Lecture Notes for PHY 405
Classical Mechanics

From Thorton & Marion's *Classical Mechanics*

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Chapter 9: Dynamics of systems of particles

To date we have focused on the motion of a single particle
(ex: the motion of a block on an inclined plane, the plane pendulum, etc.).

We also examined the 2–body central force problem by recasting it as 1–body problem.

Now we will consider N–body systems for the remainder of this course.

Our first task is to derive the relevant conservation theorems for systems of N particles (as we did earlier for N=1 systems).

Note that these results will be true for a swarm of N interacting particles, as well as for a single extended body that can be conceptually broken up into N smaller units.

Strong & Weak forms of Newton III

First consider the force $\mathbf{f}_{\alpha\beta}$ that particle β exerts on particle α:
Newton's IIIrd Law is

$$\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$$

e.g., the forces exerted by particles α and β
are equal in magnitude and opposite in direction.
This is sometimes referred to as the weak form of Newton III.

The strong form of Newton III reads:
the forces $\mathbf{f}_{\alpha\beta}$ are also parallel to the line connecting α and β.

The additional assumption is generally true in mechanical systems
(ie, the physics of solid bodies).
However it is not true for all forces, such as the magnetic force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$.
Such forces will not be considered here.

We will use the weak and strong forms of Newton III below as we derive various conservation theorems for N–body systems.

Center of Mass

The center of mass (CoM) $\mathbf{R}$ for a system of N discrete particles is

$$\mathbf{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha$$

$$\text{where} \quad M = \sum_{\alpha=1}^{N} m_\alpha = \text{ total mass}$$

To compute $\mathbf{R}$ for a continuous body,

$$d\mathbf{R} = \mathbf{r}\frac{dm}{M} = \text{ contribution by small mass } dm \text{ at } \mathbf{r}$$

$$\text{so} \quad \mathbf{R} = \int d\mathbf{R} = \frac{1}{M}\int \mathbf{r}\,dm$$

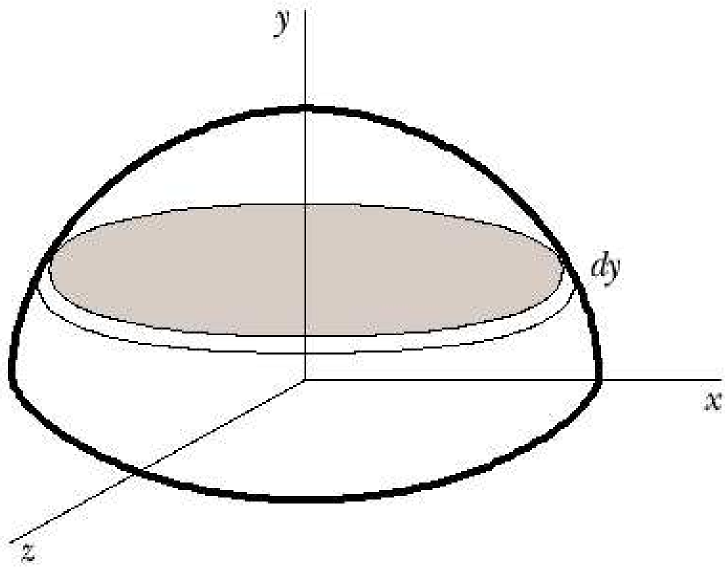

Fig. 9–3.

Example 9.1:

Calculate $\mathbf{R}$ for a hemisphere of mass M, radius a, and uniform density $\rho = 3M/2\pi a^3$:

first place the origin the center of the hemisphere's base and set

$$\begin{aligned}
\mathbf{R} &= R_x\hat{\mathbf{x}} + R_y\hat{\mathbf{y}} + R_z\hat{\mathbf{z}} \\
\text{so} \quad R_x &= \frac{1}{M}\int xdm \quad \text{where } dm = \rho dxdydx \\
&= \frac{1}{M}\int_{-a}^{a} xdx \int dy \int dz \\
&= 0 \\
\text{similarly} \quad R_y &= 0 \\
\text{however} \quad R_z &= \frac{1}{M}\int_0^a zdm \quad \text{where } dm = \rho\pi(a^2 - z^2)dz \\
&= \frac{\rho\pi}{M}\int_0^a (a^2 z - z^3)dz \\
&= \frac{3}{2a^3}\left(\frac{1}{2} - \frac{1}{4}\right)a^4 = \frac{3}{8}a \\
&\Rightarrow \mathbf{R} = \frac{3}{8}a\hat{\mathbf{z}}
\end{aligned}$$

Now add the lower hemisphere to form a complete sphere. What is $\mathbf{R}$?

