Lecture Notes for ASTR 5622
Astrophysical Dynamics

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Loosely speaking, *dynamics* = the study of the motion of matter due to internal and/or external forces, like gravity, pressure, etc.

Begin by using Newton's Laws of motion to study the two–body problem.

Newton's Laws

Law I: A body remains at rest or in uniform motion unless acted upon by a force, ie., $\mathbf{v} =$ constant provided $\mathbf{F} = 0$.

Law II: A body acted upon by a force moves such that its time rate of change of momentum equals the force, ie., $\dot{\mathbf{p}} = \mathbf{F}$ where $\mathbf{p} = m\dot{\mathbf{r}}$, where m is the particle's mass, $\mathbf{r}$ its position vector, $\dot{\mathbf{r}} = d\mathbf{r}/dt$ its velocity.
In short, $\mathbf{F} = m\ddot{\mathbf{r}}$.

Law III: If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction, ie, $\mathbf{F_{12}} = -\mathbf{F_{21}}$,
where $\mathbf{F_{12}} =$ the force on particle 1 exerted by particle 2.

Newton's laws are valid in an *inertial* reference frame.
An inertial reference frame=reference frame where Newton's laws are obeyed... yes, this is a circular argument...

Note that Law I implies that an inertial reference frame is one that is stationary or moves with velocity $\mathbf{V} =$ constant, possibly zero.

The Two–Body Problem

from Chapter 2 of Murray & Dermott's (M&D) *Solar System Dynamics*:

Lets solve the 2–body problem:

Begin with 2 gravitating masses m_1 and m_2 having position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and let $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 =$ the bodies' *relative* coordinate:

These could be two stars that orbit one another,
or a planet in orbit about one star,
or two unbound bodies that encounter each other:

Recall Newton's law of gravity:

$$|\mathbf{F}_1| = \frac{Gm_1m_2}{r^2} \tag{1.1}$$

which is an *attractive* force, so we can write

$$\mathbf{F}_1 = +\frac{Gm_1m_2}{r^3}\mathbf{r} = m_1\ddot{\mathbf{r}}_1 = \text{ grav' force on } m_1 \text{ due to } m_2 \tag{1.2}$$

$$\mathbf{F}_2 = -\frac{Gm_1m_2}{r^3}\mathbf{r} = m_2\ddot{\mathbf{r}}_2 = \text{ force on } m_2 \text{ due to } m_1 \tag{1.3}$$

$$\begin{aligned}
\text{thus} \quad \ddot{\mathbf{r}} &= \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 = -\frac{G(m_1 + m_2)}{r^3}\mathbf{r} = \text{relative acceleration} && (1.4)\\
\text{or} \quad \ddot{\mathbf{r}} &= -\frac{\mu}{r^3}\mathbf{r} && (1.5)\\
\text{where} \quad \mu &= G(m_1 + m_2) && (1.6)\\
& && (1.7)
\end{aligned}$$

This is the equation for the motion of m_2 (the secondary body) relative to m_1 (the primary body).

Note that writing our equation of motion (EOM) in terms of the relative coordinate $\mathbf{r}$ effectively places our origin on m_1.

Does this choice of a coordinate system mean that our reference frame is inertial?

The gravitational constant G is
$G = 6.67 \times 10^{-11}$ m^3/kg/sec^2 = 6.67×10^{-8} cm^3/gm/sec^2.
Although the text uses MKS units, most astronomers use cgs units...

Integrals of the Motion

The following will derive several integrals (ie, constants) of the motion that will be quite handy:

$$\text{note that} \quad \mathbf{r}\times \text{ Eqn. (1.5)} = \mathbf{r}\times\ddot{\mathbf{r}} = 0 = \frac{d}{dt}(\mathbf{r}\times\dot{\mathbf{r}}) \tag{1.8}$$

$$\text{so} \quad \mathbf{r}\times\dot{\mathbf{r}} = \mathbf{h} = \text{ constant} \tag{1.9}$$

$$\tag{1.10}$$

This is the system's *angular momentum integral* $\mathbf{h} = \mathbf{r}\times\dot{\mathbf{r}}$
which is a constant vector that is perpendicular to both $\mathbf{r}$ and $\dot{\mathbf{r}}$
$\Rightarrow m_1$ and m_2 are restricted to moving in a plane perpendicular to $\mathbf{h}$.

