CHAPTER 11

VISCOUS EVOLUTION OF A CIRCUMSTELLAR DISK

The effects of the fluid's viscosity is considered, and the Navier-Stokes equation for the fluid evolution is then obtained. Angular momentum transport in a viscous circumstellar disk is also examined, and the disk's viscous evolution is studied in considerable detail.

11.1 NAVIER-STOKES EQUATION

Viscosity is the friction that tends to resist any shearing or compression in a flowing fluid. These viscous forces also tends to smooth out gradients in the fluid's density and velocity, and it does so by transmitting momentum through the fluid in a way that reduces those gradients. To examine this in detail, Euler's equation is adapted so that it can be applied to a viscous fluid.

Recall the results of Section 10.2.3, which showed that the fluid's internal frictional forces are accounted for by adding Eqn. (10.26) to the right hand side of Euler's equation (10.12), which in Cartesian coordinates is

$$\frac{\partial v_i}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} + \frac{1}{\rho}\sum_{j=1}^{3}\frac{\partial \sigma'_{ij}}{\partial x_j}, \tag{11.1}$$

where the new term on the right is the gradient of the viscous stress tensor σ'_{ij}. This viscous stress tensor is the flux density of momentum that is transported via the fluid's viscosity. So if there is a region where there is a spatial gradient in the fluid's momentum flux density

σ'_{ij}, then that region is also gaining or losing momentum and thus experiences an additional acceleration $\rho^{-1}\sum_j \partial\sigma'_{ij}/\partial x_j$ due to this viscous transport of momentum.

Note that the viscous stress tensor must be zero when the fluid has no relative motion, so σ'_{ij} will be some function of the velocity gradients $\partial v_i/\partial x_k$. Similarly, the viscous stress tensor should be zero when the fluid is in uniform rotation; see problem 11.1. With these constraints in mind, the viscous stress tensor can be written [4] as

$$\sigma'_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\sum_{k=1}^{3}\frac{\partial v_k}{\partial x_k}\right) + \zeta\delta_{ij}\sum_{k=1}^{3}\frac{\partial v_k}{\partial x_k} \tag{11.2}$$

where η is the fluid's *shear viscosity* and ζ the *bulk viscosity*. Note also that the viscous stress tensor is symmetric, $\sigma'_{ij} = \sigma'_{ji}$. Inserting this into Euler's Eqn. (11.1) then yields the *Navier-Stokes equation*

$$\begin{aligned}\frac{\partial v_i}{\partial t} + \sum_{j=1}^{3} v_j\frac{\partial v_i}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial\Phi}{\partial x_i}\\ &+ \frac{1}{\rho}\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left[\eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\sum_{k=1}^{3}\frac{\partial v_k}{\partial x_k}\right)\right] + \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(\zeta\sum_{k=1}^{3}\frac{\partial v_k}{\partial x_k}\right).\end{aligned} \tag{11.3}$$

In general, the viscosity coefficients η and ζ can be functions of the fluid properties such as pressure, temperature, and density, and thus cannot be taken out of the gradient operators. Note that the viscous terms also require calculating second order spatial derivatives, which can make this equation too complicated to be of much use. However Section 11.2.1 will examine the viscous transport of momentum through a fluid, which can result in a much simpler set of evolutionary equations.

Nonetheless, if the η and ζ are in fact constant, then the Navier-Stokes equation can be written in the more compact vector form

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\nabla p}{\rho} - \nabla\Phi + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v}) \tag{11.4}$$

(see problem 11.2), where the vector Laplacian in the above can also be written

$$\nabla^2\mathbf{v} = \nabla(\nabla\cdot\mathbf{v}) - \nabla\times(\nabla\times\mathbf{v}), \tag{11.5}$$

which is $\sum_j(\nabla^2 v_j)\hat{\mathbf{x}}_j$ in Cartesian coordinates. Note that if the fluid is also incompressible, then ρ is constant and $\nabla\cdot\mathbf{v} = 0$ by the continuity equation (10.9). In this case it is convenient to introduce the *kinematic shear viscosity*

$$\nu = \frac{\eta}{\rho}, \tag{11.6}$$

so the acceleration on an incompressible fluid element due to viscosity is simply $\nu\nabla^2\mathbf{v}$ when ν is constant.

11.2 ANGULAR MOMENTUM TRANSPORT IN A VISCOUS DISK

Now examine the flow of angular momentum through a viscous circumstellar disk that is in circular motion about a central star. The disk is thin and axisymmetric, so all quantities are functions of radius only, and the fluid velocity is $\mathbf{v} = v_\theta(r)\,\hat{\boldsymbol{\theta}}$. If $\rho\mathbf{v}$ is the linear momentum density at some site $\mathbf{r}$ in the fluid disk, then $\boldsymbol{\ell} = \mathbf{r}\times\rho\mathbf{v}$ is the fluid's angular momentum density, which evolves at the rate $\partial\boldsymbol{\ell}/\partial t = \mathbf{r}\times\partial(\rho\mathbf{v})/\partial t$. Recall that the fluid's linear momentum density evolve at the rate given by Eqn. (10.22),

$$\frac{\partial}{\partial t}(\rho v_i) = -\nabla\cdot\mathbf{\Pi}_i - \rho\frac{\partial\Phi}{\partial x_i} \tag{11.7}$$

where the vector $\mathbf{\Pi}_i = \sum_j \Pi_{ij}\,\hat{\mathbf{x}}_j$ is the flux density of the fluid's i^{th} component of momentum that is formed from elements of the fluid's momentum flux density tensor Π_{ij}; see Section 10.2.3. The disk is thin and axisymmetric, so we only need to calculate the rate that the z component of angular momentum density evolves, which is

$$\begin{aligned}\frac{\partial\ell_z}{\partial t} &= x\frac{\partial}{\partial t}(\rho v_y) - y\frac{\partial}{\partial t}(\rho v_x) = -x\nabla\cdot\mathbf{\Pi}_2 + y\nabla\cdot\mathbf{\Pi}_1 - x\rho\frac{\partial\Phi}{\partial y} + y\rho\frac{\partial\Phi}{\partial x}\\ &= -\left[\nabla\cdot(x\mathbf{\Pi}_2) - \mathbf{\Pi}_2\cdot\nabla x\right] + \left[\nabla\cdot(y\mathbf{\Pi}_1) - \mathbf{\Pi}_1\cdot\nabla y\right] - \rho(\mathbf{r}\times\nabla\Phi)\cdot\hat{\mathbf{z}}\end{aligned} \tag{11.8}$$

