Lecture Notes for PHY 405
Classical Mechanics

From Thorton & Marion's *Classical Mechanics*

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November 9, 2004

Chapter 10: Motion in a Noninertial Reference Frame

Recall that Newton's Laws of motion are valid in an inertial (ie, non–accelerated) reference frame.

Is this classroom an inertial reference frame? Why or why not?

Since there is no reference frame that is absolutely at rest, we need to derive the laws that describe motion in a non–inertial reference frames.

In particular, we are going to focus on the motion of bodies moving in a rotating reference frame.

This will allow us to calculate the motion of bodies near the surface of a rotating planet, as well as the motion of a rotating rigid body (eg, a spinning top).

A rotating coordinate system

Let the vector $\mathbf{r}'$ point to particle P from the origin of some 'fixed' or inertial coordinate system.

Let $\mathbf{R}$ point to the origin of some *noninertial* coordinate system; this accelerated origin can be *translating* as well as *rotating*.

Let $\mathbf{r}$ = the position of particle P in this rotating ref' frame. Thus:

$$\mathbf{r}' = \mathbf{R} + \mathbf{r}$$

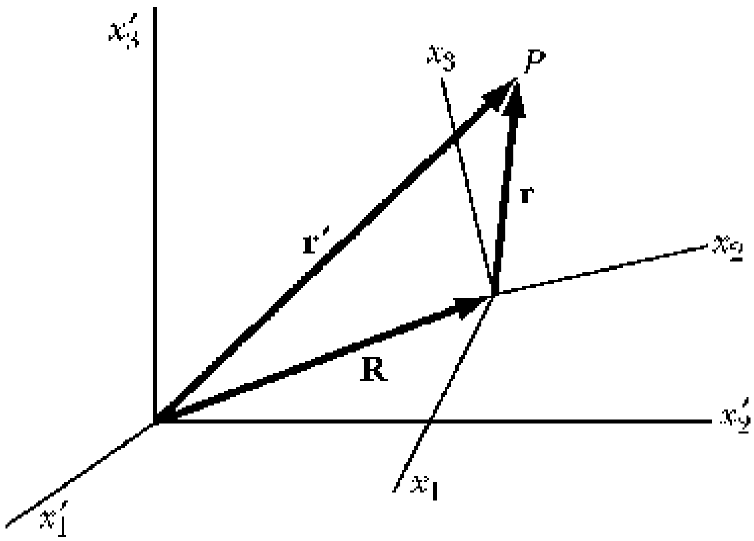

Fig. 10–1.

Now use Newton's Law, which is valid in the stationary reference frame, to derive the acceleration of point P, $\ddot{\mathbf{r}}$, as seen in the rotating ref' frame.

First, compute $\dot{\mathbf{r}}$ and then get $\ddot{\mathbf{r}}$.
We do this by first computing $\dot{\mathbf{r}'}$ = P's velocity in the *fixed* reference frame:

$$\left(\frac{d\mathbf{r}'}{dt}\right)_{fixed} = \left(\frac{d\mathbf{R}}{dt}\right)_{fixed} + \left(\frac{d\mathbf{r}}{dt}\right)_{fixed}$$

where the $()_{fixed}$ is used to indicate that we are calculating these velocities with respect to the *fixed* origin.

Note that the moving origin can be *translating* as well as *rotating*.

Thus $(d\mathbf{R}/dt)_{fixed}$ should be regarded as the *translational* velocity of the moving origin (measured in the fixed frame).

The right term will thus account for any possible rotation of the moving origin.

The position of particle P in the *moving* ref' frame is:

$$\begin{aligned} \mathbf{r} &= \sum_i x_i \hat{\mathbf{x}}_i \\ \text{which is shorthand for} \quad &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \\ \text{Thus its velocity is} \quad \left(\frac{d\mathbf{r}}{dt}\right)_{fixed} &= \sum_i \dot{x}_i \hat{\mathbf{x}}_i + \sum_i x_i \frac{d\hat{\mathbf{x}}_i}{dt} \end{aligned}$$

where the first term on the right is simply
P's velocity measured in the *rotating* frame, ie, $(d\mathbf{r}/dt)_{rot}$,

and the second term is due to the *rotation* of the moving coordinate system.

Now calculate $d\hat{\mathbf{x}}_i/dt$,
which is due to the rotation of the moving origin about the rotation axis $\vec{\omega}$.