Now derive a number of conservation theorems for systems of N particles: conservation of $\mathbf{P}$, $\mathbf{L}$, and E.

N–body forces

$$\begin{aligned}
\text{Let}\quad \mathbf{f}_{\alpha\beta} &= \text{the force on particle } \alpha \text{ due to particle } \beta \\
\text{so}\quad \mathbf{f}_{\alpha} &= \sum_{\beta=1}^{N} \mathbf{f}_{\alpha\beta} = \text{ the total force on } \alpha \text{ due to all other p's } \beta. \\
&\equiv \text{the net } \textit{internal} \text{ force on } \alpha \\
\text{note that}\quad \mathbf{f}_{\alpha\alpha} &= 0 \quad \text{and } \mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha} \quad \text{by NIII} \\
\text{also let } \mathbf{F}_{\alpha}^{e} &= \text{the } \textit{external} \text{ force on } \alpha\text{—gravity, for example} \\
\text{the total force on } \alpha \text{ is} &= \mathbf{f}_{\alpha} + \mathbf{F}_{\alpha}^{e}
\end{aligned}$$

Now recall that Newton's IInd Law, $\dot{\mathbf{p}} = \mathbf{F}$, ie,

$$\begin{aligned}
\dot{\mathbf{p}}_{\alpha} &= m_{\alpha}\ddot{\mathbf{r}}_{\alpha} = \mathbf{f}_{\alpha} + \mathbf{F}_{\alpha}^{e} \\
\text{so}\quad \frac{d^2}{dt^2} m_{\alpha}\mathbf{r}_{\alpha} &= \sum_{\beta=1}^{N} \mathbf{f}_{\alpha\beta} + \mathbf{F}_{\alpha}^{e} \\
\text{Now sum over all particles } \alpha\text{:}\ \frac{d^2}{dt^2}\sum_{\alpha=1}^{N} m_{\alpha}\mathbf{r}_{\alpha} &= \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} \mathbf{f}_{\alpha\beta} + \sum_{\alpha=1}^{N} \mathbf{F}_{\alpha}^{e} \\
\text{and note that the LHS} &= M\ddot{\mathbf{R}}
\end{aligned}$$

Now consider the first sum on the right,
which is the system's total internal force $\mathbf{F}^i$:

$$\begin{aligned} \mathbf{F}^i &\equiv \sum_\alpha \sum_\beta \mathbf{f}_{\alpha\beta} = -\sum_\alpha \sum_\beta \mathbf{f}_{\beta\alpha} \\ &= -\sum_\alpha \sum_\beta \mathbf{f}_{\alpha\beta} \quad \text{upon swapping the dummy indices } \alpha \leftrightarrow \beta \\ &= -\mathbf{F}^i \\ \Rightarrow \mathbf{F}^i &= 0 \quad \text{the internal forces sum to zero (due to weak Newton III)} \end{aligned}$$

$$\begin{aligned} \text{thus} \quad M\ddot{\mathbf{R}} &= \mathbf{F}^e \\ \text{where} \quad \mathbf{F}^e &= \sum_{\alpha=1}^{N} \mathbf{F}^e_\alpha \text{ sum of all external forces on all particles} \end{aligned}$$

This is *result I: The system's CoM* $\mathbf{R}$ *evolves as if it were a single body of mass* M *under the influence of the total external force* $\mathbf{F}^e$*.*

