CHAPTER 12

SPIRAL WAVE THEORY

This chapter examines the excitation and propagation of spiral density waves in a circumstellar or circumplanetary disk. Spiral density waves are the disk's natural response to perturbations exerted by any planets or satellites that are orbiting in or near the disk. For instance, recently-formed planets will excite spiral density waves at their Lindblad resonances that lie in the planet-forming circumstellar disk. These waves can be quite important because the perturber's gravitational attraction for the wave also transmits angular momentum from the perturber to the wave. That wave then propagates away from the resonance, and that angular momentum is ultimately deposited in the disk as the wave is damped by the disk's viscosity or shocks in the wave. And should that angular momentum transport between the disk and the perturber be vigorous enough, that can then drive a large scale adjustment of both the perturber's orbit and the disk's matter distribution. Wave-action could for instance cause the perturber's orbit to migration over time, or the perturber might instead carve open an annular gap about its orbit in the disk, due to the many resonances there.

The following section will derive a linearized theory for the spiral waves that a perturber can excite at one of its Lindblad resonances in the disk. Those results are then used to calculate the torque that the disk and perturber exert on each other, which then determines whether the perturber's orbit will drift over time, or whether the perturber shepherds open a gap in the disk.

12.1 EQUATIONS OF MOTION

This section derives the equations that govern the disk's response to the perturbations that are exerted by an orbiting secondary, both of which are in orbit about the primary mass M_p whose gravitational potential is $\Phi_p = -GM_p/r$. The secondary's potential is Φ_s, and the disk's gravitational potential is Φ_d. The disk's equation of state is an ideal gas, so the pressure in the disk is $p = c^2\rho$ where c the sound speed (see Section 3.2.2.1) and ρ is the disk's volume density. The disk is vertically thin such that $c \ll r\Omega$ where $r\Omega$ is the circular speed, and the disk's surface and volume densities are related via $\rho(r, \theta, z) = \sigma(r, \theta)\delta(z)$.

12.1.1 the disk's undisturbed state

If the disk were undisturbed, its volume density ρ would depend only on the radial coordinate, perhaps as the power law $\rho \propto r^{-\alpha}$. The disk's motions would be circular so its velocity $\mathbf{v} = r\Omega\,\hat{\boldsymbol{\theta}}$ where $\Omega(r)$ is the disk's angular velocity. Euler's equation (10.12) for a steady disk whose density is constant over time is $(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p/\rho - \nabla\Phi_p$ where the convective derivative $(\mathbf{v}\cdot\nabla)\mathbf{v} = -r\Omega^2\,\hat{\mathbf{r}}$ according to Eqn. (A.23). Solving for the disk's angular velocity then yields $\Omega^2 = \Omega_0^2 - \alpha(c/r)^2$ where $\Omega_0^2 = (\partial\Phi_p/\partial r)/r = GM_p/r^3$ would be the angular velocity squared if the disk were pressureless. So the disk will be subkeplerian when $\alpha > 0$ and the disk's density decreases outwards, as per Section 3.2.2.1. The disk's vertical scale height h is related to the gas soundspeed via $c = h\Omega$ (see problem 3.9). Since the disk is thin, $h \ll r$ and the disk's angular velocity is

$$\Omega(r) \simeq \left[1 - \frac{\alpha}{2}\left(\frac{h}{r}\right)^2\right]\sqrt{\frac{GM_p}{r^3}}. \tag{12.1}$$

12.1.2 the perturber's gravitational potential

The Fourier expansion of the secondary's potential has the form $\Phi_s(r, \theta) = \sum_{mk}\phi_m^k(r)$ $\cos m(\theta - \Omega_{mk}t)$, where the indices m, k refer to specific Lindblad resonance (see Eqn. 6.33). But as long as the resonance index m that is of interest here is sufficiently small, then these resonances are spatially segregated such that one only needs to consider the disk's response to a single m, k term in the sum, so

$$\Phi_s(r, \theta) \simeq \phi_{mk}^s(r)e^{im(\theta - \Omega_{mk}t)}. \tag{12.2}$$

Keep in mind that the switch to complex notation means that only the real parts are to be preserved in the following. In the above, the pattern speed $\Omega_{mk} = \Omega_s + k\kappa_s/m$ is the angular rate that the m, k Fourier component rotates over time (Eqn. 6.35) while Ω_s and κ_s are the secondary's angular and epicyclic frequencies. Also recall that the Fourier amplitudes $\phi_{mk}^s(r) \propto e_s^{|k|}$ where e_s is the secondary's eccentricity (see Section 6.2), which is usually small, so only the strongest $k = 0$ Lindblad resonances are considered here.

12.1.3 linearized equations of motion

When the secondary's perturbation of the disk is not too large, then linearized equations of motion may be applied, which simplifies this problem considerably. In this case, the disk's surface density and velocity have the form $\sigma = \sigma_0 + \sigma_1$ and $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$ where $\sigma_0(r)$ is the disk's unperturbed surface density and $\mathbf{v}_0(r) = r\Omega\,\hat{\boldsymbol{\theta}}$ is the disk's unperturbed circular velocity where $\Omega(r)$ is its angular velocity, while the disk's perturbed velocity is $\mathbf{v}_1 = v_{1r}\,\hat{\mathbf{r}} + v_{1\theta}\,\hat{\boldsymbol{\theta}}$. The secondary's perturbation is sinusoidal in time and azimuth, so the disk's response is also sinusoidal and of the form

$$\sigma_1(r,\theta,t) = S(r)e^{im(\theta-\Omega_{mk}t)} \tag{12.3a}$$

$$v_{1r}(r,\theta,t) = V_r(r)e^{im(\theta-\Omega_{mk}t)} \tag{12.3b}$$

$$v_{1\theta}(r,\theta,t) = V_\theta(r)e^{im(\theta-\Omega_{mk}t)} \tag{12.3c}$$

$$\Phi_d(r,\theta,t) = \phi_d(r)e^{im(\theta-\Omega_{mk}t)} \tag{12.3d}$$

where the perturbations S, V_r, V_θ, ϕ_d are all complex quantities. The disk's response will be linear when its perturbed surface density $|S| \ll \sigma_0$ and its perturbed radial and tangential speeds $|V_r|$ and $|V_\theta|$ are both small compared to the disk's undisturbed circular speed $r\Omega$.

The disk's linearized continuity equation (10.48) is $\partial\sigma_1/\partial t + \nabla\cdot(\sigma_0\mathbf{v}_1) + \nabla\cdot(\sigma_1\mathbf{v}_0) = 0$, and inserting Eqns. (12.3) into the continuity equation yields

$$i\omega_{mk}S + \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_0 V_r) + \frac{im\sigma_0}{r}V_\theta = 0, \tag{12.4}$$

where $\omega_{mk} = m(\Omega - \Omega_{mk})$ is the Doppler-shifted forcing frequency of Section 6.1.2.

The linearized Euler equation for the disk is

$$\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_0 + (\mathbf{v}_0\cdot\nabla)\mathbf{v}_1 = -\nabla\left(\frac{c^2}{\sigma_0}\sigma_1 + \Phi_d + \Phi_s\right), \tag{12.5}$$

which is from Eqn. (10.33b) where $h_1 = (dp/d\rho)(\rho_1/\rho_0) = c^2\sigma_1/\sigma_0$ is the perturbation in the disk's enthalphy. The convective derivatives are

$$(\mathbf{v}_1\cdot\nabla)\mathbf{v}_0 = \left[-\Omega V_\theta\,\hat{\mathbf{r}} + \frac{\partial(r\Omega)}{\partial r}V_r\,\hat{\boldsymbol{\theta}}\right]e^{im(\theta-\Omega_{mk}t)} \tag{12.6a}$$

and $$(\mathbf{v}_0\cdot\nabla)\mathbf{v}_1 = \Omega\left[(imV_r - V_\theta)\,\hat{\mathbf{r}} + (imV_\theta + V_r)\,\hat{\boldsymbol{\theta}}\right]e^{im(\theta-\Omega_{mk}t)}, \tag{12.6b}$$

so the radial part of Euler's equation is

$$i\omega_{mk}V_r - 2\Omega V_\theta = -\frac{\partial}{\partial r}\left(\frac{c^2}{\sigma_0}S + \phi^d + \phi^s_{mk}\right) \tag{12.7}$$

while the tangential part is

$$\left[\Omega + \frac{\partial}{\partial r}(r\Omega)\right]V_r + i\omega_{mk}V_\theta = \frac{\kappa^2}{2\Omega}V_r + i\omega_{mk}V_\theta = -\frac{im}{r}\left(\frac{c^2}{\sigma_0}S + \phi_d + \phi^s_{mk}\right) \tag{12.8}$$

since $\Omega + \partial(r\Omega)/\partial r = \kappa^2/2\Omega$ (see Eqn. 10.58).