after invoking the vector identity Eqn. (A.14). Note that $\mathbf{r}\times\nabla\Phi = 0$ in an axisymmetric disk that is a function of r only. Similarly, only the non-axisymmetric part of $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ will contribute to $\partial\ell_z/\partial t$. Also note that $\mathbf{\Pi}_2\cdot\nabla x - \mathbf{\Pi}_1\cdot\nabla y = \mathbf{\Pi}_2\cdot\hat{\mathbf{x}} - \mathbf{\Pi}_1\cdot\hat{\mathbf{y}} = \Pi_{21} - \Pi_{12} = 0$, so the above simplifies to

$$\frac{\partial\ell_z}{\partial t} = -\nabla\cdot(x\mathbf{\Pi}_2 - y\mathbf{\Pi}_1). \tag{11.9}$$

Now integrate Eqn. (11.9) over some volume V so that

$$\frac{\partial}{\partial t}\int_V \ell_z dV' = \frac{\partial\Delta\ell_z}{\partial t} = -\int_V \nabla\cdot(x\mathbf{\Pi}_2 - y\mathbf{\Pi}_1)dV' = -\int_S \mathbf{F}_z\cdot\mathbf{da}' \tag{11.10}$$

where the divergence theorem, Eqn. A.24a, was used to convert the volume integral into a surface integral and

$$\mathbf{F}_z = x\mathbf{\Pi}_2 - y\mathbf{\Pi}_1 = (x\Pi_{21} - y\Pi_{11})\,\hat{\mathbf{x}} + (x\Pi_{22} - y\Pi_{12})\,\hat{\mathbf{y}} \tag{11.11}$$

is the flux density of angular momentum that is flowing through the area $\mathbf{da}' = \hat{\mathbf{n}}\,da'$ on surface S in Eqn. (11.10). And in problem 11.5 you will show that if the disk is inviscid then the above simplifies to

$$\mathbf{F}_z = \ell_z\mathbf{v} + rp\,\hat{\boldsymbol{\theta}}\,. \tag{11.12}$$

This is the disk's *advective* angular momentum flux, and it gives the rate per area at which the inviscid fluid's motion transports angular momentum through the disk.

But lets continue to consider a viscous disk for which $\Pi_{ij} = \rho v_i v_j + \delta_{ij}p - \sigma'_{ij}$ (see Eqn. 10.24), and recall that only the non-axisymmetric terms contribute to the disk's viscous evolution. Since the disk is axially symmetric, terms due to motion and pressure have no net effect. Consequently the Π_{ij} in the above can be replaced with $-\sigma'_{ij}$ where

$$\sigma'_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{11.13}$$

since the $\sum_k \partial v_k/\partial x_k = \nabla\cdot\mathbf{v}$ terms in the viscous stress tensor (Eqn. 11.2) are zero in an axisymmetric disk.

The above expression for the disk's angular momentum flux density $\mathbf{F}_z$ is written in terms of Cartesian coordinates, which is rather awkward for use in an axially symmetric disk. To convert this to cylindrical coordinates, note that Cartesian and cylindrical coordinates are related via $x = r\cos\theta$ and $y = r\sin\theta$ while the velocities are $v_x = -v_\theta \sin\theta$ and $v_y = v_\theta \cos\theta$. These are used in problem 11.3 to write the velocity gradients $\partial v_i/\partial v_j$ in terms of cylindrical coordinates, such as

$$\frac{\partial v_x}{\partial x} = -r\frac{\partial \Omega}{\partial r}\sin\theta\cos\theta, \tag{11.14}$$

where $\Omega(r) = v_\theta/r$ is the fluid's angular velocity. Inserting those velocity gradients, Eqns. (11.61), into Eqn. (11.11) then yields the much more compact result that

$$F_z(r) = -\eta r^2\frac{\partial \Omega}{\partial r}. \tag{11.15}$$

If the disk is Keplerian, then $\partial\Omega/\partial r < 0$ and $F_z > 0$, which means that the disk's viscosity transports angular momentum radially outwards through the disk.

11.2.1 angular momentum luminosity, and the viscous acceleration

Now imagine a cylinder of radius r that is coaxial with the disk. The integrated flux of angular momentum that the disk transports across that imaginary cylinder is $\mathcal{L}_z(r) = \int_A F_z(r)da$ where A is the area where the disk and the cylinder intersect. The disk has a scale height $h(r)$, which is the vertical height that the disk extends above/below the disk's midplane, so $A(r) = 2\pi r 2h$. It is also reasonable to treat the disk's volume density as constant across the disk's vertical column, so $\rho(r,z) = \sigma/2h$ where $\sigma(r)$ is the disk's surface mass density. Then

$$\mathcal{L}_z(r) = \int_A F_z(r)da = F_z A = -2\pi\nu\sigma r^3\frac{\partial \Omega}{\partial r}. \tag{11.16}$$

where $\nu = \eta/\rho$ is the fluid's kinematic viscosity. And if the disk is in Keplerian rotation, then $\partial\Omega/\partial r = -3\Omega/2r$ and $\mathcal{L}_z = 3\pi\nu\sigma r^2\Omega$. This is the disk's *angular momentum luminosity*, which is the rate at which the disk's angular momentum flows across radius r. This then is also the torque that the faster disk material orbiting just interior to r exerts on the slower disk material that is just exterior, due to the viscous friction that is exerted across that interface.