Conservation of Linear Momentum P

$$\begin{aligned} \text{total momentum} \quad \mathbf{P} &= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \\ &= \frac{d}{dt} \sum_\alpha m_\alpha \mathbf{r}_\alpha = M\dot{\mathbf{R}} \\ \text{thus} \quad \dot{\mathbf{P}} &= M\ddot{\mathbf{R}} = \mathbf{F}^e \end{aligned}$$

II. the system's total linear momentum $\mathbf{P}$ *is the same as if the system were a single body of mass* M *located at the CoM* $\mathbf{R}$*.*

III. $\mathbf{P}$ *is conserved when* $\mathbf{F}^e = 0$*.*

Problem 9–6

Two particles of mass m start at the origin. Particle 1 feels zero force, while particle 2 feels $\mathbf{F}_2 = F\hat{\mathbf{x}}$.

What are the particle's motions & the CoM motion?
We anticipate $x_1(t) = 0$ and $x_2(t) = Ft^2/2m$ so
$x_{CoM}(t) = x_2/2 = Ft^2/4m$

Confirm:

$$\begin{aligned} M\ddot{\mathbf{R}}_{CoM} &= \mathbf{F} = F\hat{\mathbf{x}} \\ \text{so} \quad \ddot{x}_{CoM} &= \frac{F}{2m} \\ \text{and} \quad \dot{x}_{CoM} &= \frac{Ft}{2m} \\ \text{and} \quad x_{CoM} &= \frac{Ft^2}{4m} = \frac{1}{2}x_2 \quad \text{as expected} \end{aligned}$$

Angular Momentum L

Write each particle's position $\mathbf{r}_\alpha = \mathbf{R} + \mathbf{r}'_\alpha$ so that $\mathbf{r}'_\alpha =$ distance of particle α from the CoM:

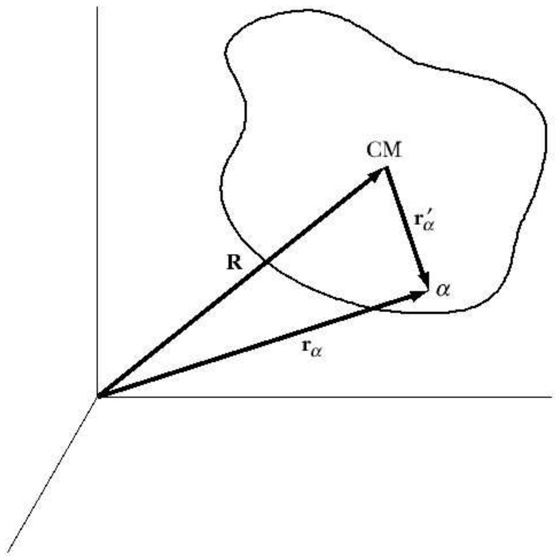

Fig. 9–5.

$$
\begin{aligned}
\text{since} \quad \mathbf{L}_\alpha &= \mathbf{r}_\alpha \times \mathbf{p}_\alpha = \text{ particle } \alpha\text{'a angular momentum,} \\
\text{total ang' mom' is} \quad \mathbf{L} &= \sum_\alpha \mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha \\
&= \sum_\alpha m_\alpha (\mathbf{R} + \mathbf{r}'_\alpha) \times (\dot{\mathbf{R}} + \dot{\mathbf{r}}'_\alpha) \\
&= \sum_\alpha m_\alpha (\mathbf{R} \times \dot{\mathbf{R}} + \mathbf{R} \times \dot{\mathbf{r}}'_\alpha + \mathbf{r}'_\alpha \times \dot{\mathbf{R}} + \mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha)
\end{aligned}
$$

Now show that the middle terms (MTs) sum to zero:

$$
\begin{aligned}
\text{MTs} &= \mathbf{R} \times \frac{d}{dt}\left(\sum_\alpha m_\alpha \mathbf{r}'_\alpha\right) + \left(\sum_\alpha m_\alpha \mathbf{r}'_\alpha\right) \times \dot{\mathbf{R}} \\
\text{but} \quad \sum_\alpha m_\alpha \mathbf{r}'_\alpha &= \sum_\alpha m_\alpha (\mathbf{r}_\alpha - \mathbf{R}) = M\mathbf{R} - M\mathbf{R} = 0 \\
\Rightarrow \text{MTs} &= 0 \\
\text{and} \quad \mathbf{L} &= \mathbf{R} \times \mathbf{P} + \sum_\alpha \mathbf{r}'_\alpha \times \mathbf{p}'_\alpha \quad \text{where } \mathbf{p}'_\alpha = m_\alpha \dot{\mathbf{r}}'_\alpha
\end{aligned}
$$

