Chapter 16: Oscillations (Sections 1-7)

oscillation = motion that repeats (ie, periodic motion)

examples: pendulum in grandfather clock
vibrating strings (on guitar, violin, etc)
drum
air molecules - occurs as they transmit a sound wave
electrons in antenna - driven by electro-magnetic waves from TV or radio station.

⇒ you are surrounded by oscillations.

Simple Harmonic motion (SHM)

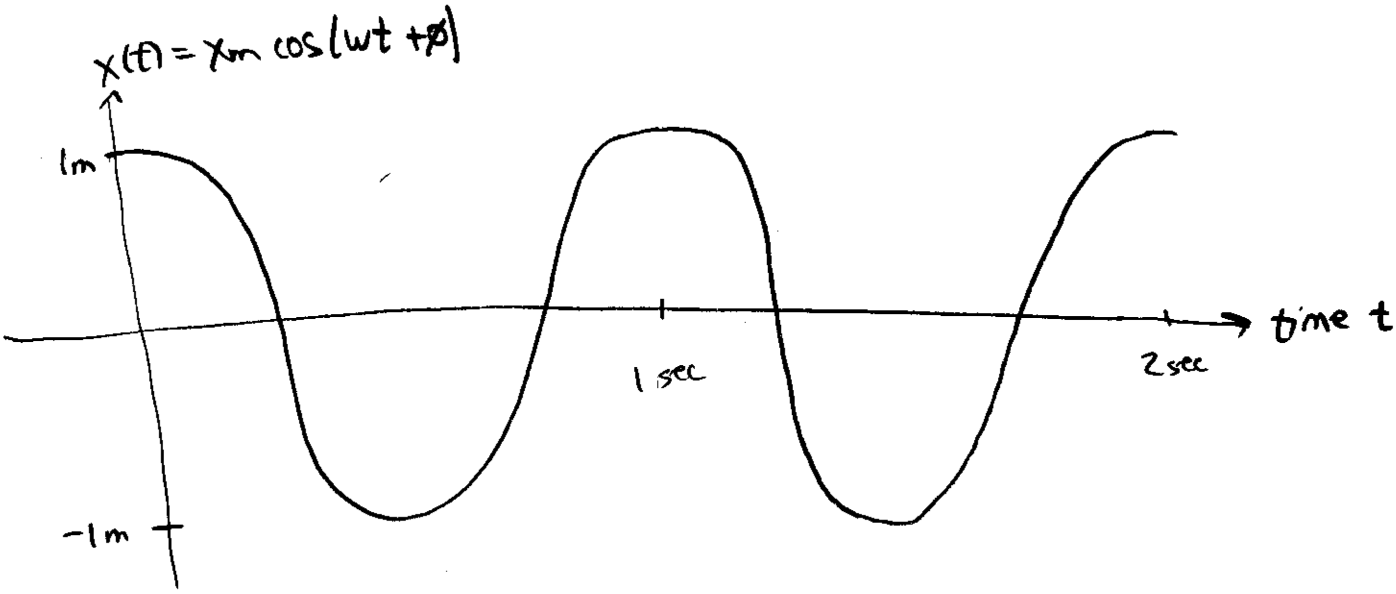
= sinusoidal motion

SHM can be expressed as a function of time t as:

$$x(t) = x_m \cos(\omega t + \phi)$$

where $x(t)$ = displacement of some particle at time t
x_m = amplitude, or maximum displacement
ω = angular frequency
ϕ = phase constant (eg, used to specify where the particle is at time t=0)

also let T = period = time for the motion to repeat
= time elapsed as the argument $(\omega t + \phi)$ increases by 2π



what is x_m ?