Now calculate the acceleration that a small fluid region experiences due to the disk's viscosity. Begin by considering a narrow annulus in the disk whose inner radius is r and outer radius is $r + \Delta r$. The torque that is exerted on this annulus due to the fluid orbiting interior to r is $\mathcal{L}_z(r)$, while the torque that that annulus exerts on the fluid just exterior to it is $\mathcal{L}_z(r+\Delta r)$, so $-\mathcal{L}_z(r+\Delta r)$ is the torque that the exterior fluid exerts on the annulus. So the net torque on that annulus is $\Delta T = \mathcal{L}_z(r) - \mathcal{L}_z(r+\Delta r) = -(\partial\mathcal{L}_z/\partial r)\Delta r$ when $\Delta r \ll r$. That annulus has mass $\Delta m = 2\pi\sigma r\Delta r$, so the torque $\Delta T = \Delta m r a_\nu$ where

$$a_\nu = \frac{\Delta T}{r\Delta m} = \frac{1}{\sigma r^2}\frac{\partial}{\partial r}\left(\nu\sigma r^3\frac{\partial \Omega}{\partial r}\right) \tag{11.17}$$

is the tangential acceleration on the fluid due to its viscosity. Note that the viscous terms in the Navier–Stokes equation (11.3) will be equivalent to a_ν. Another useful quantity is the surface density of torque in this viscous disk, which is

$$\zeta = \sigma r a_\nu = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu\sigma r^3 \frac{\partial\Omega}{\partial r}\right). \tag{11.18}$$

This becomes $\zeta = -(3/2r)\partial(\nu\sigma r^2\Omega)/\partial r$ if the disk is in Keplerian rotation. The following will show how this torque also drives a slow radial flow of the disk's fluid.

11.2.2 the disk's viscous evolution

The following derives the equation that governs the disk's viscous evolution over time. These results are drawn from the review given in [6], while reference [5] provides an even more comprehensive analysis of viscous disks.

The preceding section noted that the viscous torque will drive a radial flow, so the fluid disk's velocity becomes $\mathbf{v}(r,t) = v_r\,\hat{\mathbf{r}} + v_\theta\,\hat{\boldsymbol{\theta}}$. Keep in mind that the preceding had neglected the disk's radial velocity v_r, but those results will still be applicable when the disk's viscosity is sufficiently weak such that $|v_r| \ll v_\theta$, which is usually the case.

Now recall the mass continuity equation (10.9) for the disk's surface density, $\partial\sigma/\partial t + \nabla\cdot(\sigma\mathbf{v}) = 0$, which becomes

$$\frac{\partial\sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\sigma v_r) = 0 \tag{11.19}$$

for an axially symmetric disk that is a function of r only; see Eqn. (A.20a). Note that the source term on the right hand side is zero, which indicates that mass is conserved and not created or destroyed. Also keep in mind that σ and v_r are both functions of radius r and time t.

The surface density of angular momentum in the disk is $\ell = \sigma r v_\theta = \sigma r^2\Omega$, and this quantity also obeys a continuity equation $\partial\ell/\partial t + \nabla\cdot(\ell\mathbf{v}) = \zeta$ where ζ is the viscous torque density, Eqn. (11.18). This time the source term on the right is nonzero because angular momentum can flow into or out of a region due to the viscous torque that is exerted by adjacent regions. For an axially symmetric disk, this equation becomes

$$\frac{\partial}{\partial t}(\sigma r^2\Omega) + \frac{1}{r}\frac{\partial}{\partial r}(\sigma r^3\Omega v_r) = \frac{1}{r}\frac{\partial}{\partial r}(\nu\sigma r^3\Omega'). \tag{11.20}$$

where $\Omega' = \partial\Omega/\partial r$. The first term on the left is $r^2\Omega(\partial\sigma/\partial t)$ while the second term can be written $r\Omega\partial(\sigma r v_r)/\partial r + \sigma v_r\partial(r^2\Omega)/\partial r = -r^2\Omega(\partial\sigma/\partial t) + \sigma v_r\partial(r^2\Omega)/\partial r$ when Eqn. (11.19) is used to cancel out the $\partial\sigma/\partial t$ terms, which then yields the fluid's radial velocity

$$v_r(r,t) = \frac{\frac{\partial}{\partial r}(\nu\sigma r^3\Omega')}{r\sigma\frac{\partial}{\partial r}(r^2\Omega)}. \tag{11.21}$$

This is then inserted into Eqn. (11.19) to provide a single partial differential equation for the disk's surface density,

$$\frac{\partial\sigma}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\frac{\partial}{\partial r}(\nu\sigma r^3\Omega')}{\frac{\partial}{\partial r}(r^2\Omega)}\right], \tag{11.22}$$

which is a diffusion equation for the disk's surface density $\sigma(r,t)$. Once Eqn. (11.22) is solved, that result can be inserted into Eqn. (11.21) to obtain the disk's radial velocity $v_r(r,t)$. And if the disk is Keplerian, $\Omega = \sqrt{GM/r^3}$ and

$$\frac{\partial\sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(\nu\sigma r^{1/2})\right]. \tag{11.23}$$

11.2.3 constant viscosity disk in Keplerian rotation

Now consider a Keplerian disk having a constant viscosity ν. Begin by solving Eqn. (11.23) by separation of variables, which assumes that the disk's surface density has the form $\sigma(r,t) = R(r)T(t)$. Insert this into the above and then divide by σ so that

$$\frac{1}{T}\frac{\partial T}{\partial t} = \frac{3\nu}{rR}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(r^{1/2}R)\right]. \tag{11.24}$$

The left hand side of this equation is a function of time t only while the right hand side is a function of r only. Both terms are a function of different independent quantities, so both functions must be equal to some separation constant that will be written $-\lambda$. So the solution to the time-dependent part of Eqn. (11.24) is simply $T(t) \propto e^{-\lambda t}$ while $R(r)$ satisfies

$$\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(r^{1/2}R)\right] + k^2 rR = 0 \tag{11.25}$$

where the new separation constant is $k^2 = \lambda/3\nu$. One can manipulate this equation further until it is in the form of Bessel's equation, but it is easier to just type this equation into your favorite symbolic mathematics software, to see a computer can find the solution. For example, `MAPLE` will solve this equation via

```
eqn:= Diff(sqrt(r)*Diff(R(r)*sqrt(r), r), r) + k^2*r*R(r);
dsolve(eqn=0, R(r));
```

which yields

$$R(r) = Ar^{-1/4}J_{1/4}(kr) + Br^{-1/4}Y_{1/4}(kr) \tag{11.26}$$

where the J_n and Y_n are Bessel functions of the first and second kind of order $n = 1/4$, and A and B are integration constants that are determined by boundary conditions.

