac{1}{2}(\theta_s+\pi) = -\frac{1}{e} \qquad \text{Eqn. (1.97)} \tag{2.115}$$

$$\text{so} \quad \sin\frac{1}{2}\theta_s = \frac{1}{e} \tag{2.116}$$

$$\text{where} \quad e = \sqrt{1+b^2v_\infty^4/\mu^2} \qquad \text{Eqn. (1.95)} \tag{2.117}$$

is the eccentricity of the hyperbolic orbit, $\mu = Gm_2$ (why?), P's velocity-at-infinity relative to m_2 is $v_\infty = 3bn/2$ (why?) where $n^2 = Gm_1/a^3$ (why?).

Distant encounters (ie, large b) are fast, so $e \gg 1$ and

$$\theta_s \simeq \frac{2}{e} \simeq \frac{2Gm_2}{bv_\infty^2} = \frac{8\mu_2}{9}\left(\frac{a}{b}\right)^3 = \frac{8}{3}\left(\frac{R_H}{b}\right)^3 \tag{2.118}$$

so the scattering angle is small (as expected) when $b \gg$ Hill radius R_H.

Recall that a hyperbolic scattering event does not alter P's energy; it merely alters the *direction* of P's outbound velocity $\mathbf{v}$, giving it a radial velocity in the $\hat{\mathbf{x}}$ direction:

$$|v_x| \simeq v_\infty\theta_s \;\rightarrow\; v_x \simeq -\frac{4\mu_2}{3}\left(\frac{a}{b}\right)^2 an \tag{2.119}$$

why the $-$ sign?

Long after the encounter, P resumes *epicyclic* motion about the primary m_1:

$$r(t) \simeq (a+b) - (a+b)e\cos nt \qquad \text{from Eqn. (1.98)} \tag{2.120}$$

$$\text{so} \quad \dot{r} \simeq ean\sin(nt) \qquad \text{since } |b| \ll a. \tag{2.121}$$

Since $\dot{r} \sim \mathcal{O}(v_x)$, this tells you that m_2's perturbation pumped P's eccentricity up to

$$e \sim \left|\frac{\dot{r}}{an}\right| \sim \frac{4}{3}\left(\frac{a}{b}\right)^2\mu_2 \simeq 1.33\left(\frac{a}{b}\right)^2\mu_2 \tag{2.122}$$

Julian & Toomre (1966) solved this problem in their study of stars in nearly circular orbits in a disk galaxy that are perturbed by another massive body; they provide a more exact solution to the linearized 3-body equations (2.113), and show that long after the encounter ($t \rightarrow +\infty$), P's radial motion is:

$$x_1(t) \rightarrow -\text{sgn}(b)\frac{8f}{3b^2}\sin\tau \tag{2.123}$$

(in dimensionless Hill units), where the coefficient
$f = 2K_0(2/3) + K_1(2/3) \simeq 2.52$ depends on modified Bessel functions K_i.

Convert this to physical units, ie, multiply lengths $\times R_H$ and replace $\tau \rightarrow nt$:

$$x_1(t) \rightarrow -\text{sgn}(b)\frac{8f\mu_2}{9}\left(\frac{a}{b}\right)^2\sin(nt)a \tag{2.124}$$

where a is m_s's semimajor axis.

This result was used to examine how a giant molecular cloud orbiting in a disk galaxy can pump up the radial motions of stars.

We will use it to study how a satellite orbiting in a planetary ring,
or how a planet in a circumstellar disk will open a gap about its orbit.

This process is known as *shepherding*—see below.

The eccentricity e inferred from the above exact solution is

$$e = \left|\frac{x_1}{a}\right| = \frac{8f}{9}\left(\frac{a}{b}\right)^2\mu_2 \simeq 2.24\left(\frac{a}{b}\right)^2\mu_2. \tag{2.125}$$

which is ~ 2 times larger that our earlier estimate.

As expected, particles having a smaller impact parameter $|b|$ gets excited to higher-e orbits.

Shepherding

Shepherding is the process by which a perturbed embedded in a disk tends to open a gap. To assess this, use the preceding solution to Hill's eqn' & the Jacobi integral J to calculate $\Delta a =$ P's change in semimajor axis after each encounter.