IV. The system's total angular momentum = angular momentum of the CoM about the origin ($\mathbf{R} \times \mathbf{P}$) plus the angular momentum of the system about the CoM ($\sum_\alpha \mathbf{r}'_\alpha \times \mathbf{p}'_\alpha$).

L conservation

Particle α's angular momentum $\mathbf{L}_\alpha = \mathbf{r}_\alpha \times \mathbf{p}_\alpha$, so its time rate–of–change is

$$\begin{aligned}\dot{\mathbf{L}}_\alpha &= m_\alpha \dot{\mathbf{r}}_\alpha \times \dot{\mathbf{r}}_\alpha + m_\alpha \mathbf{r}_\alpha \times \ddot{\mathbf{r}}_\alpha \\ &= \mathbf{r}_\alpha \times \dot{\mathbf{p}}_\alpha \quad \text{where} \quad \dot{\mathbf{p}}_\alpha = \sum_{\beta=1}^{N} \mathbf{f}_{\alpha\beta} + \mathbf{F}^e_\alpha \\ &= \mathbf{r}_\alpha \times \left(\sum_{\beta=1}^{N} \mathbf{f}_{\alpha\beta} + \mathbf{F}^e_\alpha \right)\end{aligned}$$

The total rate–of–change of the system's angular momentum is then

$$\dot{\mathbf{L}} = \sum_\alpha \dot{\mathbf{L}}_\alpha = \sum_\alpha \mathbf{r}_\alpha \times \left(\sum_\beta \mathbf{f}_{\alpha\beta} + \mathbf{F}^e_\alpha \right)$$

Now show that the first term (FT) on the right is zero:

$$\begin{aligned}FT &= \sum_{\alpha=1}^{N} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \mathbf{r}_\alpha \times \mathbf{f}_{\alpha\beta} \\ &= \sum_\alpha \sum_{\beta\neq\alpha} \mathbf{N}_{\alpha\beta} \\ \text{where } N_{\alpha\beta} &= \mathbf{r}_\alpha \times \mathbf{f}_{\alpha\beta} = \text{ torque on } \alpha \text{ due to } \beta\end{aligned}$$

$$\text{now note that} \quad \sum_{\alpha=1}^{N} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \mathbf{N}_{\alpha\beta} = \sum_{\alpha=1}^{N} \sum_{\beta=\alpha+1}^{N} (\mathbf{N}_{\alpha\beta} + \mathbf{N}_{\beta\alpha}) = \sum_{\alpha<\beta} (\mathbf{N}_{\alpha\beta} + \mathbf{N}_{\beta\alpha})$$

Confirm the above for an $N = 3$ system:

$$\begin{aligned}LHS &= N_{12} + N_{13} + N_{21} + N_{23} + N_{31} + N_{32} \\ RHS &= (N_{12} + N_{21}) + (N_{13} + N_{31}) + (N_{23} + N_{32}) = LHS \quad \checkmark\end{aligned}$$

To formally prove that the above is

$$\sum_{\beta\neq\alpha} N_{\alpha\beta} \;=\; \sum_{\alpha<\beta}(\mathbf{N}_{\alpha\beta}+\mathbf{N}_{\beta\alpha}),$$

note that the LHS and RHS are the same sums over the non–diagonal matrix whose elements are $N_{\alpha\beta}$. Then