The remainder will assume that the disk's angular momentum surface density $\ell = \sigma r v_\theta \propto \sigma\sqrt{r}$ is zero at the disk's inner boundary at $r = 0$. To apply this boundary condition, we need to know how $\ell \propto R(r)\sqrt{r}$ behaves in the limit that $r \to 0$, which is easily assessed in `MAPLE` by expanding ℓ as a power series in r to order r^2,

```
ell:= R(r)*sqrt(r);
series(ell, r, 2);
```

which yields $\ell = c_1 B + c_2(A+B)\sqrt{r} + \mathcal{O}(r^2)$ where c_1 and c_2 are numerical coefficients. Consequently, $\ell \to 0$ as $r \to 0$ requires the integration constant $B = 0$, so the disk's surface density $\sigma = R(r)T(t)$ has the form

$$\sigma(r,t) \sim Ae^{-3\nu k^2 t}r^{-1/4}J_{1/4}(kr). \tag{11.27}$$

Keep in mind that Eqn. (11.27) is just one of many particular solutions to Eqn. (11.25), each of which are characterized by different values of the separation constant k, so the general solution to Eqn. (11.25) is the integrated sum of all possible solutions

$$\sigma(r,t) = r^{-1/4}\int_0^\infty A(k)e^{-3\nu k^2 t}J_{1/4}(kr)dk, \tag{11.28}$$

where the function $A(k)$ must be chosen to agree with the system's initial conditions. That determination of $A(k)$ will benefit from the orthogonality relation [3]

$$\int_0^\infty rJ_m(kr)J_m(k'r)dr = \delta(k'-k)/k, \tag{11.29}$$

where $\delta(x)$ is the delta function of Eqn. (A.27), and the order of the Bessel function obeys $m > -\frac{1}{2}$.

11.2.4 evolution of a viscous ring

Assume that all of the disk's mass M_d was initially concentrated in a circular ring of radius R at time $t = 0$, so its initial surface density is also the delta function

$$\sigma(r,0) = \frac{M_d}{2\pi R}\delta(r-R). \tag{11.30}$$

To solve for $A(k)$, insert $\sigma(r,0)$ into Eqn. (11.28) at time $t = 0$, multiply by $r^{5/4}J_m(k'r)$, and then integrate that equation over all r. Integrating over the delta functions then isolates $A(k)$, which is

$$A(k) = \frac{M_d R^{1/4}}{2\pi}kJ_{1/4}(kR), \tag{11.31}$$

so the viscous disk's surface density is

$$\sigma(x,\tau) = \frac{M_d}{2\pi R^2 x^{1/4}}\int_0^\infty se^{-\tau s^2/4}J_{1/4}(xs)J_{1/4}(s)ds \tag{11.32}$$

where $x = r/R$ is a dimensionless radial coordinate, $\tau = 12\nu t/R^2$ is the dimensionless time, and $s = kR$ is a dimensionless integration variable. Inspection of the table of integrals in [2], Eqn. (6.633) there, will show that this rather complicated integral evaluates to

$$\int_0^\infty se^{-\tau s^2/4}J_{1/4}(xs)J_{1/4}(s)ds = \frac{2}{\tau}e^{-(x^2+1)/\tau}I_{1/4}(2x/\tau), \tag{11.33}$$

so the disk's surface density simplifies to

$$\sigma(x,\tau) = \frac{M_d}{\pi R^2}\frac{e^{-(x^2+1)/\tau}}{x^{1/4}\tau}I_{1/4}(2x/\tau) \tag{11.34}$$

where $I_{1/4}$ is the modified Bessel function. Problem 11.6 notes that this equation can be problematic when the argument of the Bessel function is large, and you are asked to show that

$$\sigma(x,\tau) \simeq \frac{M_d}{\pi R^2}\frac{e^{-(x-1)^2/\tau}}{x^{3/4}\sqrt{4\pi\tau}} \tag{11.35}$$

early in the disk's evolution when and where $\tau \ll 2x$.

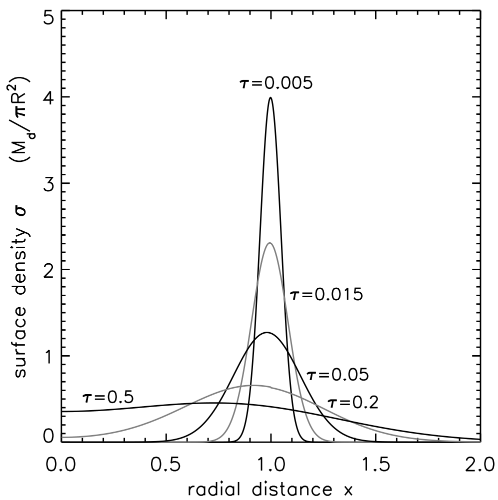

Figure 11.1 A fluid ring having an initial mass M_d and radius R spreads radially over time due to its constant shear viscosity ν. The fluid's surface density is given by Eqns. (11.34–11.35), which is plotted versus radial distance $x = r/R$ at selected times time $\tau = 12\nu t/R^2$.

11.2.5 the viscous timescale

Figure 11.1 shows the viscous ring's surface density $\sigma(x, \tau)$ as a function of distance x at selected times τ. It is evident that time $\tau = 12\nu t/R^2 \sim 1$ is the time for a narrow ring to have spread into a broad disk of radius R, so the ring's viscous spreading timescale is simply

$$t_\nu \sim \frac{R^2}{12\nu}. \tag{11.36}$$

After this time, viscosity will have erased any evidence that this fluid might once have been confined to a narrow ring. And in problem 11.7 you will consider the early evolution of an initially narrow ring, and show that the time for it to spread a small radial distance Δr is also

$$t_\nu \simeq \frac{\Delta r^2}{12\nu}, \tag{11.37}$$

where $\Delta r \ll R$ is the e-fold halfwidth of the ring's surface density profile such that $\sigma(R \pm \Delta r, t) = e^{-1}\sigma(R, t)$. Differentiating Eqn. (11.37) with respect to time t_ν yields $2\Delta r v_r \simeq 12\nu$, so the fluid's radial velocity at the ring edge where $r = R \pm \Delta r$ is

$$v_r \simeq \pm\frac{6\nu}{\Delta r}. \tag{11.38}$$

See also problem 11.8.