And lastly, the linearized Poisson equation is Eqn. (10.38c),

$$\nabla^2\Phi^d = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi_d}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi_d}{\partial\theta^2} + \frac{\partial^2\Phi_d}{\partial z^2} = 4\pi G\rho_1. \tag{12.9}$$

12.1.4 the tight winding and WKB approximations

The following will consider spiral density waves that are propagating in either a circumstellar or a circumplanetary disk, and in both cases the disk's mass is small relative to the primary's mass. The disk's low mass then suggests that the radial scale λ over which the disk responds collectively to any perturbations is likely to be small compared to the disk's physical scale r. λ is of course the radial wavelength of the wave, so a tightly-wrapped spiral density wave having $\lambda \ll r$ also has a wavenumber $|k| = 2\pi/\lambda$ such that $|kr| \gg 2\pi$. Note also that the disk potential $\Phi_d(r)$ will cycle rapidly across a small distance λ while r changes little. Consequently the first derivative in Eqn. (12.9) is simply $\partial^2\Phi_d/\partial^2 r$, which is of order $\Phi_d/\lambda^2 \sim k^2\Phi^d$. And since $\Phi_d \propto e^{im\theta}$ (see Eqn. 12.3d), the second derivative in Eqn. (12.9) is $-m^2\Phi_d/r^2$ which is small compared to the first since $|kr| \gg 2\pi$. So the linearized Poisson equation becomes

$$\frac{\partial^2\Phi_d}{\partial r^2} + \frac{\partial^2\Phi_d}{\partial z^2} \simeq 4\pi G\sigma_1\delta(z) \tag{12.10}$$

since $\rho_1 = \sigma_1\delta(z)$. This assumption that $\lambda \ll r$ that entered into Eqn. (12.10) is known as the *tight-winding approximation*.

Now recall the results of Section 10.3.2, which considered the disk's gravitational stability. It is shown there that the disk's gravitational potential varies in the vertical direction as $\Phi_d \propto e^{-|kz| = -s_z|k|z}$ where $s_z = \text{sgn}(z)$, so the Poisson equation in the $|z| > 0$ region is $\partial^2\Phi_d/\partial z^2 = k^2\Phi_d = -\partial^2\Phi_d/\partial r^2$, which is satisfied by

$$\frac{\partial\Phi_d}{\partial r} = ik\Phi_d, \tag{12.11}$$

keeping in mind that the wavenumber k may be a function of r. In this case Eqn. (12.11) may be solved via the WKB approximation that Wentzel, Kramers, and Brillioun first used to solve the Schrödinger equation:

$$\Phi_d(r,\theta,t) = A(r,\theta,t)e^{i\int_{r_0}^{r} k(r')dr'} \tag{12.12}$$

where $k(r)$ is the wavenumber, $A(r,\theta,t)$ is the amplitude of the wave in the $z = 0$ plane, and r_0 in the above is an arbitrary reference radius. Comparison to Eqn. (12.11) shows that the WKB approximation is a solution to the Poisson equation when the phase in Eqn. (12.12), $\int k(r')dr'$, varies much more rapidly than any variations in the wave amplitude such that $|\partial A/\partial r| << |kA|$.

With this in mind, integrate Eqn. (12.10) vertically across $-a < z < a$ where the small distance $a \ll \lambda$, and then take the limit $a \to 0$, which yields

$$\left.\frac{\partial\Phi_d}{\partial z}\right|_{z=+a} - \left.\frac{\partial\Phi_d}{\partial z}\right|_{z=-a} = -2|k|\Phi_d = 4\pi G\sigma_1 \tag{12.13}$$

since $\Phi_d \propto e^{-s_z|k|z}$ and $\partial\Phi_d/\partial z = -s_z|k|\Phi_d$ (e.g. Section 10.3.2). Noting that $k\Phi_d = -i\partial\Phi_d/\partial r$, the above becomes

$$\sigma_1 = -\frac{|k|\Phi_d}{2\pi G} = \frac{is_k}{2\pi G}\frac{\partial\Phi_d}{\partial r} \tag{12.14}$$

where $s_k = \text{sgn}(k)$, or

$$\Phi_d = -2\pi G\sigma_1/|k|. \tag{12.15}$$

Now recall that $\sigma_1 = S(r)e^{im(\theta-\Omega_{mk}t)}$ (e.g. Eqns. 12.3), so inserting this into the above then yields

$$S(r) = \frac{is_k}{2\pi G}\frac{\partial\phi_d}{\partial r}. \tag{12.16}$$

This is the linearized Poisson equation in the tight-winding limit after factoring out the θ and t dependencies, and it provides a relatively simple relationship between the disk's perturbed surface density $S(r)$ and the disk potential $\phi_d(r)$.

When the spiral wave is tightly wound, the disk's perturbed surface density $S(r)$ varies rapidly over the small radial scale $\lambda \ll r$, as will the perturbations in the disk's velocities V_r and V_θ. In this case, one can treat quantities that change slowly in r as constant, so derivatives like that in the equation (12.4) become $r^{-1}\partial(r\sigma_0 V_r)/\partial r \simeq \sigma_0\partial V_r/\partial r$, and so the continuity equation in the tight-winding limit simplifies to

$$i\omega_{mk}S + \sigma_0\frac{\partial V_r}{\partial r} + \frac{im\sigma_0}{r}V_\theta \simeq 0, \tag{12.17}$$

and the radial and tangential parts of Euler's equation become

$$i\omega_{mk}V_r - 2\Omega V_\theta \simeq -\frac{\partial}{\partial r}\left(\phi_d + \phi^s_{mk} + \frac{c^2}{\sigma_0}S\right) \tag{12.18a}$$

$$\frac{\kappa^2}{2\Omega}V_r + i\omega_{mk}V_\theta \simeq -\frac{im}{r}\left(\phi_d + \phi^s_{mk} + \frac{c^2}{\sigma_0}S\right) \tag{12.18b}$$

which can be solve for the velocities V_r, V_θ:

$$V_r = -\frac{i}{D}\left(\omega_{mk}\frac{\partial}{\partial r} + \frac{2m\Omega}{r}\right)\left(\phi_d + \phi^s_{mk} + \frac{c^2}{\sigma_0}S\right) \tag{12.19a}$$

$$V_\theta = \frac{1}{D}\left(\frac{\kappa^2}{2\Omega}\frac{\partial}{\partial r} + \frac{m\omega_{mk}}{r}\right)\left(\phi_d + \phi^s_{mk} + \frac{c^2}{\sigma_0}S\right) \tag{12.19b}$$

where $D(r) = \kappa^2 - \omega^2_{mk}$ is the wave's distance from resonance in frequency-squared units, Eqn. (6.20). Equations (12.16), (12.17), and (12.19) provide four coupled partial differential equations for the wave's four unknown quantities S, V_r, V_θ, and ϕ_d, and they are solved below in Section 12.3. But before tackling that problem, the following derive the waves' dispersion relation next, which is a very useful equation that reveals many of the spiral waves' properties without having solved the equations of motion.

12.2 DISPERSION RELATION FOR SPIRAL DENSITY WAVES

Assume for now that the perturber has launched a spiral density wave at a resonance in the disk, and lets focus on the downstream part of the wave that has propagated away from resonance. Downstream and far from resonance, the wave is propagating via the disk's internal forces, which are pressure and/or self-gravity. Since the the secondary's forcing of the disk is unimportant downstream of the resonance, once can set $\phi^s_{mk} = 0$ in the above equations of motion. Inserting the WKB form into those equations will then yield the dispersion relation for spiral density waves.

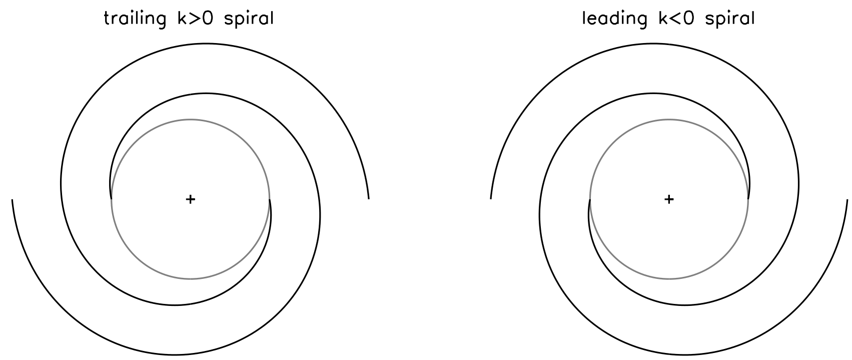

Figure 12.1 This trailing $m = 2$-armed spiral pattern has a wavenumber $k = 10/r_0$ where r_0 is the radius of the gray unit circle, and the leading $m = 2$ spiral has wavenumber $k = -10/r_0$.

12.2.1 WKB approximation is a spiral

But first confirm that the WKB form can in fact represent a spiral. Note that Eqns. (12.12) and (12.14) indicate that the perturbation in the disk's surface density also has the WKB form,

$$\sigma_1(r, \theta, t) = S(r)e^{i(m\theta - m\Omega_{mk}t)} = \mathcal{A}(r)e^{i\left[\int_{r_0}^{r} k(r')dr' + m\theta - m\Omega_{mk}t\right]} \tag{12.20}$$

where the surface density amplitude $\mathcal{A}(r)$ is again some function that varies slowly with r. Now trace the spiral along a spiral arm. Advancing a small step along the spiral will require a small radial step Δr plus a small tangential step $\Delta\theta$. Since the spiral's surface density should be constant or nearly so along that step, then $\sigma_1(r + \Delta r, \theta + \Delta\theta, t) \simeq \sigma_1(r, \theta, t)e^{i(k\Delta r + m\Delta\theta)}$, so the radial and tangential steps are related via $\Delta\theta = -k\Delta r/m$ when stepping along a spiral. Figure 12.1 shows that a spiral having a wavenumber $k > 0$ results in a trailing spiral in the sense that a positive radial step requires a negative azimuthal step in order to stay on the spiral, while $k < 0$ results in a leading spiral wave.