$\mathbf{h}$ has units of ang' mom' per unit mass,
and is sometimes call the *specific angular momentum*

Keep in mind that $\mathbf{h}$ is *not* the system's total angular momentum per unit mass, due to the fact that we used a coordinate system that is not inertial.

The total ang' mom' is calculated in Section 2.7 of the text using center–of–mass (COM) aka barycentric coordinates.

Since the motion is restricted to a plane,
we will proceed using polar coordinates $\mathbf{r}(r, \theta)$:

Your elementary Mechanics class showed that

$$\begin{aligned}
\mathbf{r} &= r\hat{\mathbf{r}} && (1.11)\\
\dot{\mathbf{r}} &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} && (1.12)\\
\ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \left[\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\right]\hat{\theta} && (1.13)
\end{aligned}$$

where $\hat{\mathbf{r}}$, $\hat{\theta}$, and $\hat{\mathbf{z}}$ are the usual unit vectors in cylindrical coordinates.

$$\begin{aligned}
\text{Thus} \quad \mathbf{h} &= \mathbf{r} \times \dot{\mathbf{r}} = r^2\dot{\theta}\hat{\mathbf{r}} \times \hat{\theta} = r^2\dot{\theta}\hat{\mathbf{z}} && (1.14)\\
\text{and} \quad h &= |\mathbf{h}| = r^2\dot{\theta} && (1.15)
\end{aligned}$$

Kepler's 2^{nd}

The above leads to Kepler's 2^{nd} law of planetary motion:
a planet's position vector $\mathbf{r}$ sweeps out equal areas in equal times.

In a short time interval Δt,
the radius vector $\mathbf{r} \rightarrow \mathbf{r} + \boldsymbol{\Delta}\mathbf{r}$ and sweeps out a small area ΔA, where

$$\begin{aligned}
\Delta A &= \frac{1}{2}\text{base} \cdot \text{height} \simeq \frac{1}{2}r^2\Delta\theta && (1.16)\\
\text{Thus} \quad \frac{dA}{dt} &= \left.\frac{\Delta A}{\Delta t}\right|_{\Delta t \rightarrow 0} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}h = \text{ a constant} && (1.17)
\end{aligned}$$

Thus $\mathbf{r}$ has a constant *areal velocity*, ie,
$\mathbf{r}$ sweeps out equal areas in equal times.

Solve for the orbit

$$\text{Recall that} \quad \ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \quad \text{(see Eqn. 1.5)} \tag{1.18}$$

$$\text{whose radial part is} \quad \ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \quad \text{(Eqn. 1.13)} \tag{1.19}$$

An easy solution is obtained by first replacing r with the variable $u = 1/r$, and assume that $u = u(\theta)$ while $\theta = \theta(t)$:

$$\text{since} \quad r = u^{-1} \tag{1.20}$$

$$\dot{r} = -u^{-2}\frac{du}{d\theta}\frac{d\theta}{dt} = -r^2\frac{du}{d\theta}\frac{h}{r^2} \quad \text{since } \dot{\theta} = h/r^2 \tag{1.21}$$

$$\text{so} \quad \dot{r} = -h\frac{du}{d\theta} \tag{1.22}$$

$$\text{and} \quad \ddot{r} = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -h^2u^2\frac{d^2u}{d\theta^2} \tag{1.23}$$

So the equation of motion (1.19) becomes

$$-h^2u^2\frac{d^2u}{d\theta^2} - u^{-1}h^2u^4 = -\mu u^2 \tag{1.24}$$

$$\text{or} \quad \frac{d^2u}{d\theta^2} = -u + \frac{\mu}{h^2} \tag{1.25}$$

This familiar EOM has the same form as that of a mass dangling from a spring and subject to gravity:

$$m\ddot{x} = -kx + mg \quad \text{where } x = \text{displacement, } k = \text{spring const'} \tag{1.26}$$

$$\text{or} \quad \ddot{x} = -\omega^2x + g \quad \text{where } \omega = \sqrt{k/m} \tag{1.27}$$

$$\text{with sol'n} \quad x(t) = A\cos(\omega t - \delta) + x_0 \tag{1.28}$$

where $x_0 = mg/k$, and δ = phase constant.