11.2.6 angular momentum conservation

Figure 11.2 shows the evolution at later times $\tau \gtrsim 1$, when the fluid has spread into a broad disk. Figures 11.1–11.2 indicates that much of the disk's mass dribbles onto the central star

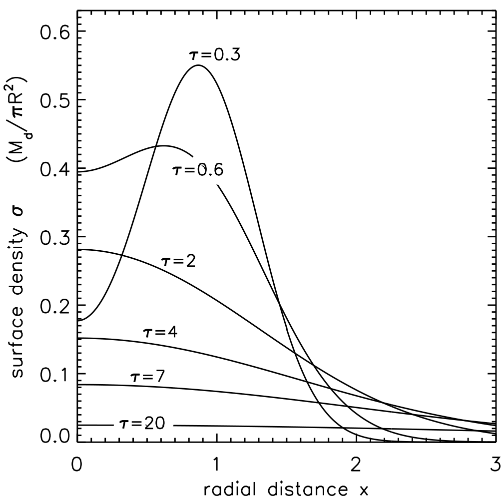

Figure 11.2 A fluid ring having an initial mass M_d and radius R spreads radially over time due to its constant shear viscosity ν. The fluid's surface density is given by Eqns. (11.34–11.35), which is plotted versus radial distance $x = r/R$ at selected times time $\tau = 12\nu t/R^2$.

over time, while some disk matter flows to larger radial distances. This is a consequence of angular momentum conservation. Recall the boundary condition, which requires the surface density of angular momentum $\ell = \sigma r^2 \Omega \rightarrow 0$ as $r \rightarrow 0$. This is known as the zero-torque boundary condition; since ℓ is always a constant zero at the disk's inner edge, this also implies that the central star does not exert a torque on the disk. And because there are no other external torques on the disk, the disk's total angular momentum L is conserved such that

$$L = \int_0^\infty 2\pi r \ell(r) dr = M_r R^2 \Omega_R \tag{11.39}$$

where Ω_R is the angular velocity of the ring at time $t = 0$; see problem 11.9

11.2.7 radial evolution of a broad viscous disk

Figure 11.2 shows that at later times when $\tau \gg 1$, the disk at $x \sim 1$ varies slowly with radial distance $x = r/R$. The disk's surface density is Eqn. (11.34), which may be simplified further via [1] because the Bessel function there becomes $I_{1/4}(2x/\tau) \simeq (2x/\tau)^{1/4}/\Gamma(5/4)$ when its argument $2x/\tau \ll 1$ is small. So at late times the disk's surface density becomes

$$\sigma(x, \tau) \simeq \frac{M_d}{\pi R^2} \frac{e^{-(x^2+1)/\tau}}{\Gamma(5/4)\tau^{5/4}} \tag{11.40}$$

where the gamma function evaluates to $\Gamma(5/4) \simeq 0.9064$.

Equation (11.21) provides the disk's radial velocity, which for a Keplerian disk having $\Omega' = -3\Omega/2r$ becomes

$$v_r = -\frac{3\nu}{2r}\left[1 + \frac{\left(\frac{r^2\Omega}{\sigma}\right)\left(\frac{\partial\sigma}{\partial r}\right)}{\left(\frac{\partial(r^2\Omega)}{\partial r}\right)}\right] = -\frac{3\nu}{2r}\left(1 + \frac{2r}{\sigma}\frac{\partial\sigma}{\partial r}\right) \tag{11.41}$$

since $\partial(r^2\Omega)/\partial r = r\Omega/2$. According to Eqn. (11.40), the derivative $\partial\sigma/\partial r = (\partial\sigma/\partial x)/R = -2\sigma x/\tau R$, so the disk's radial velocity becomes

$$v_r(r,t) = -\frac{3\nu}{2r}\left(1 - \frac{4x^2}{\tau}\right) = -\frac{3\nu}{2r}\left[1 - \left(\frac{r}{r_s}\right)^2\right] \tag{11.42}$$

where

$$r_s(t) = \sqrt{\tau R^2/4} = \sqrt{3\nu t} \tag{11.43}$$

is known as the *stagnation point*, which is the site in the disk where the fluid velocity is zero. Note that the fluid interior to the disk's stagnation point has $v_r < 0$ while $v_r > 0$ at $r > r_s$, so the disk flows away from its stagnation point.

The trajectory $r(t)$ for a fluid element in the disk is obtained from the differential equation $dr/dt = v_r = -3\nu/2r + r/2t$ (but see also problem 10.1). To solve, move the rightmost term to the left hand side and multiply by $2r/t$ to obtain $t^{-1}(dr^2/dt) - r^2/t^2 = d(r^2/t)/dt = -3\nu/t$, which is easily integrated and yields

$$r^2(t) = r_0^2(t/t_0) - 3\nu t\ln(t/t_0) = r_{s0}^2\left(\frac{t}{t_0}\right)\left[\left(\frac{r_0}{r_{s0}}\right)^2 - \ln\left(\frac{t}{t_0}\right)\right] \tag{11.44}$$

where the initial condition $r_0 = r(t_0)$ is the fluid parcel's distance from the central star at the initial time t_0 and r_{s0} is the initial location of the disk's stagnation point at time t_0. These trajectories are plotted in Fig. 11.3. The grey curve there shows the location of the disk's stagnation point $r_s(t) = r_{s0}\sqrt{t/t_0}$ that steadily advances outwards across the disk. As Fig. 11.3 shows, the stagnation point eventually sweeps sweeps across the entire disk. This reverses the motion of all the outward-evolving disk parcels that started at $r_0 > r_{s0}$, causing all portions of the disk to then evolve onto the central star after time $t_\star = t_0 e^{(r_0/r_{s0})^2}$. But do keep in mind that not all of the disk matter will be accreted. This is due to the disk's conservation of angular momentum, Eqn. (11.39), which implies that a tiny sliver of disk matter must nonetheless evolve out to great distances, carrying with it all of the disk's angular momentum L.

11.2.8 steady disk evolution

Now consider a broad viscous disk that is in steady state in the region interior to its stagnation point at $r \ll r_s$, which means that disk's surface density in this region does not change over time. The flow is inwards, so the rate that mass traverses radius r in the disk is

$$\dot{M}_d = -2\pi\sigma r v_r, \tag{11.45}$$

which is the disk's accretion rate. Note that $v_r = -3\nu/2r$ in this region (Eqn. 11.42), so the accretion rate $\dot{M}_d$ is also a constant.