We will also use the magnitude of the rotation axis to indicate the rotation rate, ie $\omega = |\vec{\omega}|$.

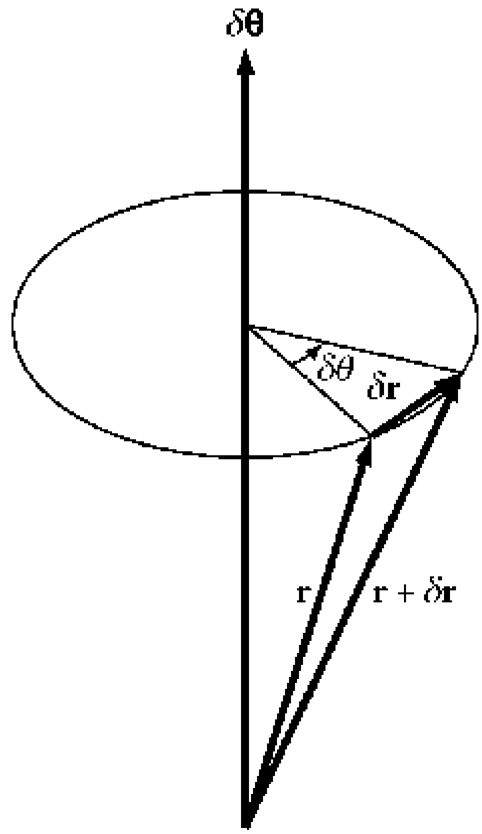

Fig. 1–19

If α is the angle between $\hat{\mathbf{x}}_i$ and $\vec{\omega}$, then $\hat{\mathbf{x}}_i$ moves a distance $\boldsymbol{\Delta}\hat{\mathbf{x}}_i$ in time Δt.

Note that $\boldsymbol{\Delta}\hat{\mathbf{x}}_i$ is perpendicular to both $\hat{\mathbf{x}}_i$ and $\vec{\omega}$, so $\boldsymbol{\Delta}\hat{\mathbf{x}}_i \propto (\vec{\omega} \times \hat{\mathbf{x}}_i)$.

We also anticipate that $\boldsymbol{\Delta}\hat{\mathbf{x}}_i \propto \Delta t$, so we write $\boldsymbol{\Delta}\hat{\mathbf{x}}_i = \gamma \Delta t \vec{\omega} \times \hat{\mathbf{x}}_i$,
where γ is some coefficient.

Now show that $\gamma = 1$. But first note that

$$|\vec{\omega} \times \hat{\mathbf{x}}_i| = \omega \sin\alpha \quad \text{since } |\hat{\mathbf{x}}_i| = 1, \quad \text{and that } \sin\alpha = \frac{\ell}{|\hat{\mathbf{x}}_i|} = \ell$$

$$\Rightarrow |\vec{\omega} \times \hat{\mathbf{x}}_i| = \omega\ell$$

And since $|\boldsymbol{\Delta}\hat{\mathbf{x}}_i|$ = distance the end of $\hat{\mathbf{x}}_i$ has traveled in time Δt, then

$$\begin{aligned} |\boldsymbol{\Delta}\hat{\mathbf{x}}_i| &= \ell\omega\Delta t = \gamma\Delta t|\vec{\omega} \times \hat{\mathbf{x}}_i| = \gamma\Delta t\omega\ell \\ \text{this implies that} \quad \gamma &= 1 \\ \text{thus} \quad \frac{d\hat{\mathbf{x}}_i}{dt} &= \frac{\boldsymbol{\Delta}\hat{\mathbf{x}}_i}{\Delta t} = \vec{\omega} \times \hat{\mathbf{x}}_i \end{aligned}$$

which is the velocity of the $\hat{\mathbf{x}}_i$ axes due to their *rotation* about the $\vec{\omega}$ axis. This velocity is measured relative to the origin of the *fixed reference frame*.