$$\begin{aligned} FT &= \sum_{\alpha<\beta}(\mathbf{N}_{\alpha\beta}+\mathbf{N}_{\beta\alpha}) \\ &= \sum_{\alpha<\beta}(\mathbf{r}_\alpha\times\mathbf{f}_{\alpha\beta}+\mathbf{r}_\beta\times\mathbf{f}_{\beta\alpha}) \\ &= \sum_{\alpha<\beta}(\mathbf{r}_\alpha-\mathbf{r}_\beta)\times\mathbf{f}_{\alpha\beta} \quad \text{since } \mathbf{f}_{\alpha\beta}=-\mathbf{f}_{\beta\alpha} \\ &= \sum_{\alpha<\beta}\mathbf{r}_{\alpha\beta}\times\mathbf{f}_{\alpha\beta} \end{aligned}$$

where $\mathbf{r}_{\alpha\beta}\equiv\mathbf{r}_\alpha-\mathbf{r}_\beta =$ points from β to α.

Now invoke the strong form of Newton III:
that $\mathbf{f}_{\alpha\beta}$ points along $\mathbf{r}_{\alpha\beta}$ $\quad\Rightarrow FT=0$.
This indicates that the total internal torque, $\sum_\alpha\sum_{\beta\neq\alpha}\mathbf{N}_{\alpha\beta}=0$,
ie, the internal torques do not alter the system's $\mathbf{L}$.

The total rate–of–change of the system's angular momentum is simply the sum of all the external torques $\mathbf{N}^e_\alpha=\mathbf{r}_\alpha\times\mathbf{F}^e_\alpha$ that are due to external forces:

$$\dot{\mathbf{L}} \;=\; \sum_\alpha \mathbf{r}_\alpha\times\mathbf{F}^e_\alpha=\sum_\alpha\mathbf{N}^e_\alpha\equiv\mathbf{N}^e$$

V. if the torque about some given axis $\hat{\mathbf{x}}$ is zero, ie. $\mathbf{N}^e\cdot\hat{\mathbf{x}}=0$, then $\mathbf{L}\cdot\hat{\mathbf{x}}=$ constant.

VI. The total internal torques sum to zero when the internal forces are central, ie, $\mathbf{f}_{\alpha\beta}=-\mathbf{f}_{\beta\alpha}$. In this case only external torques can alter the system's angular momentum.

A simple example

Let two masses m, which are attached to the unextended ends of a spring having a natural length b, rest on a frictionless plane. At time $t = 0$, give one mass a sudden velocity kick $\mathbf{V}$ perpendicular to the spring while giving the other mass a velocity kick $-2\mathbf{V}$.

Where is the CoM?

What is the motion of the CoM?

$$\begin{aligned}
\text{the total momentum is} \quad \mathbf{P} &= (-mV + 2mV)\hat{\mathbf{x}} = mV\hat{\mathbf{x}} = M\dot{\mathbf{R}} = 2m\dot{\mathbf{R}} \\
&= \text{constant} \\
\Rightarrow \dot{\mathbf{R}} &= \frac{1}{2}V\hat{\mathbf{x}} \\
\text{and} \quad \mathbf{R}(t) &= \frac{1}{2}Vt\hat{\mathbf{x}}
\end{aligned}$$

What is the system's total $\mathbf{L}$?

$$\begin{aligned}
\mathbf{L} &= m_1\mathbf{r_1} \times \mathbf{v_1} + m_2\mathbf{r_2} \times \mathbf{v_2} \\
&= m\frac{1}{2}b\hat{\mathbf{y}} \times (-V\hat{\mathbf{x}}) - m\frac{1}{2}b\hat{\mathbf{y}} \times (2V\hat{\mathbf{x}}) \\
&= \frac{1}{2}mbV\hat{\mathbf{z}} + mbV\hat{\mathbf{z}} \\
&= \frac{3}{2}mbV\hat{\mathbf{z}}
\end{aligned}$$

The total energy E at time $t = 0$ is

$$\begin{aligned}
E &= T + U = T \\
&= \frac{1}{2}mV^2 + \frac{1}{2}m(2V)^2 = \frac{5}{2}mV^2 \quad \text{which is conserved}
\end{aligned}$$