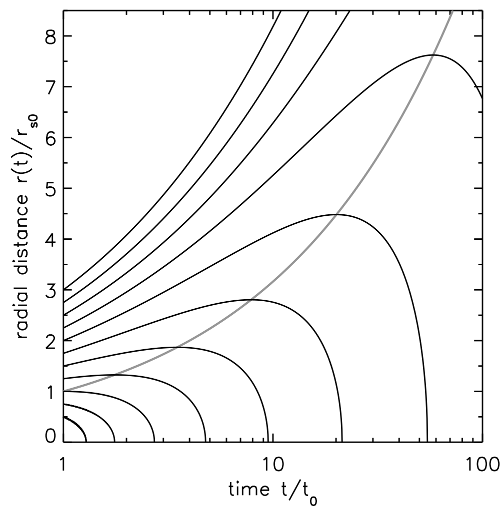

Figure 11.3 Black curves show trajectories $r(t)$ for various fluid parcels orbiting in a viscous disk, Eqn. 11.44. Curves representing different disk parcels that start at various initial radii r_0, with $0.5 < r_0/r_{s0} < 3$ where r_{s0} is the radius of the disk's stagnation point at the initial time t_0. The grey curve shows how the disk's stagnation point $r_s(t) = r_{s0}\sqrt{t/t_0}$ advances across the disk over time, which causes the initially outward-evolving trajectories to reverse their motion and evolve inwards.

The continuity equation for the surface density of the disk's angular momentum $\ell = \sigma r^2\Omega$ is Eqn. (11.20). The first term there is zero when the disk is steady, so that equation is now easily integrated which yields $\sigma r^3\Omega v_r = \nu\sigma r^3\Omega' + c/2\pi$ where the integration constant $c(r) = -2\pi\nu\sigma r^3\Omega' + 2\pi\sigma r^3\Omega v_r$ is a constant function of distance r. Note that the first term is the disk's viscous angular momentum luminosity $\mathcal{L}_z$, Eqn. (11.16), while the second term becomes $-r^2\Omega\dot{M}_d$, which is the rate of angular momentum transport due to the disk's radial flow. So the integration constant $c(r)$ is

$$c(r) = \mathcal{L}_z(r) - r^2\Omega\dot{M}_d = 3\pi\nu\sigma r^2\Omega - r^2\Omega\dot{M}_d, \tag{11.46}$$

which is the net rate that viscosity plus radial motion transmits angular momentum through the disk. Now evaluate $c(r)$ at the disk's inner edge which lies at some small radial distance $r_\star$. Recall the zero-torque boundary condition, which says that no torque is exerted on the disk's inner edge, so $\mathcal{L}_z(r_\star) = 0$ there and thus $c = -\dot{M}_d r_\star^2\Omega_\star$ where $\Omega_\star$ is the angular velocity at $r_\star$.

Inserting c into Eqn. (11.46) then yields a simple expression that relates the disk's surface density σ to its viscosity ν and its accretion rate $\dot{M}_d$:

$$\sigma(r) = \frac{\dot{M}_d}{3\pi\nu}\left(1 - \frac{r_\star^2\Omega_\star}{r^2\Omega}\right) = \frac{\dot{M}_d}{3\pi\nu}\left(1 - \sqrt{\frac{r_\star}{r}}\right). \tag{11.47}$$

The radius $r_\star$ will be the site where the disk's angular velocity $\Omega(r)$ must rapidly transition from Keplerian rotation having $\Omega \propto r^{-3/2}$ to the central star's angular velocity, which is a constant. This transition region is known as the *boundary layer*, and it is the narrow zone between the stellar surface and the disk's inner edge where $\Omega'(r_\star)$ and $\mathcal{L}_z(r_\star)$ are zero.

Far from the boundary layer at $r \gg r_\star$ the disk's surface density is $\sigma \simeq \dot{M}_d/3\pi\nu$ and its radial velocity is $v_r \simeq -3\nu/2r$ at $r \ll r_s$. The timescale for a steady disk's viscous evolution is then

$$t_\nu \simeq \frac{r}{v_r} \simeq \frac{2r^2}{3\nu}, \tag{11.48}$$

which is the timescale for viscosity to slowly drain a steady disk of its mass by dumping it onto the central star.

11.2.9 α viscosity

The above shows that the pace of the disk's evolution is governed by the disk's viscosity ν. Unfortunately, the precise nature of the physical process that generates a disk's viscous friction is not known. For instance, it is possible that a circumstellar disk is convecting in the vertical direction, and that turbulence generated by those vertical motions is responsible for the disk's viscosity. Alternatively it is argued that a circumstellar disk is susceptible to the magnetorotational instability (MRI), wherein the magnetic field in a partly ionized disk is magnified by an instability that then drives turbulent motions within the disk. So the suspicion is that some kind of turbulence is responsible for the disk's viscosity, though the details remain unclear.

Inspection of Eqn. (11.48) shows that the viscosity is the product of a velocity and a scale length. Since turbulence appears to be the generator of disk viscosity, then the disk's sound speed c is likely the maximum speed at which this process operates. Similarly, the largest length scale over which turbulence can operate in the disk is simply the disk's shortest dimension, which is the disk's vertical scale height h. Noting that c and h are related via $c = h\Omega$, this then leads to the α viscosity law

$$\nu = \alpha c h = \alpha h^2 \Omega, \tag{11.49}$$

where all the unknown physical details are hidden within the dimensionless parameter $\alpha < 1$. This is also known as the Shakura and Sunyaev viscosity law, who first applied this formula to accretion disks around black holes. Inserting this into Eqn. (11.48) then yields $\alpha \sim r^2/h^2 t_\nu \Omega$. A typical circumstellar disk orbiting a solar mass star has a radius $r \sim 50$ AU, a fractional scale height $h/r \sim 0.1$, and a lifetime of $t_\nu \sim 10^6$ yrs, which suggests a typical value of $\alpha \sim 5 \times 10^{-3}$.