Plug this result back into our earlier equation:

$$\begin{aligned} \left(\frac{d\mathbf{r}}{dt}\right)_{fixed} &= \sum_i \dot{x}_i\hat{\mathbf{x}}_i + \sum_i x_i\frac{d\hat{\mathbf{x}}_i}{dt} \\ &= \left(\frac{d\mathbf{r}}{dt}\right)_{rot} + \sum_i x_i\vec{\omega} \times \hat{\mathbf{x}}_i \\ &= \left(\frac{d\mathbf{r}}{dt}\right)_{rot} + \vec{\omega} \times \mathbf{r} \end{aligned}$$

The above result is of course true for any arbitrary vector $\mathbf{Q}$:

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{Q}}{dt}\right)_{rot} + \vec{\omega} \times \mathbf{Q}$$

ie, the velocity of $\mathbf{Q}$ as measured in the fixed reference frame = the velocity of $\mathbf{Q}$ in the rotating frame + $\vec{\omega} \times \mathbf{Q}$.

Keep this result handy...we will use it again.

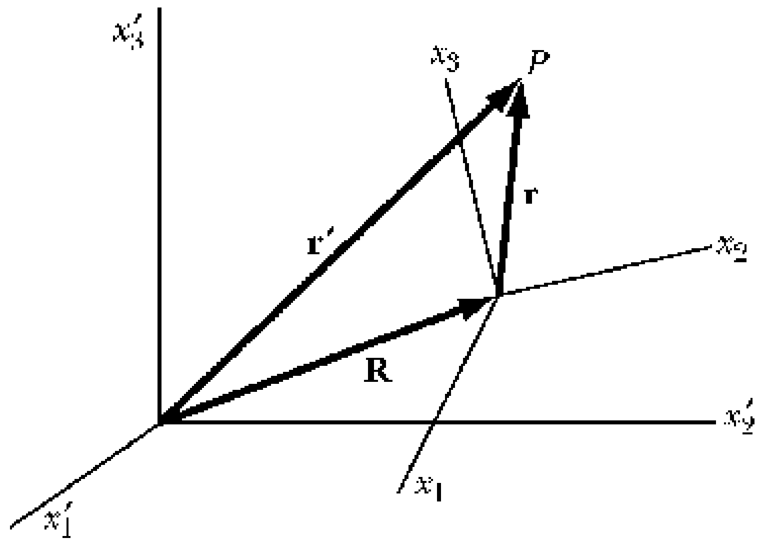

Fig. 10–1.

Or if we again adopt $\mathbf{r}' = \mathbf{R} + \mathbf{r}$,
where $\mathbf{r}'$ is particle P's position vector in the fixed ref' frame,
$\mathbf{R}$ is the position vector of the moving origin,
and $\mathbf{r} =$ P's position relative to the moving origin, then

$$\left(\frac{d\mathbf{r}'}{dt}\right)_{fixed} = \left(\frac{d\mathbf{R}}{dt}\right)_{fixed} + \left(\frac{d\mathbf{r}}{dt}\right)_{rot} + \vec{\omega} \times \mathbf{r}$$

Or if we define
$\mathbf{v}_f =$ particle P's velocity relative to the fixed origin
$\mathbf{V} =$ *translational* velocity of the *moving* origin (relative to the fixed origin)
$\mathbf{v}_r =$ P's velocity relative to the rotating axes
$\vec{\omega} =$ angular velocity of the rotating axes
$\vec{\omega} \times \mathbf{r} =$ translational velocity due to the rotating axes,

Then

$$\mathbf{v}_f = \mathbf{V} + \mathbf{v}_r + \vec{\omega} \times \mathbf{r}$$

Now calculate particle P's acceleration $\mathbf{a}_f$ as measured in the fixed frame:

$$\begin{aligned}\mathbf{a}_f &= \left(\frac{d\mathbf{v}_f}{dt}\right)_{fixed} = \left(\frac{d\mathbf{V}}{dt}\right)_{fixed} + \left(\frac{d\mathbf{v}_r}{dt}\right)_{\text{fixed}} + \dot{\vec{\omega}} \times \mathbf{r} + \vec{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} \\ &= \frac{\mathbf{F}}{m}\end{aligned}$$

where $\mathbf{F}$ is the force on particle P which has mass m.

set $\quad \ddot{\mathbf{R}}_f = \left(\dfrac{d\mathbf{V}}{dt}\right)_{fixed} =$ the *translational* acceleration of the moving origin

The $(d\mathbf{v}_r/dt)_{fixed}$ term is obtained using our formula for $d\mathbf{Q}/dt$:

$$\begin{aligned}\left(\frac{d\mathbf{v}_r}{dt}\right)_{fixed} &= \left(\frac{d\mathbf{v}_r}{dt}\right)_{rot} + \vec{\omega} \times \mathbf{v}_r \\ &= \mathbf{a}_r + \vec{\omega} \times \mathbf{v}_r\end{aligned}$$

where $\mathbf{a}_r$ is P's acceleration as measured in the rotating reference frame.