Thus Eqn. (1.25) has solution

$$u(\theta) = A\cos(\theta - \tilde{\omega}) + B \tag{1.29}$$

where $A, B, \tilde{\omega}$ are constants determined by initial conditions.

Since

$$\frac{d^2u}{d\theta^2} = -A\cos(\theta - \tilde{\omega}) = -A\cos(\theta - \tilde{\omega}) - B + \frac{\mu}{h^2} \tag{1.30}$$

$$\Rightarrow B = \frac{\mu}{h^2} \tag{1.31}$$

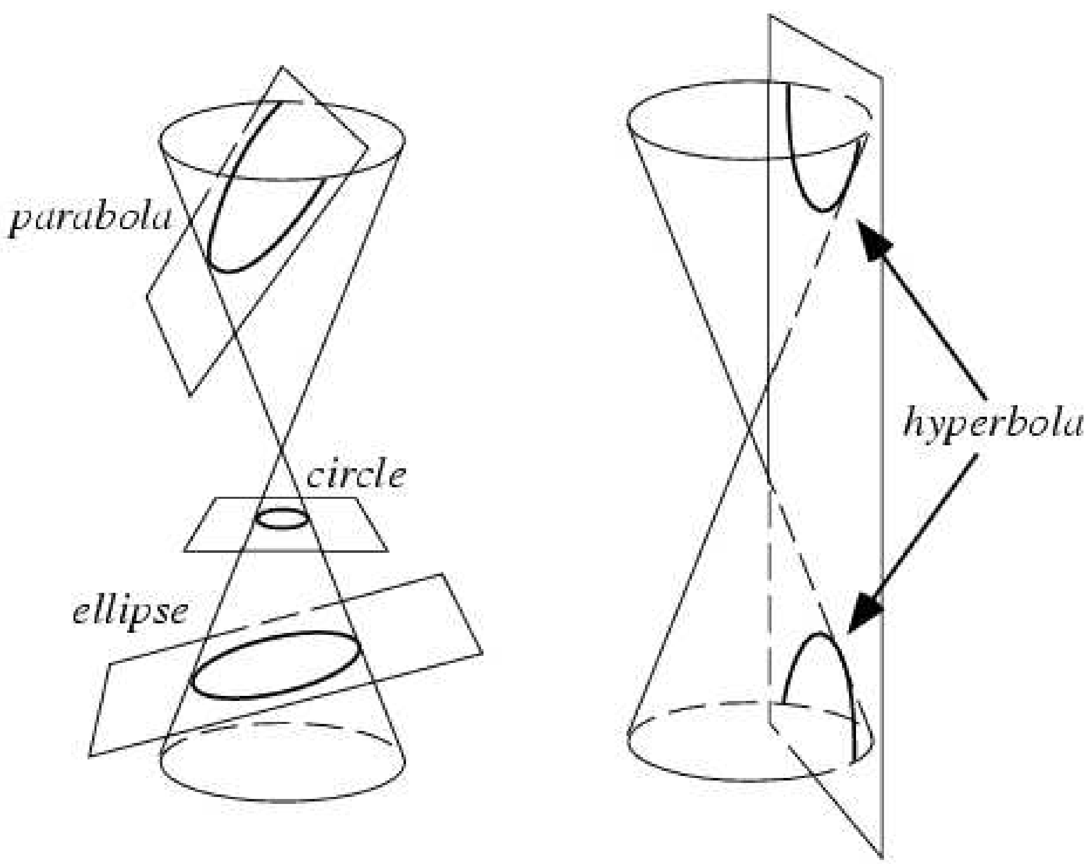
Also, set $A = eB$ where e is another constant. Then