11.3 ENERGY TRANSPORT IN A STEADY DISK

The disk's inward flow implies a loss of orbital energy due to friction within the disk, and this dissipation of energy also heats the disk. To calculate the rate of energy dissipation within the disk, consider the fluid's density of kinetic energy, $e = \rho v^2/2$. This quantity evolves at the rate $\partial e/\partial t = \rho \mathbf{v} \cdot (\partial \mathbf{v}/\partial t)$. The Navier-Stokes equation (11.4) provides the derivative

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla\Phi - \frac{\nabla p}{\rho} + \frac{1}{\rho} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial \sigma'_{ij}}{\partial x_j} \hat{\mathbf{x}}_i, \tag{11.50}$$

noting that the viscous acceleration (the right term in Eqn. 11.1) is instead written in terms of Cartesian coordinates. The following will consider a steady disk, so the disk's density ρ

is constant over time and the disk behaves as if incompressible such that $\nabla\cdot\mathbf{v} = 0$. In this case the fluid's energy evolves at the rate

$$\frac{\partial e}{\partial t} = -\rho\mathbf{v}\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}] - \mathbf{v}\cdot\nabla(\rho\Phi + p) + \sum_{i,j}\left[\frac{\partial(v_i\sigma'_{ij})}{\partial x_j} - \sigma'_{ij}\frac{\partial v_i}{\partial x_j}\right] \tag{11.51}$$

where the $v_i(\partial\sigma'_{ij}/\partial x_j)$ was replaced with $\partial(v_i\sigma'_{ij})/\partial x_j - \sigma'_{ij}(\partial v_i/\partial x_j)$. Setting $\alpha_j = \sum_i v_i\sigma'_{ij}$ also allows the first double sum in Eqn. (11.51) to be written as $\sum_j \partial\alpha_j/\partial x_j = \nabla\cdot\boldsymbol{\alpha}$ where the vector $\boldsymbol{\alpha}$ has Cartesian components $\alpha_j = \sum_i v_i\sigma'_{ij}$. Problem 11.11 then uses a number of vector identities to show that $\partial e/\partial t$ has the form

$$\frac{\partial e}{\partial t} = -\nabla\cdot\mathbf{F}_e + \delta \tag{11.52}$$

where

$$\mathbf{F}_e = \left(\frac{1}{2}\rho v^2 + \rho\Phi + p\right)\mathbf{v} - \boldsymbol{\alpha} \tag{11.53}$$

is the fluid's energy flux density and

$$\delta = -\sum_{i,j}\sigma'_{ij}\frac{\partial v_i}{\partial x_j} = -\frac{1}{2}\sum_{i,j}\sigma'_{ij}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = -\frac{1}{2}\sum_{i,j}\eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 \tag{11.54}$$

is the fluid's dissipation density.

To illustrate the meaning of these quantities, integrate Eqn. (11.52) over some volume V and apply the divergence theorem so that

$$\frac{\partial}{\partial t}\int_V e dV = -\int_S \mathbf{F}_e\cdot\mathbf{da} + \int_V \delta dV. \tag{11.55}$$

This shows that $\mathbf{F}_e\cdot\mathbf{da}$ is the rate that energy is transported across area $\mathbf{da}$ on the surface S that bounds volume V, while δ is the disk's three dimensional dissipation density. To get the two dimensional dissipation that is more appropriate for a thin disk, integrate Eqn. (11.54) through the disk's vertical column, which yields

$$d = \int \delta dz = -\frac{1}{2}\nu\sigma\sum_{i,j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 = -\nu\sigma(r\Omega')^2 \tag{11.56}$$

when Eqns. (11.61) are used to write d in terms of cylindrical coordinates; see problem 11.12. This is the rate per area that a patch in the disk is losing kinetic energy due to viscous friction. Hence $-d$ is the rate that the molecules in the disk are heated as viscous friction converts the disk's ordered orbital energy into a disordered form of energy. Of course, the disk will then radiate that heat into space, which is assessed below.

11.3.1 disk radiation

The viscous dissipation in a Keplerian disk is

$$d = -\frac{9}{4}\nu\sigma\Omega^2 = -\frac{3GM_\star\dot{M}_d}{4\pi r^3}\left(1 - \sqrt{\frac{r_\star}{r}}\right). \tag{11.57}$$

where $M_\star$ is the mass of the central star and $\dot{M}_d$ is the disk's mass accretion rate. This energy is converted into heat in the disk which is then radiated into space. The total luminosity of the disk's radiation is

$$L_d = \int_{r_\star}^{r_{out}} d(r')2\pi r' dr' \simeq \frac{GM_\star \dot{M}_d}{2r_\star} \tag{11.58}$$

where $r_{out} \gg r_\star$ is outer edge of where the disk is still steady. So by measuring a disk's accretion luminosity L_d one can infer the disk's accretion rate $\dot{M}_d$, which is also related to the combination $\sigma\nu$ via Eqn. (11.47).

Consider an annulus in the disk of area δA. That annulus generates heat in the disk at the rate $\delta A|d|$, but both sides of the disk will radiate that heat at the rate $2\delta A\sigma_{sb}T^4$ where σ_{sb} is the Stefanâ^Boltzmann constant, so the temperature $T(r)$ of an annulus of radius r is

$$T^4(r) = \frac{|d|}{2\sigma_{sb}} = -\frac{3GM_\star \dot{M}_d}{8\pi\sigma_{sb}r^3}\left(1 - \sqrt{\frac{r_\star}{r}}\right), \tag{11.59}$$

assuming that the central star does not contribute to any significant heating of the disk.

The monochromatic intensity of the radiation emitted by an annulus in the disk is $\pi B_\lambda(T)$ where $B_\lambda(T)$ is Plank's blackbody law and πB_λ is the rate per area per wavelength interval that the disk emits thermal energy of wavelength λ. So the disk's thermal spectrum, which is rate that the viscous disk radiates energy of wavelength λ, is the disk integral

$$I(\lambda) = \int_{r_\star}^{r_{out}} 2\pi^2 B_\lambda[T(r')]r' dr. \tag{11.60}$$

Figure 11.4 illustrates this concept with the spectrum of the young star GM Aurigae, whose spectral energy distribution shows the classic signature of a circumstellar disk, which is an excess of thermal radiation at infrared and longer wavelengths (which is the white curve in Fig. 11.4) that is not be accounted by the star's own blackbody radiation (black curve).

Problems

11.1 A viscous fluid is flowing in uniform rotation about rotation axis $\boldsymbol{\omega} = \omega\hat{\boldsymbol{z}}$ where ω is the fluid's constant angular rate of rotation, so the uniformly rotating fluid's velocity is $\mathbf{v} = \boldsymbol{\omega} \times \boldsymbol{r}$. Show that this fluid's viscous stress tensor $\sigma'_{ij} = 0$.