The last term is:

$$\begin{aligned}\vec{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{fixed} &= \vec{\omega} \times \left[\left(\frac{d\mathbf{r}}{dt}\right)_{rot} + \vec{\omega} \times \mathbf{r}\right] \\ &= \vec{\omega} \times \mathbf{v}_r + \vec{\omega} \times (\vec{\omega} \times \mathbf{r})\end{aligned}$$

Thus

$$\mathbf{a}_f = \ddot{\mathbf{R}}_f + \mathbf{a}_r + \dot{\vec{\omega}} \times \mathbf{r} + 2\vec{\omega} \times \mathbf{v}_r + \vec{\omega} \times (\vec{\omega} \times \mathbf{r}) = \frac{\mathbf{F}}{m}$$

Now suppose you are an observer that co–moves and co–rotates with this moving/rotating reference frame.

What is P's acceleration that you would measure in this moving ref' frame?

You would thus see particle P move as if it were driven by the *effective* force

$$\mathbf{F}_{eff} \;=\; m\mathbf{a}_r = \mathbf{F} - m\ddot{\mathbf{R}}_f - m\dot{\vec{\omega}} \times \mathbf{r} - 2m\vec{\omega} \times \mathbf{v}_r - m\vec{\omega} \times (\vec{\omega} \times \mathbf{r})$$

Keep in mind that $\mathbf{F}_{eff}$ is a *fictitious* force!

It is merely the invention of the observer who insists writing a Newton–esque law of motion $\mathbf{F}_{eff} = m\mathbf{a}_r$.

The only *real* force in this problem is the *inertial* force $\mathbf{F}$;
the other "noninertial" force terms are a consequence of the observer employing a noninertial reference frame.

The first two noninertial force terms, $m\ddot{\mathbf{R}}_f$ and $m\dot{\vec{\omega}} \times \mathbf{r}$, are due to:
translational acceleration of the moving origin &
changes in the orientation of the rotation axis $\vec{\omega}$.

However these terms are zero if the moving frame is merely rotating about a fixed axis with a constant angular velocity ω.
This will be typical of the problems studied in this class.

The remaining two terms are more important:
$-m\vec{\omega} \times (\vec{\omega} \times \mathbf{r})$ = centrifugal force
$-2m\vec{\omega} \times \mathbf{v}_r$ = Coriolis force

Which direction is the centrifugal force pointing?

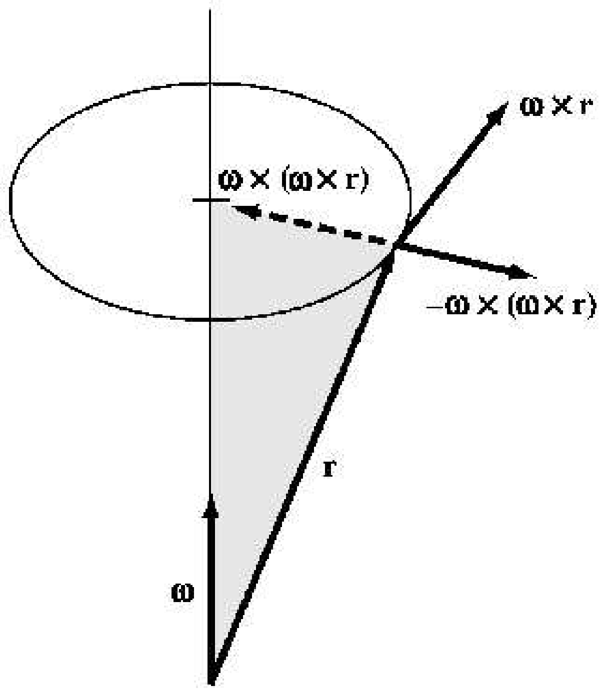

Fig. 10–3

The centrifugal force is perpendicular to & points away from the $\vec{\omega}$ axis.

Example: A hockey puck is initially stationary on a frictionless merry–go–round that rotates counter–clockwise with an angular velocity ω.