$$u(\theta) = B[e\cos(\theta - \tilde{\omega}) + 1] = \frac{\mu}{h^2}[1 + e\cos(\theta - \tilde{\omega})] \tag{1.32}$$

$$\text{so} \quad r(\theta) = u^{-1} = \frac{h^2/\mu}{1 + e\cos(\theta - \tilde{\omega})} \tag{1.33}$$

$$\equiv \frac{p}{1 + e\cos(\theta - \tilde{\omega})} \tag{1.34}$$

This is the equation for a *conic section*=intersection of a plane & cone, where the constant $p = h^2/\mu$ is known as the semilatus rectum.



The type of conic section depends upon the orbit's eccentricity e:

$$\begin{array}{llll} \text{circle} & e = 0 & p = a & E < 0 \\ \text{ellipse} & 0 < e < 0 & p = a(1-e^2) & E < 0 \\ \text{parabola} & e = 1 & p = 2q & E = 0 \\ \text{hyperbola} & e > 0 & p = a(1-e^2) & E > 0 \end{array} \tag{1.35}$$

where the constant a is the orbit's *semimajor axis*,
while $b = a\sqrt{1-e^2}$ = the *semiminor axis*.

The constants $a, e, \tilde{\omega}$ are known as *orbit elements*.

Later we will show that orbits with eccentricities $e < 1$ are *bound*,
ie, they have energies $E < 0$,
while a parabolic orbit with $e = 1$ is marginally bound with $E = 0$,
and a hyperbolic orbit with $e > 1$ is unbound with $E > 0$.

The bound, elliptic orbit is most relevant to planetary problems:

$$r(f) = \frac{a(1-e^2)}{1 + e\cos f} \tag{1.36}$$

$$\text{where} \quad f = \theta - \tilde{\omega} = m_2\text{'s } \textit{true anomaly} \tag{1.37}$$

$$\theta = \text{its } \textit{longitude} \tag{1.38}$$

$$\tilde{\omega} = \text{its } \textit{longitude of periapse} \tag{1.39}$$

m_2 is closest to m_1 at *periapse*, when $f = 0$ and $r(0) \equiv q = a(1-e)$,
and furthest at *apoapse* when $f = \pi$ and $r(\pi) \equiv Q = a(1+e)$.

Note that the primary m_1 is not at the center of the ellipse; rather it lies at a *focus*.

We have also recovered Kepler's 1^{st} Law of planetary motion: a planet moves along an ellipse with the Sun at one focus.

Since $h = \sqrt{\mu p}$, this also means that the system's angular momentum (actually, its angular momentum integral) depends on e, too. For an elliptical orbit, $h = \sqrt{\mu a(1 - e^2)}$.

Kepler's 3^{rd} Law: $T^2 \propto a^3$

Recall that

$$\frac{dA}{dt} = \frac{1}{2}h = \quad \mathbf{r}\text{'s areal velocity, Eqn. (1.17)} \tag{1.40}$$

If $T =$ the planet's orbital period, then

$$\begin{aligned}
\text{total orbit area} &= \int_0^T \frac{dA}{dt}dt = A = \int_0^T \frac{1}{2}hdt = \frac{1}{2}hT && (1.41)\\
\Rightarrow T &= \frac{2A}{h} && (1.42)\\
\text{where} \quad A &= \pi ab = \text{ area of ellipse} && (1.43)\\
&= \pi a^2\sqrt{1 - e^2} && (1.44)\\
\text{and} \quad T &= \frac{2\pi a^2\sqrt{1 - e^2}}{\sqrt{\mu a(1 - e^2)}} && (1.45)\\
&= 2\pi\sqrt{\frac{a^3}{\mu}} = 2\pi\sqrt{\frac{a^3}{G(m_1 + m_2)}} && (1.46)
\end{aligned}$$

which confirm's Kepler's 3^{rd}.