ϕ ?

period T ? = time for argument to increment by 2π:

$$\omega(t+T) + \phi = \omega t + \phi + 2\pi$$

$$\Rightarrow \omega T = 2\pi$$

so $T = \frac{2\pi}{\omega}$ = period (units of time, usually seconds)

or $\omega = \frac{2\pi}{T}$ = angular frequency (units of radians per second)

Sometimes we also use

frequency $f = \frac{\omega}{2\pi} = \frac{1}{T}$ = # of cycles completed per second
units = Hz (Hertz)

or $\omega = 2\pi f$

Please dont confuse angular frequency ω with frequency f

Velocity of SHM:

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}\left[x_m \cos(\omega t + \phi)\right]$$

$$= x_m \frac{d}{dt} \cos(\omega t + \phi)$$

Use Chain Rule: $$= -x_m \sin(\omega t + \phi) \frac{d}{dt}(\omega t + \phi)$$

$$= -x_m \omega \sin(\omega t + \phi)$$

Likewise,

acceleration $a(t) = \dfrac{dv}{dt} = -x_m \omega^2 \cos(\omega t + \phi)$

$\boxed{a(t) = -\omega^2 x(t)}$ ⇐ equation of motion for a body in SHM

(we will come back to this later)

So force $F = ma$

$= -m\omega^2 x$ = force on particle that executes SHM

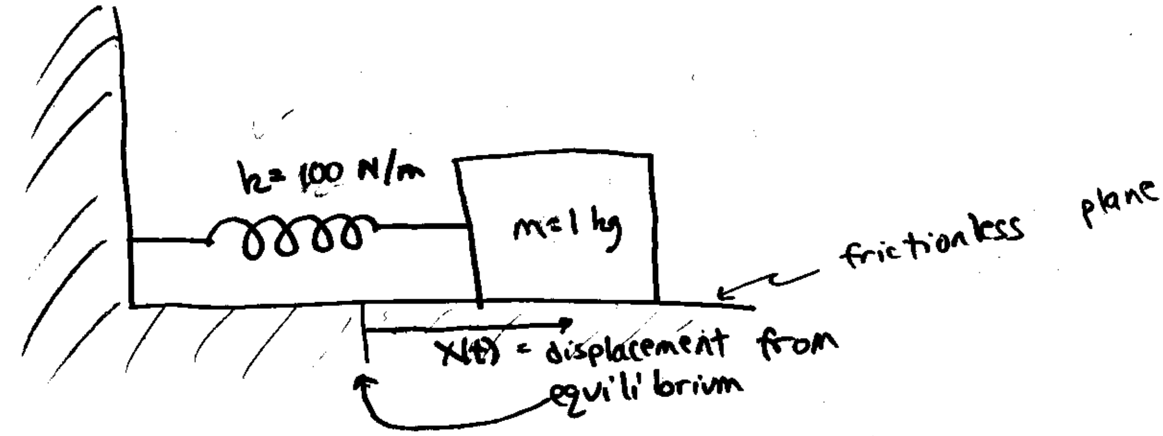
What force law does this resemble?

$$F = -m\omega^2 x = -kx$$

⇒ spring constant $\boxed{k = m\omega^2}$

⇒ a body that is executing SHM behaves as if subject to a spring-force.

Example



What is mass m's angular frequency ω in terms of k and m?

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100\ N/m}{1\,kg}}$$

recall $1\,N = 1\ kg \cdot m/sec^2$

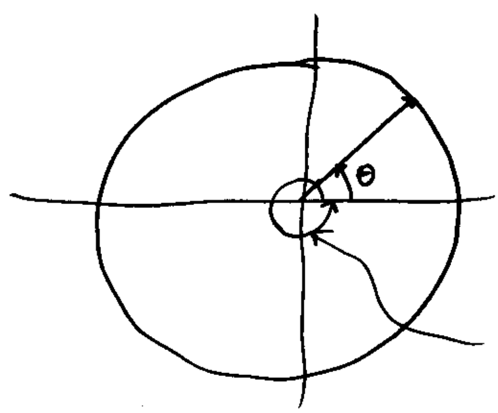
a dimensionless unit of angle

so $\omega = \sqrt{\frac{100}{sec^2}} = 10$ radians/sec

frequency $f = \frac{\omega}{2\pi} = \frac{10\ radians/sec}{2 \times 3.14\ radians} \approx 1.6\ sec^{-1}$