An observer on the merry–go–round releases the puck from an off–axis spot. In which directions does the puck initially move in due to centrifugal & Coriolis forces?

First: what is $\mathbf{F} =$?. $\ddot{\mathbf{R}}_f$? $\dot{\vec{\omega}} \times \mathbf{r}$?

Another observer sitting in a fixed reference frames also watches the puck—what does he see?

Motion at the Surface of the Earth

Earth spins about its rotation axis with an angular frequency

$$\omega = \frac{2\pi}{1 \text{ day}} = 7.3 \times 10^{-5} \text{ radians/sec}$$

Place the fixed (inertial) coordinates $(\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}')$ at the center of the Earth, and the moving coordinates $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ at the surface such that $\hat{\mathbf{x}}$ points South, and $\hat{\mathbf{y}}$ points East, and $\hat{\mathbf{z}}$ points Up:

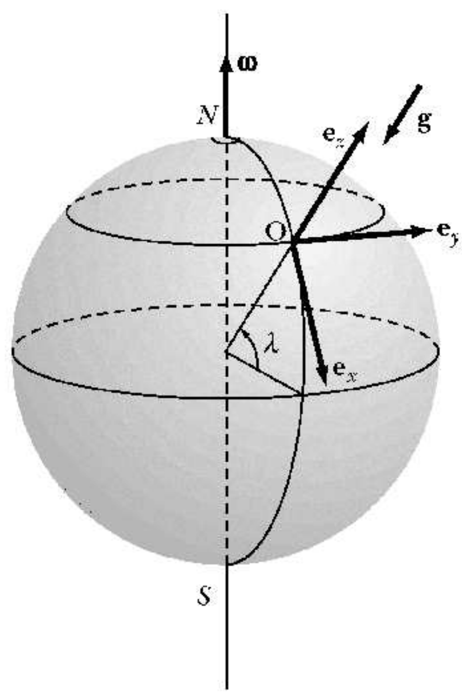

Fig. 10–9

Examine how the centrifugal force alters our perception of gravity $\mathbf{F} = m\mathbf{g}$. Also assume that the Earth's rotation pole $\vec{\omega} =$ constant. Then

$$\begin{aligned} \mathbf{a}_r &= \mathbf{g}_0 - \ddot{\mathbf{R}}_f - \dot{\vec{\omega}} \times \mathbf{r} - 2\vec{\omega} \times \mathbf{v}_r - \vec{\omega} \times (\vec{\omega} \times \mathbf{r}) \\ \text{where} \quad \mathbf{g}_0 &= -\frac{GM_\oplus}{R_f^2}\hat{\mathbf{R}}_f \quad \text{is the downward pull due to gravity} \end{aligned}$$

Is $\ddot{\mathbf{R}}_f = 0$?

$$\begin{aligned} \text{recall that} \quad \left(\frac{d\mathbf{Q}}{dt}\right)_{fixed} &= \left(\frac{d\mathbf{Q}}{dt}\right)_{rot} + \vec{\omega} \times \mathbf{Q} \\ \text{thus} \quad \ddot{\mathbf{R}}_f &= \left(\frac{d\dot{\mathbf{R}}_\mathbf{f}}{dt}\right)_{fixed} = \left(\frac{d\dot{\mathbf{R}}_\mathbf{f}}{dt}\right)_{rot} + \vec{\omega} \times \dot{\mathbf{R}}_\mathbf{f} \\ \text{but} \quad \mathbf{R}_f &= R_f\hat{\mathbf{z}} \quad \text{so} \quad \left(\frac{d\dot{\mathbf{R}}_\mathbf{f}}{dt}\right)_{rot} = 0 \\ \text{and} \quad \ddot{\mathbf{R}}_f &= \vec{\omega} \times \dot{\mathbf{R}}_\mathbf{f} \\ \text{similarly} \quad \dot{\mathbf{R}}_f &= \left(\frac{d\mathbf{R}_\mathbf{f}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{R}_\mathbf{f}}{dt}\right)_{rot} + \vec{\omega} \times \mathbf{R}_\mathbf{f} \\ \Rightarrow \ddot{\mathbf{R}}_f &= \vec{\omega} \times (\vec{\omega} \times \mathbf{R}_\mathbf{f}) \end{aligned}$$

Plug this result into $\mathbf{a}_r$:

$$\begin{aligned} \mathbf{a}_r &= \mathbf{g}_0 - 2\vec{\omega} \times \mathbf{v}_r - \vec{\omega} \times [\vec{\omega} \times (\mathbf{R}_f + \mathbf{r})] \\ &\simeq \mathbf{g}_0 - 2\vec{\omega} \times \mathbf{v}_r - \vec{\omega} \times (\vec{\omega} \times \mathbf{R}_f) \end{aligned}$$

for motion near the Earth's surface where $|\mathbf{r}| \ll |\mathbf{R}_f|$.