Particle P's dimensionless Jacobi integral written in terms of its osculating orbit elements a_P, e_P, i_P is Eqn. (2.69)

$$J' = \frac{a}{a_P} + 2\sqrt{\left(1+\frac{m_2}{m_1}\right)\frac{a_P}{a}(1-e_P^2)}\cos i_P + \frac{m_2 a}{m_1 \Delta} \quad (2.126)$$

Calculate J' long before & after the encounter,
when P's distance from m_2 is large, $\Delta \sim \mathcal{O}(a)$, $m_2 \ll m_1$, $i_P = 0$, $a_P = a+b$.
This allows us to ignore μ_2's gravitational contribution to J':

$$J' \simeq \left(1+\frac{b}{a}\right)^{-1} + 2\sqrt{\left(1+\frac{b}{a}\right)(1-e_P)^2} \quad (2.127)$$

If P is initially on a circular $e_P = 0$ orbit with small impact parameter, $|b| \ll a$ and

$$J'_{init} \simeq 3 + \frac{3}{4}\left(\frac{b}{a}\right)^2 + \mathcal{O}(b/a)^3 \quad (2.128)$$

After the encounter, P will have been nudged into an eccentric orbit ($e_P > 0$), and a new semimajor axis $a_p = a + b + \Delta a \equiv a + b'$, so

$$J'_{final} \simeq \left(1+\frac{b'}{a}\right)^{-1} + 2\sqrt{\left(1+\frac{b'}{a}\right)}\sqrt{(1-e_P)^2} \quad (2.129)$$

$$\simeq \left(1+\frac{b'}{a}\right)^{-1} + 2\sqrt{\left(1+\frac{b'}{a}\right)} - e_P^2 + \text{ small terms}^3 \quad (2.130)$$

$$\simeq 3 + \frac{3}{4}\left(\frac{b'}{a}\right)^2 - e_P^2 \quad (2.131)$$

$$\simeq 3 + \frac{3}{4}\left(\frac{b}{a}+\frac{\Delta a}{a}\right)^2 - e_P^2 \quad (2.132)$$

Assume the nudge is small compared to impact parameter, $|\Delta a| \ll |b|$. Then

$$J'_{final} \simeq J'_{init} + \frac{3}{2}\frac{b}{a}\frac{\Delta a}{a} - e_P^2 \tag{2.133}$$

Since J' is conserved, this implies that

$$\Delta a \simeq \text{sgn(b)}\frac{2}{3}\left|\frac{a}{b}\right| e_P^2 a \tag{2.134}$$

This is sometimes called the *impulse approximation*,
since μ_2 is impulsively nudging P's semimajor axis *away*,
and nudging is eccentricity e_P upwards as per Eqn. 2.125.

Problem 3 of Assignment #1 showed that P will encounter μ_2 again after one synodic period:

$$\Delta t = T_{syn} = \frac{4\pi}{3n}\left|\frac{a}{b}\right| \tag{2.135}$$

which is many orbital periods later when $|b| \ll a$.

Suppose particle P is orbiting in a disk that is crowded with many particles: (eg, a planetary ring, or a disk galaxy composed of many stars).

Subsequent interactions can damp out P's eccentricity:
via collisions among particles in a planetary ring,
or via dynamical friction in a galaxy due to interactions with other stars.

A suffiently massive planet still orbiting in its natal circumstellar gas disk can also open a gap in the gas disk—why?

In such cases, P will approach μ_2 again in a circular orbit
which will nudge its semimajor axis away at the average rate

$$\dot{a} = \frac{\Delta a}{\Delta t} = \text{sgn(b)}\frac{32 f^2}{81\pi}\left|\frac{a}{b}\right|^4 \mu_2^2 a n \tag{2.136}$$

This is shepherding: the process by which perturber μ_2 embedded in a dense disk tends to open a gap in the disk about μ_2's orbit. This occurs provided the particle's e's get damped prior to each encounter with μ_2.