$= 1.6$ Hz

period $T = \frac{1}{f} = \frac{2\pi}{\omega} = 0.63\ Hz^{-1} = 0.63$ sec



Recall that a circle encloses an angle of 2π radians

Energy when in SHM:

Total energy $E = K + U$ (from chapter 8)

The potential energy for a spring is

$$U = \frac{1}{2}kx^2 \qquad \text{(Eqn 8-11)}$$

$$= \frac{1}{2}m\omega^2 x^2$$

$$= \frac{1}{2}m\omega^2 x_m^2 \cos^2(\omega t + \phi)$$

The system's kinetic energy is

$$K = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m x_m^2 \omega^2 \sin^2(\omega t + \phi)$$

so the system's total energy is

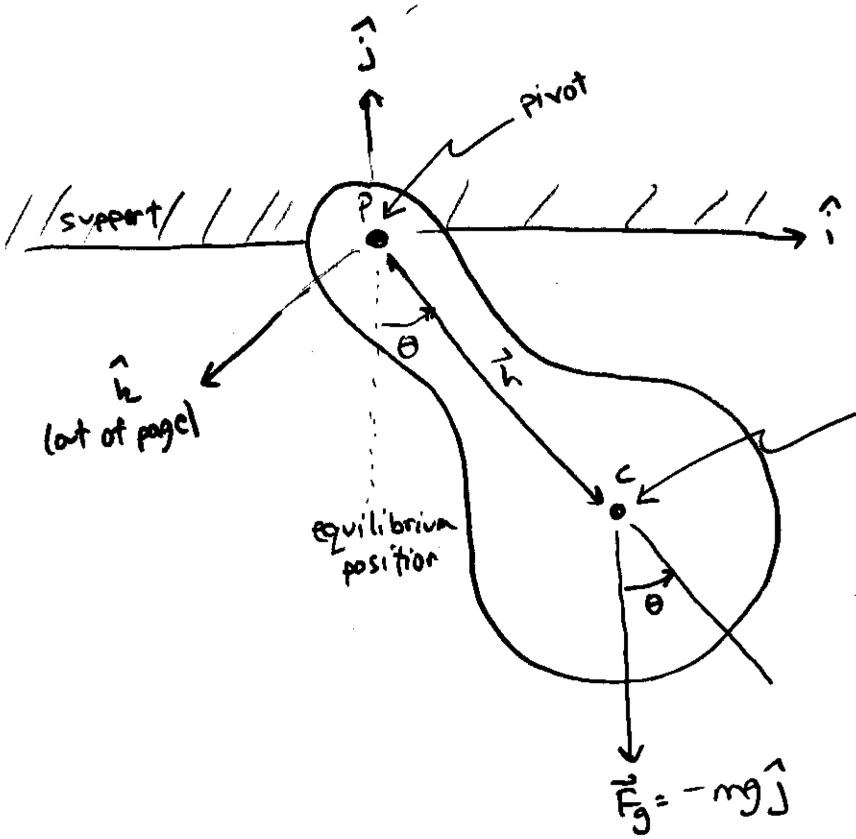
$$E = K + U = \frac{1}{2}m\omega^2 x_m^2 \left[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right]$$

$$= \frac{1}{2}m\omega^2 x_m^2$$

= constant, independant of time t

What if we added friction to our system?

Physical Pendulum = a body that is free to rotate about some pivot point P:



consider the torque on the pendulum due to gravity:

(formula I use)

torque $\vec{\tau} = \vec{h} \times \vec{F}_g$

vector "cross-product"

force

"lever-arm" from pivot P to center-of-mass C

$= h F_g \sin\theta \, (-\hat{k})$

$= -mgh\sin\theta \, \hat{k}$ ← this is a vector

so $\tau = -mgh\sin\theta$ is the magnitude (scalar) of the torque text uses alternate formula (see Chapter 11)

$\tau = h F_g \sin(-\theta)$ ← scalar

$= -mgh\sin\theta$ since $\sin(-\theta) = -\sin\theta$

minus sign indicates that this torque strives to reduce θ.