Consequently, a plumb line (eg, a mass dangling from a string) does not point directly towards the center of the Earth,
due to the centrifugal acceleration $-\vec{\omega} \times (\vec{\omega} \times \mathbf{R}_f)$.

And if you cut the string, the Coriolis acceleration $-2\vec{\omega} \times \mathbf{v}_r$
will cause an additional deflection as the mass falls.

Example 10.3

Estimate the deflection caused by the Coriolis force as a particle falls a distance h due to gravity.

The particle's acceleration in the rotating reference frame is

$$\begin{aligned} \mathbf{a}_r &\simeq \mathbf{g}_0 - 2\vec{\omega} \times \mathbf{v}_r - \vec{\omega} \times (\vec{\omega} \times \mathbf{R}_f) \\ &\equiv \mathbf{g} - 2\vec{\omega} \times \mathbf{v}_r \end{aligned}$$

where $\mathbf{g} \equiv \mathbf{g}_0 - \vec{\omega} \times (\vec{\omega} \times \mathbf{R}_f)$ is the effective gravity that includes the constant centrifugal term.

Note that the centrifugal force causes $\mathbf{g}$ to deviate slightly from the normal direction, $\hat{\mathbf{z}}$.

However we will ignore this slight deviation and write $\mathbf{g} =\simeq g\hat{\mathbf{z}}$.

From the diagram,

$$\begin{aligned} \vec{\omega} &= -\omega \sin(\pi/2 - \lambda)\hat{\mathbf{x}} + \omega \cos(\pi/2 - \lambda)\hat{\mathbf{z}} \\ &= -\omega \cos \lambda \hat{\mathbf{x}} + \omega \sin \lambda \hat{\mathbf{z}} \\ \text{and} \quad \mathbf{v}_r &= v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}} \simeq v_z\hat{\mathbf{z}} \simeq -gt\hat{\mathbf{z}} \end{aligned}$$

since we anticipate the horizontal deflections and velocities will be small compared to the vertical motions.

Thus

$$\vec{\omega} \times \mathbf{v}_r = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ 0 & 0 & -gt \end{vmatrix} = -\omega gt\cos\lambda\hat{\mathbf{y}}$$

$$\begin{aligned} \text{So} \quad \mathbf{a}_r &\simeq +2\omega gt\cos\lambda\hat{\mathbf{y}} - g\hat{\mathbf{z}} \\ &= \ddot{x}\hat{\mathbf{x}} + \ddot{y}\hat{\mathbf{y}} + \ddot{z}\hat{\mathbf{z}} \\ \Rightarrow \ddot{x} &= 0 \\ \ddot{y} &= 2\omega gt\cos\lambda \\ \ddot{z} &= -g \end{aligned}$$

How do I obtain the particle's vertical motion $z(t)$?
$\dot{z} = -gt$ so $z(t) = h - \frac{1}{2}gt^2$.

How do I obtain the free–fall time until impact, t_f?
$z(t_f) = 0 \Rightarrow t_f = \sqrt{2h/g}$.

How do I solve for the particle's deflection due to the Coriolis force?

$$\begin{aligned} \dot{y}(t) &= \omega gt^2\cos\lambda \\ y(t) &= \frac{1}{3}\omega gt^3\cos\lambda. \\ &= \frac{\omega}{3}\sqrt{\frac{8h^3}{g}}\cos\lambda \end{aligned}$$

Thus a person at latitude $\lambda = 45^\circ$ who drops a pebble from a height of $h = 100$ m will see the pebble deflected a distance $y = 1.6$ cm away from the downward direction.

Which way was he pebble deflected—N,S,E, or W?

Note that the angular deflection due to Coriolis is small,
$\phi \simeq \frac{y}{h} = 1.6 \times 10^{-4}$ radians $\simeq 0.01$ degrees.