12.2.2 dispersion relation

Now derive the dispersion relation for spiral density waves. The derivative of Eqn. (12.20) in the tight winding approximation yields

$$\frac{\partial S}{\partial r} \simeq ikS. \tag{12.21}$$

The disk's other perturbed quantities also obey the tight winding approximation so $\partial V_r/\partial r \simeq ikV_r$ and the continuity equation (12.17) becomes $i\omega_{mk}S + (i\sigma_0/r)(krV_r + mV_\theta) \simeq 0$ but the second term is much larger than the third since $|kr| \gg 2\pi$ when the resonance index is of order $m \sim 1$, so the continuity equation becomes

$$V_r \simeq -\frac{\omega_{mk}}{k\sigma_0}S = \frac{s_k\omega_{mk}}{2\pi G\sigma_0}\phi_d \tag{12.22}$$

since the Poisson equation (12.16) becomes

$$S(r) = \frac{is_k}{2\pi G}\frac{\partial \phi_d}{\partial r} = -\frac{|k|\phi_d}{2\pi G} \tag{12.23}$$

with $s_k = \mathrm{sgn}(k)$. Inserting Eqns. (12.22–12.23) into Eqn. (12.19a) with $\phi^s_{mk} = 0$ yields

$$V_r = \frac{s_k\omega_{mk}}{2\pi G\sigma_0}\phi_d = -\frac{i\omega_{mk}}{D}\left(\frac{\partial}{\partial r} + \frac{2m\Omega}{r\omega_{mk}}\right)\left(1 - \frac{c^2|k|}{2\pi G\sigma_0}\right)\phi_d \simeq \frac{k\omega_{mk}}{D}\left(1 - \frac{c^2|k|}{2\pi G\sigma_0}\right)\phi_d \tag{12.24}$$

in the tight-winding limit, which simplifies to

$$D(r) = \kappa^2 - \omega^2_{mk} = 2\pi G\sigma_0|k| - c^2k^2. \tag{12.25}$$

This is the dispersion relation for spiral density waves in a gravitating and pressure supported disk. Note that when we write $\omega_{mk} = m\Omega - \omega$ where $\omega = m\Omega_{mk}$ is the forcing frequency, then the dispersion relation is

$$D(r) = \kappa^2 - (m\Omega - \omega)^2 = 2\pi G\sigma_0|k| - c^2k^2. \tag{12.26}$$

which is equivalent to the dispersion relation for axisymmetic instabilities in a disk when $m = 0$, Eqn. (10.59).

12.2.3 group velocity

Figure 12.2 plots the right-hand side of Eqn. (12.26) versus wavenumber $|k|$, which initially grows linearly with small wavenumber $|k|$ but then turns over as $-|k|^2$ at larger wavenumbers. Of course the left-hand side of Eqn. (12.26) is the wave's distance from resonance in frequency-squared units, so the wavenumber $|k|$ must adjust as the waves propagate away and $D(r)$ varies with distance from resonance (see Eqn. 6.20). Now suppose $D(r)$ takes some value D_0. Figure 12.2 shows that in this case the dispersion relation yields two possible solutions for the wavenumber: a smaller $|k|$ solution that corresponds to *long waves* (because wavelength $\lambda \propto |k|^{-1}$), and a larger $|k|$ solution that corresponds to *short waves*. .

Now recall Eqn. (12.20), which indicates that the wave's surface density varies as $\sigma_1 \propto e^{i(kr-\omega t)}$. Appendix F.4 of reference [2] shows that a wave of this form propagates radially at the rate $v_g = \partial\omega/\partial k$, which is the waves' *group velocity*; see that reference for a rigorous derivation of v_g. The forcing frequency ω is now to be regarded as a function of wavenumber, so $\partial D/\partial|k| = 2\omega_{mk}s_kv_g = 2\pi G\sigma_0 - 2c^2|k|$ where $s_k = \mathrm{sgn}(k)$, so the group velocity of spiral density waves is

$$v_g = (\pi G\sigma_0 s_k - c^2k)/\omega_{mk} \simeq \epsilon s_k(\pi G\sigma_0 - c^2|k|)/\kappa, \tag{12.27}$$

where the right hand side assumes that the density waves remain in the vicinity of the resonance where $D(r) \simeq 0$ and $\omega_{mk} \simeq \epsilon\kappa$ with $\epsilon = +1(-1)$ for waves launched at an inner (or outer) Lindblad resonance.

Since $|v_g| \propto \partial D/\partial|k|$, the waves' group speed is proportional to the slope of the curve seen in Fig. 12.2. Note that $D(r)$ has a local maximum, so the peak in Fig. 12.2 also represents a turning point in the disk where long waves can reflect as short waves and vice versa. That site in the disk where $v_g = 0$ is where the density waves have wavenumber

$$k_Q = \frac{\pi G\sigma_0}{c^2}, \tag{12.28}$$

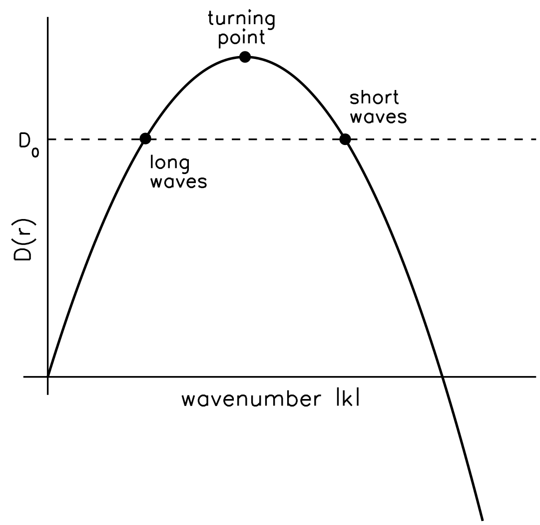

Figure 12.2 The black curve is the dispersion relation for spiral density waves, Eqn. (12.26), plotted versus wavenumber $|k|$. The vertical axis also represents the waves' frequency distance from resonance $D(r)$ that is the left-hand side of Eqn. (12.26). Now consider a density wave whose $D(r)$ takes the value D_0 as indicated by the dashed line. The values of $|k|$ where the solid curve intersects the dash are the solutions to the dispersion relation, with one solution corresponding to a longer-wavelength solution with wavenumbers $|k| < k_Q$ where $k_Q = \pi G\sigma_0/c^2$ is the wavenumber at the turning point, and the other a shorter-wavelength solution that has $|k| > k_Q$.

and in problem 12.1 you will show that this corresponds to a wavelength

$$\lambda_Q \simeq 2\pi Q h \tag{12.29}$$

where $Q \simeq c\Omega/\pi G\sigma_0$ is disk's stability parameter in a nearly keplerian disk (from Eqn. 10.61) and $h = c/\Omega$ is the disk's vertical scale height.

Long waves that have wavenumber $|k| < |k_Q|$ can also be called gravity waves because $\pi G\sigma_0 > c^2|k|$ so self gravity is the dominant restoring force in the disk that allows the density wave to propagate. Likewise, disk pressure is the dominant restoring force that allows short waves to propagate, and that occurs when $c^2|k| > \pi G\sigma_0$. Note also that the sign of the group velocity differs between gravity and pressure waves, so gravity and pressure waves propagate in opposite directions through the disk.

12.2.3.1 long gravity waves A gravity–dominated (*i.e.* long) density wave will have $\pi G\sigma_0 \gg c^2|k|$, so the waves' dispersion relation is $D(x) \simeq 2\pi G\sigma_0|k|$. When the disk is nearly keplerian, $D(x) \simeq \mathcal{D}x = 3\epsilon(m-\epsilon)\Omega^2 x$ (from Eqns. 6.23–6.25), so the wavelength $\lambda = 2\pi/|k| \simeq 4\pi^2 G\sigma_0/3(m-\epsilon)\Omega^2|x|$ shrinks as the waves propagate away, where $x = (r - r_r)/r_r$ is the fractional distance from the resonance radius r_r.

Note also that the dispersion relation for gravity waves requires $D(x) > 0$, so long waves only propagate in regions of the disk where $\epsilon x > 0$. Thus waves launched at an $\epsilon = +1$ inner Lindblad resonance (ILR) propagate radially outwards while those launched at an $\epsilon = -1$ outer Lindblad resonance (OLR) propagate inwards. This is illustrated in Fig. 12.3, which shows that long gravity-dominated density waves that are launched from a

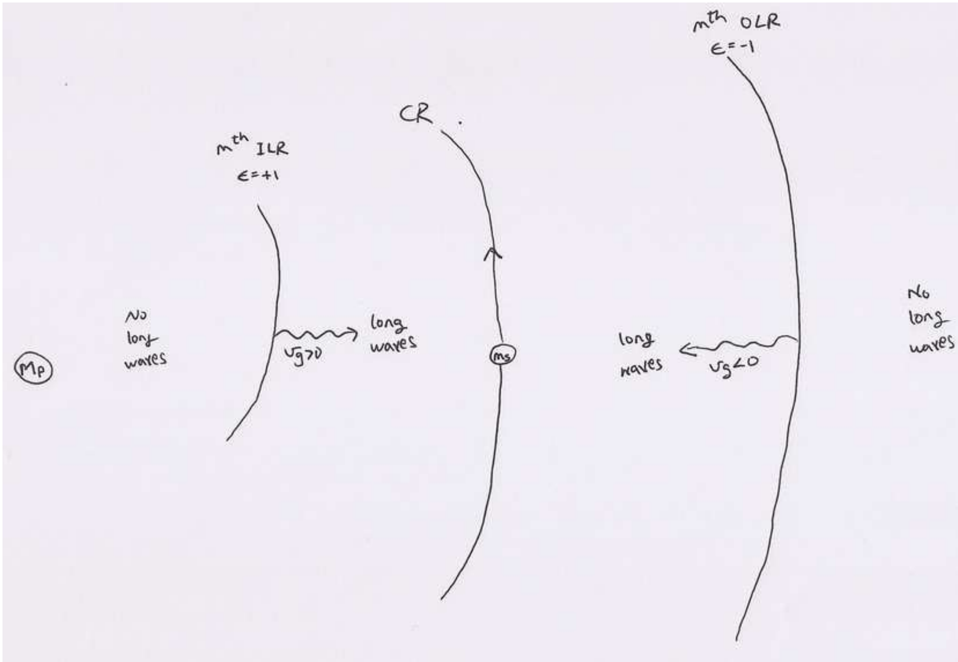

Figure 12.3 This schematic illustrates how the secondary m_s launches long spiral density waves at its m^{th} inner Lindblad resonance (ILR) where $\epsilon = +1$, and its m^{th} outer $\epsilon = -1$ Lindblad resonance (OLR) in a gravity-dominated disk. As section 12.2.3.1 shows, long density waves only propagate where $D(x) > 0$, which requires the sign of the group velocity to obey $\text{sgn}(v_g) = \epsilon$, so long density waves propagate towards the corotation (CR) circle, which is the secondary's orbit about the primary M_p.