Energy conservation

First look at the system's kinetic energy T:

$$T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha^2$$

And write the particles positions $\mathbf{r}_\alpha$ and velocities $\dot{\mathbf{r}}_\alpha$ as:

$$\begin{aligned} \mathbf{r}_\alpha &= \mathbf{R} + \mathbf{r}'_\alpha \\ \text{and} \quad \dot{\mathbf{r}}_\alpha &= \dot{\mathbf{R}} + \dot{\mathbf{r}}'_\alpha \end{aligned}$$

where $\mathbf{r}'_\alpha$ and $\dot{\mathbf{r}}'_\alpha$ are α'a position and velocity relative to the CoM.

$$\begin{aligned} \text{thus} \quad T &= \sum_\alpha \frac{1}{2} m_\alpha (\dot{\mathbf{R}}^2 + 2\dot{\mathbf{R}} \cdot \dot{\mathbf{r}}'_\alpha + \dot{\mathbf{r}'}^2_\alpha) \\ \text{note the middle term} \quad &= \dot{\mathbf{R}} \cdot \frac{d}{dt} \sum_\alpha m_\alpha \mathbf{r}'_\alpha = 0 \\ \text{so} \quad T &= \frac{1}{2} M \dot{\mathbf{R}}^2 + \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}'}^2_\alpha \end{aligned}$$

VII: the system's total KE is the sum of the KE due to the motion of the CoM + the KE due to internal motions.

Now the system's total potential energy U:

Section 9.5 shows that the total potential energy U is the sum of the potential due to the external forces plus the potential due to the particles' interactions:

$$U = \sum_{\alpha=1}^{N} U_\alpha^e + \sum_{\alpha=1}^{N} \sum_{\beta=\alpha+1}^{N} U_{\alpha\beta}^i$$

Note that the right sum is *not* over *all* particles α & β
(which would overcounting the internal potential energy!).

As usual, this is true for a *conservative* system, which means that these forces can be written in terms of potential energies that depend only on the coordinates $\mathbf{r}_\alpha$, and *not* on velocities $\dot{\mathbf{r}}_\alpha$ or time t.

The total force $\mathbf{F}_\gamma$ on particle γ is then

$$\mathbf{F}_\gamma = -\nabla_\gamma U = -\nabla_\gamma \left(\sum_{\alpha=1}^{N} U_\alpha^e + \sum_{\alpha=1}^{N} \sum_{\beta=\alpha+1}^{N} U_{\alpha\beta}^i \right)$$

where ∇_γ = gradient with respect to $\mathbf{r}_\gamma$
(ex: $\nabla_3 = \frac{\partial}{\partial x_3}\hat{\mathbf{x}} + \frac{\partial}{\partial y_3}\hat{\mathbf{y}} + \frac{\partial}{\partial z_3}\hat{\mathbf{z}}$ in Cartesian coordinates).

The ∇_γ operator selects the $\alpha = \gamma$ and $\beta = \gamma$ terms from these sums:

$$\begin{aligned} \mathbf{F}_\gamma &= -\nabla_\gamma U_\gamma^e - \nabla_\gamma \sum_{\beta=\gamma+1}^{N} U_{\gamma\beta}^i - \nabla_\gamma \sum_{\alpha=1}^{\gamma-1} U_{\alpha\gamma}^i \\ &= -\nabla_\gamma U_\gamma^e - \nabla_\gamma \sum_{\beta\neq\gamma}^{N} U_{\gamma\beta}^i \quad \text{since } U_{\alpha\gamma}^i = U_{\gamma\alpha}^i \text{ by Newton III} \end{aligned}$$

Suppose we have a 3–particle system.
What is the total force on particle $\gamma = 2$?

$$\begin{aligned}\mathbf{F}_2 &= -\nabla_2 U_2^e - \nabla_2 \sum_{\beta \neq 2} U_{2\beta}^i \\ &= \mathbf{F}_2^e + \mathbf{f}_{21} + \mathbf{f}_{23}\end{aligned}$$

VIII. The system's total energy $E = T + U$ is a constant for a conservative system.

This is rigorously proven in Section 9.5, but we will not do this here since the proof is similar to that for a one–particle system we did in Chapter 2.