Energy Integral E

$$
\begin{aligned}
\text{Recall that} \quad \ddot{\mathbf{r}} &= -\frac{\mu}{r^3}\mathbf{r} \quad \text{(Eqn. 1.5)} && (1.47)\\
\text{so} \quad \dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} + \frac{\mu}{r^3}\dot{\mathbf{r}}\cdot\mathbf{r} &= 0 && (1.48)\\
\text{since} \quad \mathbf{r} &= r\hat{\mathbf{r}} \quad \text{and} \quad \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} && (1.49)\\
\dot{\mathbf{r}}\cdot\mathbf{r} &= \dot{r}r && (1.50)\\
\text{and} \quad \dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} + \frac{\mu\dot{r}}{r^2} &= 0 && (1.51)\\
\text{now note that} \quad v^2 &= \dot{\mathbf{r}}\cdot\dot{\mathbf{r}} && (1.52)\\
\text{so} \quad \frac{dv^2}{dt} &= 2\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} && (1.53)\\
\text{or} \quad \dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} &= \frac{1}{2}\frac{dv^2}{dt} && (1.54)\\
\text{Next note that} \quad \frac{d}{dt}\left(\frac{1}{r}\right) &= -\frac{\dot{r}}{r^2} && (1.55)\\
\text{so} \quad \dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} + \frac{\mu\dot{r}}{r^2} &= \frac{d}{dt}\left(\frac{1}{2}v^2 - \frac{\mu}{r}\right) = 0 && (1.56)\\
\text{ie} \quad E &= \frac{1}{2}v^2 - \frac{\mu}{r} = \text{constant energy integral} && (1.57)\\
&= \text{kinetic + potential E per mass} && (1.58)
\end{aligned}
$$

Again, this is not the system's *total* energy since our coordinate system is not inertial (ie, the relative velocity v would need to be replaced with COM velocities $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$).

Since E is constant, we can evaluate it at any site in m_2's orbit about m_1. Lets evaluate E when m_2 is at periapse, when it is closest to m_1.
Then $r = q = a(1-e)$, $\dot{r} = 0$ so $v = r\dot{\theta}$ (why?)
where $\dot{\theta} = h/r^2$, so $v = h/r = h/a(1-e)$ where $h = \sqrt{\mu a(1-e^2)}$ and

$$E = \frac{\mu a(1-e^2)}{2a^2(1-e)^2} - \frac{\mu}{a(1-e)} = \frac{\mu}{2a(1-e)}\left[\frac{(1+e)(1-e)}{1-e} - 2\right] \quad (1.59)$$

$$= \frac{\mu}{2a(1-e)}(-1+e) \quad (1.60)$$

$$= -\frac{\mu}{2a} = -\frac{G(m_1+m_2)}{2a} \quad (1.61)$$

which of course is a constant since our 2-body system is conservative.

The other important integral is the angular momentum integral h

$$h = \sqrt{\mu a(1-e^2)} = \sqrt{G(m_1+m_2)a(1-e^2)} \quad (1.62)$$

Section 2.7 of the text also computes the system's *total* energy and angular momentum in *barycentric* (ie, COM) coordinates:

$$\text{Let} \quad \mu^\star = \frac{m_1 m_2}{m_1+m_2} = \text{system's } \textit{reduced mass} \quad (1.63)$$

$$\text{Sec' 2.7 shows that} \quad E^\star = \mu^\star E = -\frac{G m_1 m_2}{2a} = \text{total energy} \quad (1.64)$$

$$\text{and} \quad L^\star = \mu^\star h = \frac{m_1 m_2}{m_1+m_2} h = \text{total ang' mom'} \quad (1.65)$$

$$(1.66)$$

please don't confuse $\mu^\star$ and μ...

The Orbit in Space

In general, m_2's orbit plane will differ from
your preferred $x - y$ "reference plane".

If you were studying the motion of, say, a comet, then your reference plane would likely be the ecliptic (the plane of Earth's orbit). If studying the motion of a star, then your $x - y$ plane might instead be the Galactic plane.

Three additional angles are needed to describe the orientation of m_2's orbit:
inclination i, the tilt of orbit plane relative to the reference plane
longitude of ascending node Ω
argument of periapse ω

The set a, e, i, ω, Ω the shape and orientation of m_2's orbit, and f gives its angular location in that orbit.