Lindblad resonance will propagate towards the the secondary's orbit, which is also known as the corotation circle. That figure also shows that the sign of these waves' group velocity is $\text{sgn}(v_g) = \epsilon$. But the group velocity for gravity waves, Eqn. (12.27), is $v_g = \epsilon s_k \pi G \sigma_0$, so $s_k = \text{sgn}(k)$ must then be $+1$. This means that the density waves that a perturber launches at its Lindblad resonances in a gravitating disk are all long trailing $s_k = +1$ density waves, according to Section 12.2.1.

The most well-known examples of long spiral density waves are those that Saturn's satellites launch in Saturn's main A ring, and a spacecraft image of one such density wave train is shown in Fig. 12.4. This wave is launched at the satellite's inner Lindblad resonance in the A ring, and such waves propagate towards the perturber with a wavelength that shrinks with distance from resonance. So the resonance lies in the lower quarter of this image, and the waves are propagating radially outwards which in this image is upwards and towards the satellite. Density waves in planetary rings are ultimately damped downstream by the ring's viscosity, usually after propagating a few tens of wavelengths, and wave damping is also evident in Fig. 12.4, and the viscous damping of spiral density waves is assessed below in Section 12.2.4.

Lasty, note that the above results might seem to break down for the $m = 1$, $\epsilon = +1$ inner Lindblad resonance. This is because the approximation $D(x) \simeq 3\epsilon(m - \epsilon)\Omega^2 x$ is not appropriate for this particular resonance. In a nearly keplerian system, the $m = \epsilon = 1$

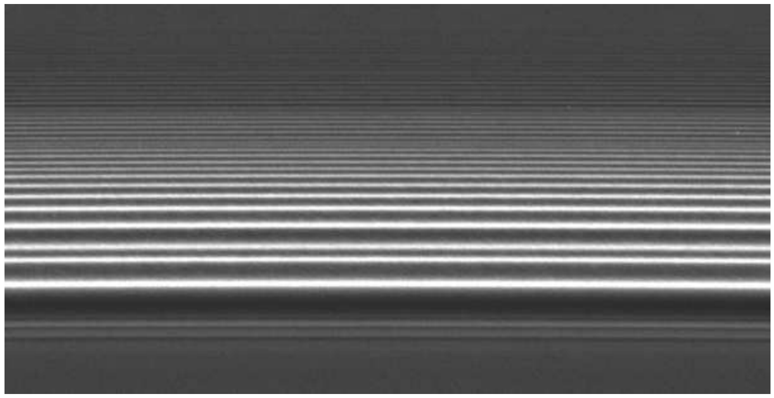

Figure 12.4 Cassini spacecraft image of a spiral density wave that the Saturnian satellite Janus excites at its $m = 4$ inner Lindblad resonance in Saturn's main A ring. This resonance is also known as Janus' 4:3 resonance since the ratio of satellite and ring orbit periods is nearly such. In this closeup view is of the ring's sunlit side, Saturn is far away in the downward direction, the satellite is far away upwards in the radial direction, and the ring particles' orbital motion carries them in the horizontal direction. Bright zones indicate crests in the density wave that are overdense with ring matter while the darker regions are underdense. Of course the spiral density pattern turns about itself like a wound-up watch spring, but the density crests appear straight rather than curved in this very close-up image. This image is from the CICLOPS website at `http://www.ciclops.org/view.php?id=4932`.

resonance is instead a secular resonance, of the kind that is analyzed in Chapter 8. To find the location of this particular resonance, one also has to account for how all of the system's perturbations (such as disk gravity or pressure, or the secondary's gravity, or the primary's oblateness) alters a particle's precession rate $\dot{\tilde{\omega}}$. And in problem 12.2 you will show that the frequency distance from the $m = \epsilon = 1$ secular resonance resonance is $D(r) \simeq 2\Omega(\omega - \dot{\tilde{\omega}})$, so the $D = 0$ resonance is the site where a particle's precession rate $\dot{\tilde{\omega}}(r)$, which is a function of radius r in the system, matches some slow disturbing frequency ω that is often associated with the secondary' precession rate.

12.2.3.2 short pressure waves A pressure dominated disk will have $c^2|k| \gg \pi G\sigma_0$, so the dispersion relation for pressure waves is $D(x) \simeq -c^2|k|^2 = 3\epsilon(m-\epsilon)\Omega^2 x < 0$. Thus pressure waves propagate in a direction opposite to that of gravity waves, *i.e.*, they propagate inwards from an ILR and outwards from an OLR; see Fig. 12.3. In other words, short pressure waves propagate away from the corotation circle CR. The wave's group velocity is $v_g = -\epsilon s_k c^2|k|/\kappa$, and this behavior also requires $s_k = +1$, so short pressure waves that are launched at a Lindblad resonance in the disk are also trailing $s_k = +1$ waves. This condition is also known as the *radiative boundary condition*, because trailing spiral density waves always propagate away from the resonance that launched them.

The wavelength of a pressure wave also shrinks with distance $|x|$ from resonance, and in problem 12.5 you will also show that the wavelength of a pressure wave is

$$\lambda = \frac{2\pi h}{\sqrt{3(m-\epsilon)|x|}} \tag{12.30}$$

in a nearly keplerian pressure-dominated disk.

12.2.3.3 the Q barrier and the forbidden zone Analysis of the dispersion relation (12.26) has shown that a perturber will launch a gravity-dominated long trailing spiral

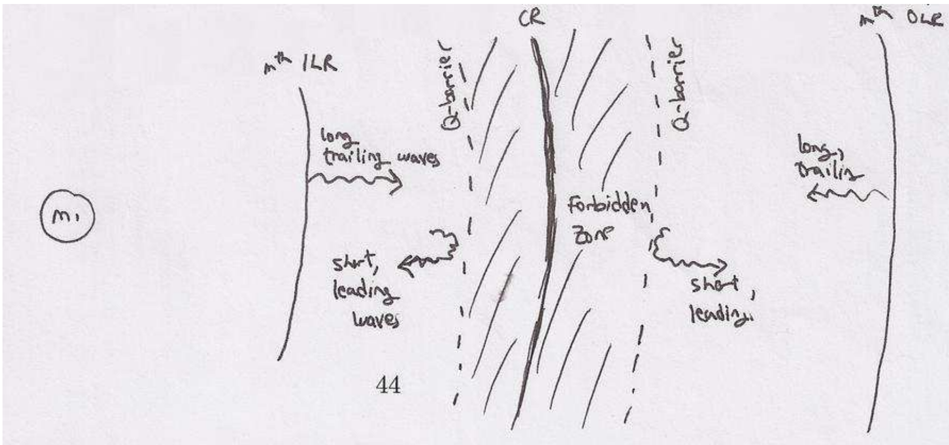

Figure 12.5 The orbit of the secondary m_2 is indicated by the corotation circle (CR), and this secondary also launched long trailing spiral density waves at its m^{th} inner and outer Lindblad resonances (ILR and OLR) in the circumprimary disk. Long spiral density waves propagate towards CR until they reach the Q-barrier, which denotes the inner and outer edge of a forbidden zone in which spiral density waves cannot propagate; instead, waves reflect at the Q-barrier and propagate back towards (and possibly beyond) the Lindblad resonances.

density waves at its Lindblad resonance in the disk, and that these waves propagate towards the corotation circle. The wavenumber $|k|$ will increase as the wave propagates away from resonance until $|k| = k_Q$, which is the site in the disk where the wave's group velocity (12.27) changes sign and the wave reflects. The site where the wave reflects is known as the *Q-barrier* because it depends on the disks's stability parameter $Q = c\kappa/\pi G\sigma_0$ from Eqn. (10.61). The frequency distance from resonance evaluated at the Q barrier is $D(k_Q) = (\pi G\sigma_0/c)^2 = (\kappa/Q)^2 \equiv D_Q$. And when the disk is nearly keplerian, $D(x_Q) \simeq x_Q\mathcal{D} = D_Q$ there. When $D = D_Q$, the wave's group velocity v_g changes sign, so the wave reflects at the fractional distance

$$x_Q = \frac{\Delta r}{r} = \frac{\kappa^2}{\mathcal{D}Q^2} \simeq \frac{1}{3\epsilon(m-\epsilon)Q^2} \tag{12.31}$$

downstream of the resonance. The reflected wave then propagates back towards the launching resonance as a short trailing pressure-dominate density wave.