Pan orbits in the narrow Encke gap in Saturn's rings.
What then prevents Pan from making the Encke gap ever wider?

Assignment #3
due Tuesday February 14
at the start of class

1. Eqn. (2.134) was derived by assuming that the displacement $|\Delta a| \ll |b|$. Use this equation to prove that our assumption is indeed valid when $|b| \gg 1.8$, ie, P's impact parameter is larger than about 1.8 Hill radii.

2. Note the Encke gap's wavy edge. Show that these so–called edge waves have a longitudinal wavelength (ie, peak-to-peak distance) of

$$\lambda = 3\pi x \tag{2.137}$$

where x is Pan's distance from the gap edge.

3. Pan orbits $x = 160$ km from edge of the Encke gap. What is the *radial* amplitude of these edge waves, in units of km? Explain your calculation.

4. A small particle of mass m is in a circular orbit, and it is subject to torque T that is normal to its orbit plane. Show that this torque drives the particle's semimajor axis at the rate

$$\dot{a} = \frac{2T}{man}. \tag{2.138}$$

Aside: recall that a small body's total angular momentum is $\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}$, and that the torque on that body is

$$\mathbf{T} = \frac{d\mathbf{L}}{dt} = m\mathbf{r} \times \ddot{\mathbf{r}} \tag{2.139}$$

Torque Density

Suppose μ_2 is now disturbing a dense *ring* of particles having a mass surface density σ, orbital radius $r = a + b$, and narrow radial width Δr.
Assume all e's gets damped downstream via particle–particle interactions.

What is the mass of this ring? $\Delta m = \sigma \Delta A = 2\pi\sigma r \Delta r$.

What is the total torque ΔT that μ_2 exerts on the ring?

$$\Delta T = \frac{1}{2} a n \dot{a} \Delta m \tag{2.140}$$

$$\text{so} \quad \frac{dT}{dr} = \text{sgn}(b)(\pi\sigma r^2)\frac{32 f^2}{81\pi}\left(\frac{a}{r-a}\right)^4 \mu_2^2 a n^2 \tag{2.141}$$

$$= \text{sgn}(b)\frac{32 f^2}{81\pi}\left(\frac{a}{r-a}\right)^4 \mu_d \mu_2^2 m_1 a n^2 \tag{2.142}$$

$$\text{where} \quad \mu_d \equiv \frac{\pi\sigma r^2}{m_1} \quad \text{(the so–called normalized disk mass)} \tag{2.143}$$

is the torque *radial density*,
ie $\frac{dT}{dr}\Delta r =$ total torque that μ_2 exerts on a ring of radius r and width Δr.

Assignment #3, continued
due Tuesday February 14
at the start of class

5. Suppose μ_2 orbits a radial distance x beyond a disk having a constant surface density σ. Show that the total torques it exerts on the disk is

$$T \;=\; \int_{disk} \frac{dT}{dr}dr \simeq -\frac{32f^2}{243\pi}\left|\frac{a}{x}\right|^3 \mu_d\mu_2^2 m_1(an)^2 \qquad (2.144)$$