11.2 Show that Eqn. (11.4) follows from Eqn. (11.3) when the viscosity coefficients η and ζ are constant.

11.3 Consider the axisymmetric fluid disk of Section 11.2 and show that the disk's velocity gradients are

$$\frac{\partial v_x}{\partial x} = -r\frac{\partial \Omega}{\partial r}\sin\theta\cos\theta = -\frac{\partial v_y}{\partial y} \tag{11.61a}$$

$$\frac{\partial v_x}{\partial y} = -\frac{\partial v_\theta}{\partial r}\sin^2\theta - \Omega\cos^2\theta \tag{11.61b}$$

$$\frac{\partial v_y}{\partial x} = \frac{\partial v_\theta}{\partial r}\cos^2\theta + \Omega\sin^2\theta. \tag{11.61c}$$

Then insert these into Eqn. (11.11) to obtain Eqn. (11.15).

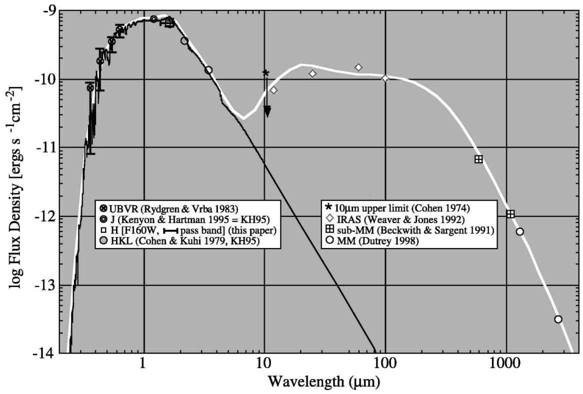

Figure 11.4 The spectral energy distribution (SED) for GM Aurigae, which is a relatively young million year-old star that also has a circumstellar disk. The black curve shows the star's contribution to the system's SED, whose shape at wavelengths $\lambda \lesssim 5\ \mu$m is a blackbody with temperature $T_\star = 3970$ K. The white curve shows a model that is fit to the excess radiation at longer wavelengths, which are the data points in the above. The model's agreement with the data indicates that the circumstellar disk has an inner radius of $r_{in} = 4$ AU and an outer radius $r_{out} = 300$ AU. Additional details are given in Schneider *et. al.* (2003), *Astronomical Journal*, v. 125, p. 1467. Note that the radius of the disk's inner edge is much larger than the stellar radius, which suggests that the disk's central region may have been cleared out by one or more planets; this possibility is explored further in Chapter 12.

11.4 a.) Section 10.2.3 shows that the flux density of a fluid's i^{th} component of linear momentum is $\mathbf{\Pi}_i$. Consider a fluid whose motions are restricted to the horizontal $\hat{\mathbf{x}}$–$\hat{\mathbf{y}}$ plane, and show that the fluid's momentum flux in the radial $\hat{\mathbf{r}}$ direction can be written

$$\mathcal{F}_r = \cos^2\theta\Pi_{11} + \sin\theta\cos\theta(\Pi_{12} + \Pi_{21}) + \sin^2\theta\Pi_{22} \tag{11.62}$$

b.) Keep only the contributions to Eqn. (11.62) that are due to viscosity, and show that fluid's viscous momentum flux in the radial direction is

$$F_r^\nu = -\left(\frac{4}{3}\eta + \zeta\right)\frac{\partial v_r}{\partial r} - \left(\zeta - \frac{2}{3}\eta\right)\left(\frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}\right) \tag{11.63}$$

when written in terms of cylindrical coordinates and velocities. This is the fluid's two dimensional viscous momentum flux density, and it has units of momentum per area per time. To apply this result to a thin fluid layer of surface density σ, integrate vertically across the fluid layer to obtain

$$f_r^\nu = \int F_r^\nu dz = -\sigma\left(\frac{4}{3}\nu_s + \nu_b\right)\frac{\partial v_r}{\partial r} - \sigma\left(\nu_b - \frac{2}{3}\nu_s\right)\left(\frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}\right) \tag{11.64}$$

where $\nu_s = \eta/\rho$ is the fluid's kinematic shear viscosity and $\nu_b = \zeta/\rho$ is the kinematic bulk viscosity.

c.) Consider the viscous transfer of radial momentum into as well as out of a small fluid patch that has area $dA = rd\theta \times dr$. Show that this viscous transfer of momentum results in the radial acceleration

$$a_r^\nu = -\frac{1}{\sigma}\frac{\partial f_r^\nu}{\partial r} \tag{11.65}$$

being exerted on that patch.

11.5 Show that when the disk of Section 11.2 inviscid, its angular momentum flux due to advection, Eqn. (11.11), is simply $\mathbf{F}_z = \ell_z \mathbf{v} + p\,\hat{\boldsymbol{\theta}}$. *Hint:* see the results of problem 1.7, which is conceptually similar.

11.6 Equation (11.34) provides the surface density σ of the viscous disk that is considered in Section 11.2.3, and that formula is plotted in Fig. 11.1. But that equation can be problematic early in the disk's evolution when time τ is small, which makes the argument of the exponential small while the argument of the Bessel function is large. Equation (11.34) is then the product of a very small number and a very large number, and evaluating these on a computer can lead to underflow and overflow errors. To avoid this, use the analysis of Bessel functions in reference [1] to obtain Eqn. (11.35), which avoids this problem and is valid when and where $\tau \ll 2x$.

11.7 Consider the early evolution of an initially narrow ring that is described in Section 11.2.3. Let Δr be the ring's e-fold halfwidth where $\sigma(R + \Delta r, t) = e^{-1}\sigma(R, t)$. From this requirement derive Eqn. (11.37), which is the time for a narrow ring to spread a radial distance Δr due to its viscosity.

11.8 Insert the viscous ring's surface density, Eqn. (11.35), into the general expression for the fluid's radial velocity, Eqn. (11.21). Evaluate v_r at the ring's edge at $r = R \pm \Delta r$, and show that it recovers Eqn. (11.38).

11.9 Insert Eqn. (11.34) into Eqn. (11.39) and evaluate the integral to show that the viscous disk of Section 11.2.3 preserves its total angular momentum over time.

11.10 Show that when a viscous disk is steady, then the tangential acceleration on a fluid element in the disk is simply

$$a_\nu = \frac{1}{2}\Omega v_r \tag{11.66}$$

where v_r is the fluid element's radial velocity.

11.11 Use the vector identities in Appendix A to derive the fluid's energy flux density $\mathbf{F}_e$ and the dissipation density δ, Eqns. (11.53–11.54), from Eqn. (11.51).

11.12 Insert Eqns. (11.61) into (11.56) to show that the disk's dissipation rate per area is $d = -\nu\sigma(r\Omega')^2$.

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