Also keep in mind that for each m there is an inner and an outer Lindblad resonance. Problem 12.3 also shows that each wave's Q-barrier straddles the corotation circle, so there is a *forbidden zone* surrounding the perturber's orbit where density waves are excluded; see Fig. 12.5 .

12.2.3.4 gravity versus pressure waves According to Eqn. (12.20), a spiral density has the form $\sigma_1 = \mathcal{A}e^{i\phi}$ where $\phi(r) = \int_{r_0}^{r} k(r')dr'$ accounts for how the wave's phase changes with radius r. Evaluating the wave's phase at the Q-barrier will indicate whether the wave is gravity dominated or pressure dominated. To proceed, write $\phi = s_k \int^{x_Q} |k| r dx$ where $s_k = \text{sgn}(k)$ and differentiate Eqn. (12.26), noting that the left-hand side is a function of distance x while the right-hand side is a function of wavenumber $|k|$, so $dx = \mathcal{D}^{-1}(2\pi G\sigma_0 - 2c^2|k|)d|k|$ when the disk is nearly keplerian. Inserting this into ϕ and evaluating that at the Q-barrier yields $\phi = \int_0^{k_Q} s_k r \mathcal{D}^{-1}(2\pi G\sigma_0 - 2c^2|k|)d|k| = s_k(\pi G\sigma_0)^3 r/3\mathcal{D}c^4$ after inserting Eqn. (12.28); this is the phase of the wave as it hits the Q-barrier. Note that if the magnitude of this phase is $\gg 2\pi$ at the Q-barrier then this is a gravity-dominated wave, because the long gravity wave will have cycled through many wavelengths before striking the Q-barrier. If however $|\phi| \ll 2\pi$ at the Q-barrier, then the long wave will already have reflected off the Q-barrier before completing even one cycle, which is the signature of a pressure dominated density wave. Thus the quantity of interest is

$$f = \frac{|\phi|}{2\pi} = \frac{r(\pi G\sigma_0)^3}{6\pi|\mathcal{D}|c^4} = \frac{1}{18\pi(m-\epsilon)Q^3(h/r)} \tag{12.32}$$

If $f \gg 1$ then the wave is gravity dominated, and one can then ignore the effects of disk pressure since other effects will likely damp the wave before it hits the Q barrier. If however $f \ll 1$, then the spiral wave is pressure dominated, and one can ignore disk self-gravity. Problem 12.4 will use f to confirm that the density waves launched in Saturn's main rings are gravity dominated, and that the density waves launched by a protoplanet orbiting in the solar nebula are pressure dominated.

12.2.4 viscous damping of spiral density waves

Viscosity will damp spiral density waves, and accounting for that dissipation introduces the additional terms

$$\left(\frac{4}{3}\nu_s + \nu_b\right)\nabla(\nabla\cdot\mathbf{v}) - \nu_s\nabla\times(\nabla\times\mathbf{v}) \tag{12.33}$$

to the right hand side of Euler's equation. The disk's kinematic shear viscosity is $\nu_s = \eta/\rho$ where η is the shear viscosity and ρ is the disk's volume density, and $\nu_b = \zeta/\rho$ is the kinematic bulk viscosity, and these new terms come from the Navier-Stokes equation for a constant density disk, Eqns. (11.4–11.5). But note that this chapter is considering linearized equations for which any density variations are small, so the use of Eqn. (12.33) here is legitimate. The viscous disk's velocity is $\mathbf{v} = v_r\,\hat{\mathbf{r}} + v_\theta\,\hat{\boldsymbol{\theta}}$ where $v_r = (v_0 + V_r e^{im(\theta-\Omega_{mk}t)})\,\hat{\mathbf{r}}$ and $v_\theta = (r\Omega + V_\theta e^{im(\theta-\Omega_{mk}t)})\,\hat{\boldsymbol{\theta}}$ where $v_0 = -3\nu_s/2r$ is the disk's steady inwards flow rate that is due to the disk's shear viscosity (e.g. Eqn. 11.42 with $r \ll r_s$).

In the tight winding limit the $\nabla\cdot\mathbf{v}$ factor in Eqn. (12.33) is $\nabla\cdot\mathbf{v} \simeq r^{-1}\partial(rv_r)/\partial r \simeq ikV_r e^{im(\theta-\Omega_{mk}t)}$ so $\nabla(\nabla\cdot\mathbf{v}) \simeq -k^2 V_r e^{im(\theta-\Omega_{mk}t)}\,\hat{\mathbf{r}}$ while $\nabla\times\mathbf{v} \simeq ikV_\theta e^{im(\theta-\Omega_{mk}t)}\,\hat{\mathbf{z}}$ so $\nabla\times(\nabla\times\mathbf{v}) \simeq k^2 V_\theta e^{im(\theta-\Omega_{mk}t)}\,\hat{\boldsymbol{\theta}}$, and the sinusoidal part of Eqn. (12.33) is

$$-\left[\left(\frac{4}{3}\nu_s + \nu_b\right)k^2 V_r\,\hat{\mathbf{r}} + \nu_s k^2 V_\theta\,\hat{\boldsymbol{\theta}}\right]e^{im(\theta-\Omega_{mk}t)} \tag{12.34}$$

to lowest order. These are then added to the right hand side of Euler's equation (12.18) whose radial and tangential parts become

$$i\omega_{mk}V_r - 2\Omega V_\theta \simeq -\frac{\partial\phi}{\partial r} - \left(\frac{4}{3}\nu_s + \nu_b\right)k^2 V_r \tag{12.35a}$$

$$\frac{\kappa^2}{2\Omega}V_r + i\omega_{mk}V_\theta \simeq -\frac{im}{r}\phi - \nu_s k^2 V_\theta \tag{12.35b}$$

where $\phi = \phi_d + \phi^s_{mk} + c^2 S/\sigma_0$. But this can be written in same form as Eqn. (12.18) with

$$i\omega_1 V_r - 2\Omega V_\theta \simeq -\frac{\partial \phi}{\partial r} \tag{12.36a}$$

$$\frac{\kappa^2}{2\Omega} V_r + i\omega_2 V_\theta \simeq -\frac{im}{r}\phi \tag{12.36b}$$

where the new doppler-shifted frequencies $\omega_1 = \omega_{mk} - i(4\nu_s/3 + \nu_b)k^2$ and $\omega_2 = \omega_{mk} - i\nu_s k^2$ also have imaginary components. And when the procedure that was outlined in Section 12.2 is again used to derive the dispersion relation for waves in a viscous disk, which is the subject of problem 12.6, that results in a complex dispersion relation

$$\kappa^2 - \omega_1\omega_2 = (2\pi G\sigma_o s_k k - c^2k^2)\omega_2/\omega_{mk}, \tag{12.37}$$

so the wavenumber k is also complex. Since the wave amplitude varies as $e^{i\int k dr}$, the imaginary part of k thus damps the wave.

Problem 12.6 considers weakly damped spiral density waves that have $(\nu_s + \nu_b)|k|^2 \ll \Omega$, which means that the waves travel many wavelengths before damping due to viscosity. In this weak damping limit, the dispersion relation (12.37) simplifies to

$$D(r) \simeq \kappa^2 - \omega^2_{mk} \simeq 2\pi G\sigma_o s_k k - c^2k^2 - i\nu_e\omega_{mk}k^2 \tag{12.38}$$

where $\nu_e = 7\nu_s/3 + \nu_b$ is the disk's effective viscosity. Inserting $k = k_R + ik_I$ into the above shows that the real part of the complex wavenumber is still $D = 2\pi G\sigma_o|k_R| - (ck_R)^2$ while the imaginary part is

$$k_I(x) \simeq \frac{s_k\epsilon\nu_e\kappa k_R^2}{2\pi G\sigma_o - 2c^2|k_R|} \tag{12.39}$$

(see problem 12.6).

Assessing viscous effects in a gravitating disk is straightforward since $c = 0$ and $k_R = x\mathcal{D}/2\pi G\sigma_0$ while $k_I = \epsilon\nu_e\kappa(x\mathcal{D})^2/(2\pi G\sigma_o)^3$, so the imaginary part of the wavenumber damps the wave by the factor $e^{-\int_0^x k_I r dx'} = e^{-(\epsilon x/x_\nu)^3}$ after traveling a fractional distance x from the resonance, where the damping length scale is

$$x_\nu = \frac{2\pi G\sigma_o}{(\nu_e\kappa r\mathcal{D}^2/3)^{1/3}} \tag{12.40}$$

in fractional units. This equation, when combined with observations of spiral density waves in a planetary ring, can be used to determine the ring's physical properties. For instance the observed wavelength readily provides an estimate of the ring surface density σ_0 since $\lambda = 2\pi/|k_R| \propto \sigma_0$, and that coupled with an observation of the wave's damping length scale x_ν yields the ring's effective viscosity ν_e via Eqn. (12.40).

12.3 SPIRAL WAVE SOLUTION

The following calculates the amplitude of the density wave that an orbiting perturber can excite in a disk, as well as the torque that the disk and the perturber exert on each other due to the perturber's gravitational attraction for the wave's spiral density pattern. Chapter 13 will then show how these disk-perturber torques can also drive an early episode of planet migration.