Now recall that $\tau = I\alpha$ (Eqn 11-36)

where $I = \int r^2 dm$ (Eqn 11-28)

= the body's rotational inertia

and $\alpha = \frac{d^2\theta}{dt^2}$ = the body's angular acceleration

So $\tau = -mgh\sin\theta = I\alpha$

or $\alpha = \frac{d^2\theta}{dt^2} = -\frac{mgh}{I}\sin\theta$

Now lets consider small oscillations, ie $|\theta| \ll 1$

so $\sin\theta \simeq \theta$

Aside:
you can check that this approximation is pretty good for angular displacements of $|\theta| \lesssim 0.25$ radians; using a calculator will show that the errors are $\lesssim 1\%$.

How many degrees corresponds to this angle?

2π radians = 360 degrees so 1 rad = $\frac{360}{2\pi}$ deg $\simeq 60$ deg

so 0.25 rad $\simeq$ 0.25×60 deg $\simeq 15$ deg

Our equation of motion becomes

$$\alpha = \frac{d^2\theta}{dt^2} \approx -\frac{mgh}{I}\theta$$

solution:
$\theta(t) = \theta_{max}\cos(\omega t + \phi)$

$$\boxed{\alpha \equiv -\omega^2\theta}$$

angular displacement

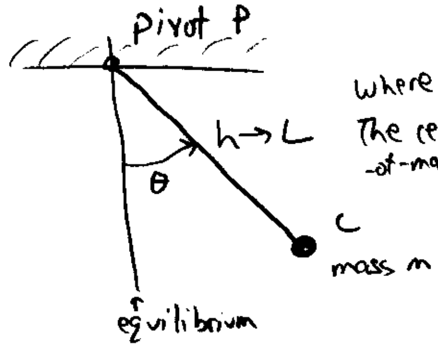
angular acceleration

where $\omega = \sqrt{\frac{mgh}{I}}$ = physical pendulum's angular frequency.

recall acceleration $a = -\omega^2 x$ where x = displacement of a body is SHM; the above is the angular analog of our earlier equation-of-motion.

Note that the rotational inertia I is provided by Table 11-2 for a variety of objects (disks, spheres, cylinders, etc). Use that table to avoid laborious calculations of $I = \int r^2 dm$

<u>Simple Pendulum</u> – mass m resides at the end of a massless string of length L:



where is The center-of-mass?

$$I = \int r^2 dm = L^2 m$$

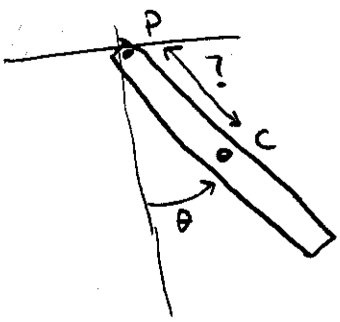
so $$\omega = \sqrt{\frac{mgL}{L^2 m}} = \sqrt{\frac{g}{L}}$$

= simple pendulum's angular frequency.

and period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$ = period for

Example: consider a physical pendulum that is comorised of a rod of length l and mass m:

<u>Where is its center-of-mass C?</u>



This physical pendulum has a moment of inertia $I = \frac{1}{3}ml^2$ (see Table 11-2 & Eqn 11-29)

its period of oscillation is $T_P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mg\,l/2}}$

so $$T_P = 2\pi\sqrt{\frac{2}{3}\frac{l}{g}}$$

Now suppose a <u>simple</u> pendulum has mass m and length L. What must its length be in order to have the same period?

$$L_S = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{2}{3}\frac{l}{g}} \Rightarrow L = \frac{2}{3}l$$

⇒ The simple pendulum is shorter by $\frac{2}{3}$, due to its higher inertia