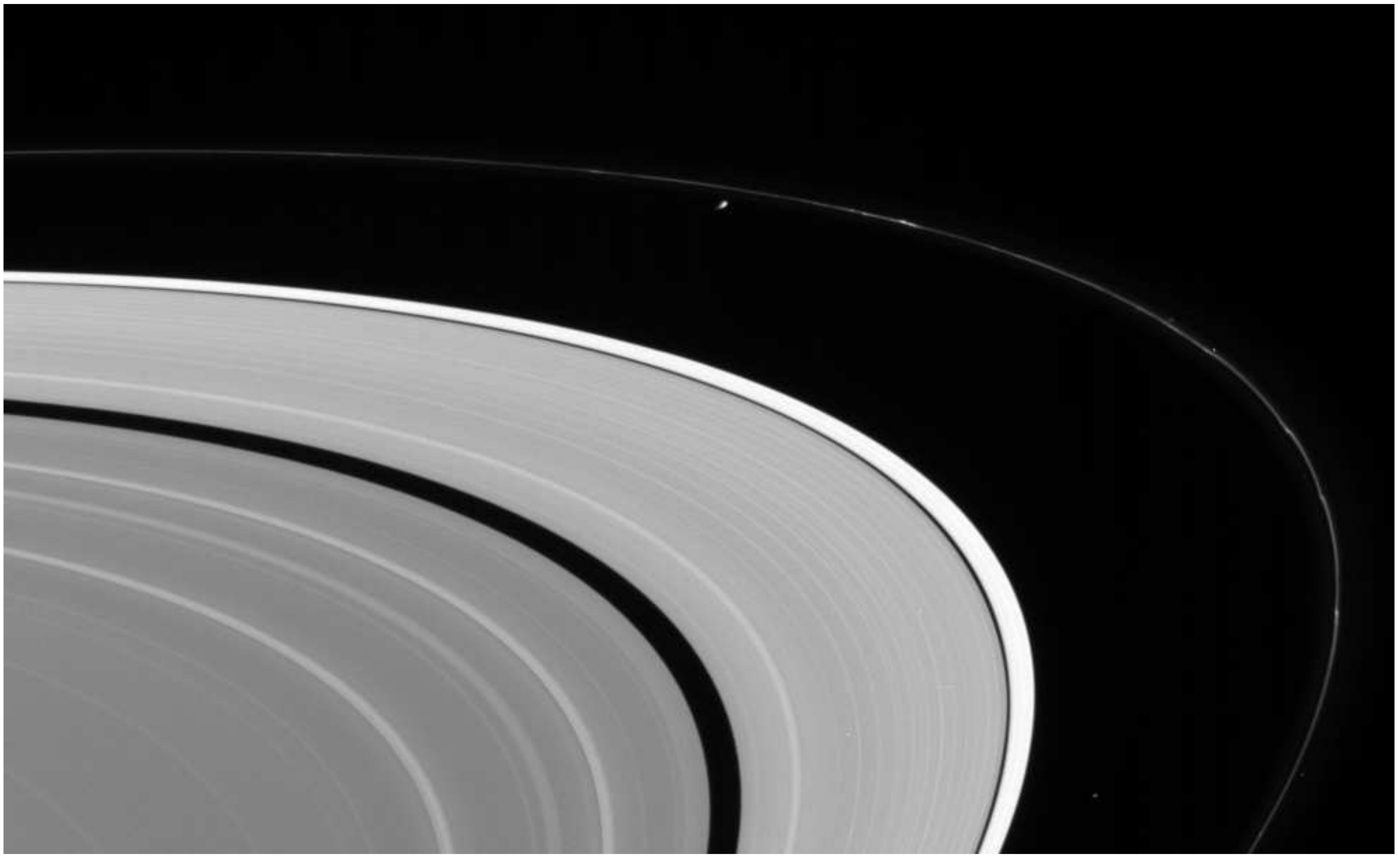
where $|x|$ is small compared to μ_2's semimajor axis a. How will the disk respond to μ_2's perturbations? How will μ_2's orbit to evolve over time?

6. Suppose μ_2 formed at the outer edge of a disk of radius a, and that the shepherding torques have since driven radially outwards a small distance $x \ll a$. Show that μ_2's travel–time t is

$$t \simeq \frac{243\pi}{256f^2}\left|\frac{x}{a}\right|^4 \frac{1}{\mu_d\mu_2 n} \qquad (2.145)$$

7. The small satellite Prometheus orbits $x = 2570$km beyond the outer edge of Saturn's main A ring, which has a mass surface density of $\sigma \sim 100$ gm/cm^2. Suppose Prometheus formed at the ring edge. How long ago did that happen, in years? How many orbital periods is that? What does this calculation tell you about the age of this part of the ring system and the nearby satellite? (Actually, your age estimate is merely a lower limit on the true age...). See Appendix A of M&D for planet & satellite data.

We will use the above calculations again later when we study the dynamics of recently–formed planets that are interacting with a circumstellar gas disk.



Saturn's A ring and two gaps: Encke & Keeler gap (narrower one).
Beyond that is the narrow F ring, which is straddled by the two shepherd satellites Prometheus (seen here) & Pandora.