12.3.1 gravitating spiral density waves

Solve for the amplitude of tightly-wound spiral density waves that a perturber launches at its Lindblad resonance in a gravitating, inviscid, and pressureless disk. In the tight-winding limit, the $\partial V_r/\partial r$ term in the continuity equation (12.17) is large in comparison to the V_θ term so

$$S \simeq \frac{i\sigma_0}{\omega_{mk}}\frac{\partial V_r}{\partial r} = \frac{is_k}{2\pi G}\frac{\partial \phi_d}{\partial r} \tag{12.41}$$

by the Poisson equation (12.16). Integrating in r then gives

$$V_r = \frac{\omega_{mk}}{2\pi G\sigma_0}\phi_d \tag{12.42}$$

since the integration constant must be zero (why?) and $s_k = +1$. The radial velocity is Eqn. (12.19a) with $c = 0$ since the disk is pressureless so

$$V_r = \frac{i\omega_{mk}}{D}\left(-\frac{\partial}{\partial r} - \frac{2m\Omega}{r\omega_{mk}}\right)(\phi_d + \phi_s) \simeq \frac{i\omega_{mk}}{D}\left(-\frac{\partial \phi_d}{\partial r} + \Psi_s\right) \tag{12.43}$$

in the tight-winding limit, where $\Psi_s = -\partial\phi_s/\partial r - 2m\Omega\phi_s/r\omega_{mk}$ is the satellite's forcing function, Eqn. (6.17). Equating Eqns. (12.42–12.43) then yields

$$\frac{\partial \phi_d}{\partial r} - \frac{iD}{2\pi G\sigma_0}\phi_d = \Psi_s \tag{12.44}$$

but $D = \mathcal{D}x$ and $\partial\phi_d/\partial r = r^{-1}(\partial\phi_d/\partial x)$ where x is the fractional distance from resonance, so

$$\frac{\partial \phi_d}{\partial x} - i\epsilon\alpha x\phi_d = r\Psi_s, \tag{12.45}$$

where the constant $\alpha = r|\mathcal{D}|/2\pi G\sigma_0 \gg 1$ (see Problem 12.7). In problem 12.8 you will show that Eqn. (12.45) is easily solved using an integrating factor; see also [3]. That solution is

$$\phi_d(\xi) = \epsilon\sqrt{\frac{2\pi}{\alpha}}r\Psi_s H_\epsilon(\xi) = \epsilon\sqrt{\frac{4\pi^2 G\sigma_0}{r|\mathcal{D}|}}r\Psi_s H_\epsilon(\xi) \tag{12.46}$$

where $\epsilon = +1(-1)$ at an inner (outer) Lindblad resonance and the dimensionless wave form $H_\epsilon(\xi)$ is

$$H_\epsilon(\xi) = \frac{e^{i\epsilon\xi^2}}{\sqrt{\pi}}\int_{-\infty}^{\xi} e^{-i\epsilon\eta^2}d\eta \tag{12.47}$$

where

$$\xi = \epsilon\sqrt{\frac{\alpha}{2}}x = \epsilon\sqrt{\frac{r|\mathcal{D}|}{4\pi G\sigma_0}}x \tag{12.48}$$

is the new distance from resonance. $H_\epsilon(\xi)$ is a complex function of distance ξ, and it is plotted in Fig. 12.6, which shows that the wave amplitude $|H_\epsilon| \to 1$ far downstream of the resonance where $\xi \gg 1$ and $|H_\epsilon| \to 0$ far away on the non-wave side where $\xi \ll -1$.

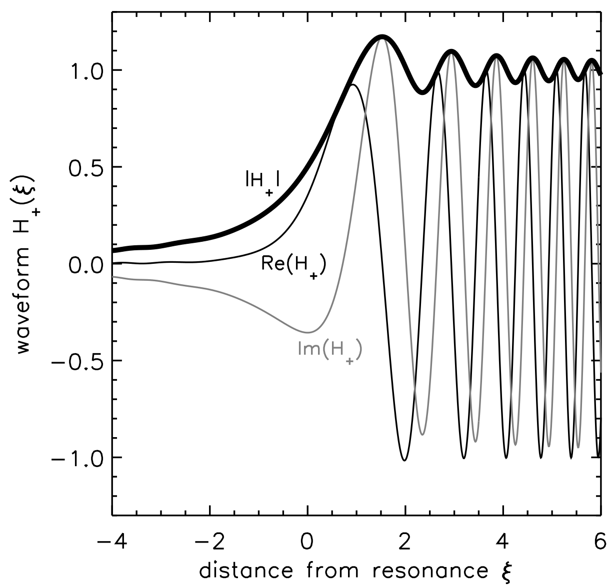

Figure 12.6 Thin black and grey curves give the real and imaginary parts of the dimensionless waveform $H_\epsilon(\xi)$ as a function of dimensionless distance ξ from resonance; see Eqns. (12.47–12.48). These curves are for spiral density waves launched at an $\epsilon = +1$ ILR, and the thick black curve is the magnitude of this complex function $|H_\epsilon(\xi)|$.

Inspection of Eqn. (12.47) and Fig. 12.6 shows that the first wavelength is where $\xi^2 = 2\pi$ or $x = \sqrt{8\pi^2 G\sigma_0/r|\mathcal{D}|}$, and the physical length of the first wavelength can be written

$$\lambda = xr = \sqrt{\frac{8\pi\mu_d}{3(m-\epsilon)}}r \tag{12.49}$$

where $\mu_d = \pi\sigma_0 r^2/M_p$ is the normalized disk mass (see just below Eqn. 4.41). This result will be confirmed in problem 12.9.

The variations in the disk's surface density due to the wave is obtained from the Poisson Eqn. (12.41), and dividing by σ_0 then gives the fractional variation in the disk's surface density,

$$\frac{S(x)}{\sigma_0} = \frac{i}{2\pi G\sigma_0 r}\frac{\partial\phi_d}{\partial x} = \frac{i\Psi_s}{2\pi G\sigma_0}\left(1 + ix\sqrt{2\pi\alpha}H_\epsilon(\xi)\right), \tag{12.50}$$

where $\partial\phi_d/\partial x$ was eliminated via Eqn. (12.45). Problem 12.10 also shows that the lead coefficient $\Psi_s/2\pi G\sigma_0 \simeq 0.8\epsilon m\mu_s/\mu_d$, which will be useful below.

Equation (12.50) shows that the wave's fractional amplitude also grows with distance x downstream of the resonance. Which will become a concern since if $|S(x)/\sigma_0| \gtrsim 1$ then the disturbance in the disk is no longer small and these linearized equations will have broken down. And in problem 12.11 you will show that these waves go nonlinear after traveling a fractional distance

$$x_{NL} \simeq \frac{0.4}{\mu_s}\left(\frac{\mu_d}{m}\right)^{3/2} \tag{12.51}$$

from resonance. Alternatively, the secondary can instead launch a wave that is nonlinear at the outset at $x = 0$. This happens when the first term in Eqn. (12.50) exceeds unity, which occurs when the satellite's fractional mass is not much more than the disk's normalized mass, $\mu_s \gtrsim 1.3m\mu_d$; see problem 12.12.

12.3.2 pressure waves

For spiral density waves propagating in an inviscid, pressure-supported, but non-gravitating disk, the fluid disk's motions are still governed by the continuity equation Eqn. (12.17) in the tight-winding limit,

$$S \simeq \frac{i\sigma_0}{\omega_{mk}} \frac{\partial V_r}{\partial r}. \tag{12.52}$$

The disk's radial velocity is again Eqn. (12.19a) but this time with the disk's potential ϕ_d set to zero, so

$$V_r = \frac{i\omega_{mk}}{D}\left(-\frac{\partial}{\partial r} - \frac{2m\Omega}{r\omega_{mk}}\right)\left(\phi_{mk}^s + \frac{c^2}{\sigma_0}S\right) \simeq \frac{i\omega_{mk}}{D}\left(\Psi_s - \frac{c^2}{\sigma_0}\frac{\partial S}{\partial r}\right) \tag{12.53}$$

in the tight-winding limit. Inserting Eqn. (12.53) into (12.52) and replacing $D = x\mathcal{D}$ and $\partial/\partial r = r^{-1}\partial/\partial x$ yields

$$S - \frac{c^2}{r^2\mathcal{D}}\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial S}{\partial x}\right) = \frac{\sigma_0 \Psi_s}{x^2 r\mathcal{D}}. \tag{12.54}$$

This can be expressed in an integrable form when writing $S(x)$ in terms of a new function $z(x)$ that satisfies $S = \sigma_0(\partial z/\partial x)$ so that Eqn. (12.54) becomes

$$\frac{\partial}{\partial x}\left[z(x) - \frac{\epsilon\beta}{x}\frac{\partial^2 z}{\partial x^2}\right] = \frac{\Psi_s}{x^2 r\mathcal{D}} \tag{12.55}$$

where $\beta = c^2/r^2|\mathcal{D}|$. This is integrated in x, so $z - (\epsilon\beta/x)(\partial^2 z/\partial x^2) = -\Psi_s/xr\mathcal{D}$ since the integration constant must be zero at large $|x|$, and is rearranged as

$$\frac{\partial^2 z}{\partial x^2} - \frac{\epsilon x}{\beta}z = \frac{r\Psi_s}{c^2}. \tag{12.56}$$

Then change variables to $y = \epsilon\beta^{-1/3}x$ so $\partial^2 z/\partial x^2 = \beta^{-2/3}(\partial^2 z/\partial y^2)$ and Eqn. (12.56) then resemble's Airy's equation with a driving term on the right hand side,

$$\frac{\partial^2 z}{\partial y^2} - yz = \beta^{2/3} r\Psi_s/c^2, \tag{12.57}$$

whose solution is

$$z(y) = \frac{\pi\Psi_s}{\beta^{1/3} r\mathcal{D}}\left[iA_i(y) - \epsilon G_i(y)\right], \tag{12.58}$$

but see reference [5] for a more thorough treatment of the boundary conditions that are not discussed here. Integral representations of the Airy functions $A_i(z)$ and $G_i(y)$ are given in Eqns. (A.28)

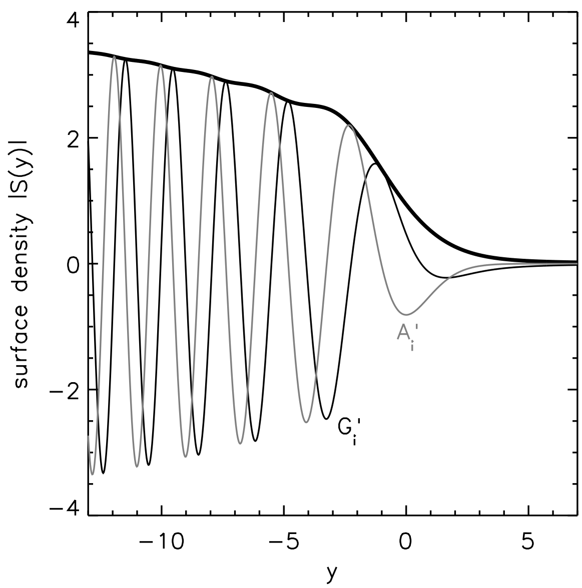

Figure 12.7 The upper thick black curve is the amplitude of a spiral density wave $|S(y)|$ in a pressure-supported disk, in units of $\pi\sigma_0\Psi_s/\beta^{2/3}r|\mathcal{D}|$ (see Eqn. 12.59), while the narrower curves are the real and imaginary components that are proportional to derivatives of Airy functions $A_i'(y) = \partial A_i/\partial y$ and $G_i'(y) = \partial G_i/\partial y$, where $y = \epsilon\beta^{-1/3}x$ is the distance from resonance defined in Section 12.3.2.

The perturbation in the disk's surface density due to this density wave is then

$$S(y) = \sigma_0 \frac{\partial z}{\partial y}\frac{\partial y}{\partial x} = \frac{\pi\sigma_0\Psi_s}{\beta^{2/3}r\mathcal{D}}\left[i\epsilon A_i'(y) - G_i'(y)\right], \tag{12.59}$$

where $A_i'(y) = \partial A_i/\partial y$ and $G_i'(y) = \partial G_i/\partial y$ are plotted in Fig. 12.7, as well as the wave amplitude $|S(y)|$. Recall that pressure waves propagate away from corotation (Section 12.2.3.2 and Fig. 12.3) where $y \propto \epsilon x < 0$ and the Airy functions are oscillatory, while these functions tend to zero on the non-wave side of the resonance. That figure also shows that the first cycle of the pressure waves occurs at $|y| \simeq 4 = \beta^{-1/3}\lambda/r$, so the pressure wave's initial wavelength is

$$\lambda \simeq 4\beta^{1/3}r = 4(c^2/r^2|\mathcal{D}|)^{1/3}r. \tag{12.60}$$

Problem 12.14 will show that the wave's downstream surface density varies as

$$|S(x)| = \frac{\sqrt{\pi}\sigma_0\Psi_s}{\beta^{3/4}r|\mathcal{D}|}|x|^{1/4} \tag{12.61}$$

far from resonance where $y \ll -1$.

And problem 12.17 will use the waves' dispersion relation to show that the time for a pressure wave to propagate a radial distance Δr through the disk is

$$t_p = \frac{2r\Omega}{c}\sqrt{\frac{\Delta r}{r|\mathcal{D}|}}. \tag{12.62}$$

12.4 ANGULAR MOMENTUM TRANSPORT BY SPIRAL DENSITY WAVES

As the following will show, a perturber that excites a spiral density wave at its resonance in a disk also deposits angular momentum there, which waves then transport away as the wave propagates. But these density waves are ultimately damped to the disk, which deposits the waves' angular momentum content elsewhere in the disk. So the perturber and the disk exert a torque on each another, and those torques are assessed below. These torques can also drive planet migration and/or cause the perturber to shepherd open a gap in the disk, and Chapter 13 will use the results obtained here to examine those phenomena further.

12.4.1 gravitational flux

The angular momentum that flows across a gravitating disk is partly due to the spiral density pattern's attraction for itself. The flux of angular momentum due to disk gravity is Eqn. (1.42) whose radial component is $F_g = rg_r g_\theta/4\pi G$ where $g_r = -\Re e(\partial\Phi_d/\partial r) = -\Re e(ik\Phi_d)$ is the radial acceleration due to disk gravity and $g_\theta = -\Re e(\partial\Phi_d/\partial\theta)/r = -\Re e(im\Phi_d/r)$ is the tangential acceleration, so $F_g(r,\theta,z) = mk\Im m(\Phi_d)^2/4\pi G$ where Φ_d is the disk potential. The disk's gravitational angular momentum luminosity is the flux F_g integrated across an infinite cylinder of radius r,

$$\mathcal{L}_g(r) = \int_{-\pi}^{\pi} rd\theta \int_{-\infty}^{\infty} dzF_g. \tag{12.63}$$

This is the rate that the disk transmits angular momentum outwards and across this imaginary cylinder via self gravity, and thus is the gravitational torque that the disk interior to r exerts on the disk exterior to r. To evaluate this integral, recall that the disk potential has the form

$$\Phi_d(r,\theta,z,t) = \phi_d(r)e^{-s_z|k|z}e^{im(\theta-\Omega_{mk}t)} \tag{12.64}$$

(see Eqn. 12.3d and note the discussion just below Eqn. 12.13). Inserting this into the above and evaluating the integrals in Eqn. (12.63) is the subject of problem 12.15, which yields

$$\mathcal{L}_g = \frac{s_k mr|\phi_d|^2}{4G} \tag{12.65}$$

where $s_k = \mathrm{sgn}(k)$.

12.4.2 advective flux

The advective angular momentum flux is due to the flow of disk matter. The surface density of angular momentum in the disk is $\ell = \sigma r v_\theta$, so the advective angular momentum flux in the radial direction is $F_a = \ell v_r$ where $\sigma = \sigma_0 + \Re e[Se^{im(\theta-\Omega_{mk}t)}]$ is the disk's total surface density, $v_r = \Re e[V_r e^{im(\theta-\Omega_{mk}t)}]$ its radial velocity, and $v_\theta = r\Omega + \Re e[V_\theta e^{im(\theta-\Omega_{mk}t)}]$ its azimuthal velocity, so the advective luminosity through a cylinder or radius r is

$$\mathcal{L}_a(r) = \int_{-\pi}^{\pi} \ell v_r rd\theta = \int_{-\pi}^{\pi} \sigma r^2 v_r v_\theta d\theta \tag{12.66a}$$

$$\simeq \int_{-\pi}^{\pi} [\sigma_0 r^2 \Re e(V_r e^{im\theta})\Re e(V_\theta e^{im\theta}) + r^3\Omega\Re e(Se^{im\theta})\Re e(V_r e^{im\theta})]d\theta \tag{12.66b}$$

to lowest order in the small nonzero quantities, and with t set to zero in the above since it is arbitrary.

But lets also consider the disk's radial flux of matter, $F_m = \sigma v_r$, so the luminosity of matter through radius r is

$$\mathcal{L}_m(r) = \int_{-\pi}^{\pi} \sigma v_r r d\theta \simeq \int_{-\pi}^{\pi} [r\sigma_0 \Re e(V_r e^{im\theta}) + r\Re e(S e^{im\theta})\Re e(V_r e^{im\theta})] d\theta. \quad (12.67)$$

This quantity must be zero since, if the disk is inviscid, there is no radial flow in the disk and $\mathcal{L}_m(r) = 0$. Note that the first term in Eqn. (12.67) is obviously zero, so the second term must also integrate to zero. But the second term in Eqn. (12.66b) is proportional to the second term in the above, so that integral is also zero. Thus

$$\mathcal{L}_a(r) = \sigma_0 r^2 \int_{-\pi}^{\pi} \Re e(V_r e^{im\theta})\Re e(V_\theta e^{im\theta}) d\theta = \pi\sigma_0 r^2 [\Re e(V_r)\Re e(V_\theta) + \Im m(V_r)\Im m(V_\theta)] \quad (12.68)$$

when the angular integration is performed; see problem 12.16. Further simplification is straightforward but a bit laborious and as usual is left for problem 12.16 where you will show that the above can be written

$$\mathcal{L}_a = -\frac{m\pi\sigma_0 r^2 k}{rD} \left|\phi_d + c^2 S/\sigma_0\right|^2 . \quad (12.69)$$

If the disk is gravitating then S is related to ϕ_d via Eqn. (12.23) and the above becomes

$$\mathcal{L}_a = -\frac{s_k m r}{2G}\left(1 - \frac{c^2|k|}{\pi G\sigma_0}\right)|\phi_d|^2. \quad (12.70)$$

But if the disk is non-gravitating, then the disk's advective angular momentum luminosity is Eqn. (12.69) with $\phi_d = 0$, which becomes

$$\mathcal{L}_a = \frac{m\pi r c^2}{k\sigma_0}|S|^2. \quad (12.71)$$

See problem 12.16.

12.4.3 total angular momentum flux

A gravitating disk's total angular momentum luminosity is

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_a = -\frac{s_k m r}{4G}\left(1 - \frac{c^2|k|}{\pi G\sigma_0}\right)|\phi_d|^2 = -\frac{s_k m r}{4G}\left(1 - \frac{|k|}{k_Q}\right)|\phi_d|^2 \quad (12.72)$$

where k_Q is the wavenumber at the turning point, Eqn. (12.28). The is the rate at which spiral density waves in the disk transport angular momentum across a cylinder of radius r. Note that $\mathcal{L} < 0$ for trailing $s_k = +1$ gravity waves that have $|k| < k_Q$, so gravity waves launched at a Lindblad resonance in the disk carry negative angular momentum. Or in other words, gravity waves transport their angular momentum inwards through the disk to smaller radii r.

If however the disk is non-gravitating, then the wave's total angular momentum luminosity is just the advective part, Eqn. (12.71), which is the rate that pressure in the spiral density wave communicates angular momentum across radius r.

12.4.4 torque exerted on a gravitating disk

The square of the amplitude of a gravity wave is $|\phi_d|^2 = 4\pi^2 G\sigma_0 r\Psi_s^2/|\mathcal{D}|$, from Eqn. (12.46). If the wave is still far from the edge of the forbidden zone then $|k| \ll k_Q$ and Eqn. (12.72) becomes $\mathcal{L} = -m\pi^2\sigma_0(r\Psi_s)^2/|\mathcal{D}|$, so the density wave transports angular momeuntum inwards across radius r. Now consider the wave that the perturbing secondary in Fig. 12.3 excites at its ILR. That wave carries angular momentum inwards and towards the ILR at the rate $|\mathcal{L}|$, which is then communicated to the secondary via its gravitational attraction for the wave there. So the spiral pattern at the $\epsilon = +1$ ILR exerts a positive torque $T_{d,s} = \epsilon|\mathcal{L}|$ on the secondary. Alternatively, the secondary exerts the torque $T = -T_{d,s} = -\epsilon|\mathcal{L}|$ on the disk at its ILR. Likewise, density waves propagating inwards from the OLR transport angular momentum inwards and away from the disk-matter at that resonance, so the secondary must exert the positive torque $T = -\epsilon|\mathcal{L}|$ at its $\epsilon = -1$ OLR in order to sustain those waves. Thus the torque that the secondary exerts at either of its m^{th} Lindblad resonances in a gravitating disk is

$$T = -\epsilon|\mathcal{L}| = -\frac{m\pi^2\sigma_0 r^2\Psi_s^2}{\mathcal{D}}. \tag{12.73}$$

This is the Goldreich-Tremaine formula [4], and it figures prominently in studies of disk-perturber interactions and theories of planet migration.

12.4.5 torque exerted on a pressure-supported disk

Inserting the pressure wave amplitude, Eqn. (12.61), into Eqn. (12.71) and using the dispersion relation for pressure waves $D = x\mathcal{D} = -c^2k^2$ (e.g. Eqn. 12.26 with $G = 0$) to eliminate x and k then yields the luminosity of angular momentum transported by spiral density waves in a pressure-supported disk,

$$\mathcal{L} = \mathcal{L}_a = \frac{m\pi^2\sigma_0(r\Psi_s)^2}{|\mathcal{D}|}. \tag{12.74}$$

Since $\mathcal{L} > 0$, pressure waves transport angular momentum radially outwards through the disk, in the opposite direction as gravity waves. Pressure waves propagate away from the corotation circle and ultimately damp their angular momentum content to gas disk, so the perturber exerts a positive torque on the disk at its OLR and a negative torque on the disk at its ILR. Interestingly, the torque that the perturber exerts on the disk, $T = -\epsilon|\mathcal{L}| = -m\pi^2\sigma_0(r\Psi_s)^2/\mathcal{D}$, is the same, regardless of whether the disk is pressure or gravity dominated.

Problems

12.1 Show that when the disk is nearly keplerian, a density wave's group velocity $v_g = 0$ when they have wavelength λ_Q given by Eqn. (12.29).

12.2 Insert Eqn. (5.17) into Eqn. (12.26) and show that the frequency distance from the $m = \epsilon = 1$ resonance is $D(r) \simeq 2\Omega(\omega - \dot{\tilde{\omega}})$ when the system is nearly keplerian, and thus this resonance is the site r where $\dot{\tilde{\omega}}(r) = \omega$ where $|\omega| \ll \Omega$ is some slow perturbing frequency.

12.3 Show that in a gravitationally stable disk that is nearly keplerian, the radial distance from a Lindblad resonance to the Q barrier is less than the distance from resonance to the corotation circle, and that all spiral density waves would hit a Q barrier before reaching the corotation circle.

12.4 Calculate the f of Eqn. (12.32) for the solar nebula and for Saturn's A ring, and demonstrate that a Saturnian satellite will launch gravity-dominated spiral density waves in the A ring, and that a protoplanet would launch pressure-dominated density waves in the solar nebula.

12.5 Use the dispersion relation to derive the wavelength of spiral density waves in a nearly keplerian pressure-dominated disk, Eqn. (12.30), and confirm that such waves propagate away from the corotation circle.

12.6 Derive the complex dispersion relation for damped spiral density waves propagating in a viscous disk.

a.) Combine the continuity equation with Poisson's equation and Euler's equation (12.36) in the tight winding limit to obtain the complex dispersion relation for spiral density waves in a viscous disk, Eqn. (12.37).

b.) Show that Eqn. (12.37) becomes Eqn. (12.38) when the disk is non-gravitating and viscosity is weak, $(\nu_s + \nu_b)|k|^2 \ll \Omega$.

c.) Show that the imaginary part of wavenumber k is Eqn. (12.39) when gravity waves are weakly damped by viscosity.

12.7 Show that the α in Eqn. (12.45) is

$$\alpha = \frac{r|\mathcal{D}|}{2\pi G\sigma_0} = \frac{3(m-\epsilon)}{2\mu_d} \gg 1 \tag{12.75}$$

when the disk is nearly keplerian, where $\mu_d = \pi\sigma_0 r^2/M_p$ is the normalized disk mass of Section 4.4.2.

12.8 Solve Eqn. (12.45) for the perturbation in the gravitating disk's potential ϕ_d.

a.) Review the method of integrating factors, and show that the integrating factor for Eqn. (12.45) is $e^{-i\epsilon\alpha x^2/2}$.

b.) Use this integrating factor to show that the solution to Eqn. (12.45) can be written

$$\phi_d(x) = r\Psi_s e^{i\epsilon\alpha x^2/2}\int_c^x e^{-i\epsilon\alpha y^2/2}dy \tag{12.76}$$

where the integration limit c is chosen to satisfy the boundary condition, namely, that the amplitude of the spiral density wave is zero far away on the non-wave side of the resonance.

c.) Use Eqn. (12.48) to recast Eqn. (12.76) in the form of Eqns. (12.46–12.47). Explain how you chose c.

12.9 Derive Eqn. (12.49) from the condition $\xi^2 = 2\pi$.

12.10 Show that the lead coefficient in Eqn. (12.50) is $\Psi_s/2\pi G\sigma_0 \simeq 0.8\epsilon m\mu_s/\mu_d$ when the disk is nearly keplerian.

12.11 The second term in Eqn. (12.50) will dominate over the first once the spiral density wave has propagated a few wavelengths. Show that the wave will go nonlinear, $|S/\sigma_0| > 1$, once the wave has traveled a distance x_{NL} given by Eqn. (12.51).

12.12 Show that spiral density waves launched by a secondary of fractional mass μ_s are nonlinear at the $x = 0$ resonance when $\mu_s \gtrsim 1.3m\mu_d$.

12.13 Show that the fluid disk's motions far away and on the non-wave side of the resonance resembles the solution for an isolated particle orbiting near a Lindblad resonance, Chapter 6.

a.) Show that $H_\epsilon(\xi) \rightarrow i\epsilon/2\sqrt{\pi}\xi$ when $\xi \ll -1$ and thus the wave surface density $S/\sigma_0 \rightarrow 0$ far on the non-wave side of resonance, as expected.

b.) Insert this result into Eqn. (12.19a) to obtain the disk's radial velocity V_r on the non-wave side of the resonance. Then estimate the disk's eccentricity there from the magnitude of $e \simeq |V_r/r\Omega|$, and show that $e \sim |\Psi_s/r\mathcal{D}x|$, which of course is the eccentricity of an isolated particle near resonance, Eqn. (6.28).

12.14 Use the analysis of Airy functions in reference [1] to show that the magnitude of the factor $|i\epsilon A_i'(y) - G_i'(y)| \simeq |y|^{1/4}/\sqrt{\pi}$ in Eqn. (12.59) far downstream of resonance where $y \ll -1$, and then confirm the pressure wave amplitude, Eqn. (12.61)

12.15 Insert the wave potential, Eqn. (12.64), into Eqn. (12.63) and evaluate the integrals there to obtain Eqn. (12.65), which is the luminosity of angular momentum that results from the spiral wave's gravitational attraction for itself.

12.16 Perform the angular integration to obtain the right hand side of Eqn. (12.68). Then insert Eqns. (12.19) into (12.68) to obtain Eqn. (12.69), which is the spiral density wave's angular momentum luminosity far downstream of the resonance where the secondary's forcing is negligable. Then use the dispersion relation to obtain Eqns. (12.70–12.71).

12.17 Use the dispersion relation and the group speed for spiral density waves to calculate the time for waves propagating in a pressure-supported disk to travel a radial distance Δr from resonance, Eqn. (12.62).

REFERENCES

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