The Orbit in Time

We still need to specify where m_2 is as a function of time, ie, we need to solve for m_2's position r and true anomaly f as functions of time t.

This is done in Section 2.4 of M&D; the derivation is straightforward calculus and geometry, but laborious, and is not repeated here.

Section 2.4 shows that r and f can be parameterized as functions of the *eccentric anomaly* E_c:

$$r(E_c) = a(1 - e\cos E_c) \qquad (1.67)$$

$$M \equiv n(t - \tau) = E_c - e\sin E_c \quad \text{(known as Kepler's equation)} \qquad (1.68)$$

where M = *mean anomaly,*
$n = \sqrt{\mu/a^3} = \sqrt{G(m_1 + m_2)/a^3}$ = the mean motion
(ie, m_2's mean angular velocity about m_1),
and τ = time of periapse passage.

It is recommended that you work through Section 2.4
and confirm the above equations.

Suppose you wish to know what r and f are for m_2 at some time t:

1. calculate n and M (Note: you can set your clock so that $\tau = 0$).

2. solve Kepler's equation *numerically* for E_c, then get $r(E_c)$.

3. solve the ellipse equation $r = a(1 - e^2)/(1 + e\cos f)$ for f,
taking care to get the sign of f correct (or use Eqn' 2.46 of M&D) .

This might seem laborious,
but the above steps are easily automated on a computer.

Elliptic expansions of the orbit

Kepler's eqn. (KE) relates m_2's position↔time via

$$M = n(t-\tau) = E - e\sin E \tag{1.69}$$
$$r = a(1 - e\cos E). \tag{1.70}$$

However the relationship between t and $r(f)$ is difficult to use in analytic studies, principally because KE is a transcendental function of E.

But useful analytic formulas describing m_2's motion are possible when the orbit is nearly circular, ie, in the limit that $e \ll 1$.

This approximation is often accurate enough when describing the motion of most of the planets, satellites, and the orbits of dust grains, planetary ring particles, as well as for binary stars whose orbits have been tidally circularized.

The following approximation will be marginally true for asteroids having $e \sim 0.1$.

However it is not useful for cometary orbits, and stars in clusters or galaxies, which often have substantially larger e's.

Consider an orbit with $e \ll 1$, and derive expressions for the true anomaly f and the radial coordinate r/a as power series in e.

The following derives those expressions to an accuracy of $\mathcal{O}(e)$.
In your homework, you will rederive these expressions to accuracy $\mathcal{O}(e^2)$.

The following trig identities will be useful:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \tag{1.71}$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \tag{1.72}$$

$$\sin^2 A = \frac{1}{2}(1-\cos 2A) \tag{1.73}$$

$$\cos^2 A = \frac{1}{2}(1+\cos 2A) \tag{1.74}$$

Also useful is the binomial expansion:

$$(1+x)^n = 1 + nx + \frac{1}{2!}n(n-1)x^2 + \frac{1}{3!}n(n-1)(n-2)x^3 + \cdots \tag{1.75}$$

and the Taylor series expansions

$$\sin x \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7) \tag{1.76}$$

$$\cos x \simeq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \mathcal{O}(x^6) \tag{1.77}$$

Start by inserting

$$E = M + e\sin E \tag{1.78}$$

$$\text{into} \quad \frac{r}{a} = 1 - e\cos E = 1 - e\cos(M + e\sin E) \tag{1.79}$$

$$\text{thus} \quad \frac{r(t)}{a} = 1 - e\cos M\cos(e\sin E) + e\sin M\sin(e\sin E) \tag{1.80}$$

$$\simeq 1 - e\cos(M) + \mathcal{O}(e^2) \tag{1.81}$$

since $M(t) = nt$ and $e \ll 1$.

The series expansion for $f(t)$ is obtained from

$$h = r^2\dot{\theta} = r^2\frac{df}{dt} = na^2\sqrt{1-e^2} \tag{1.82}$$

$$\text{so} \quad df = \frac{a^2}{r^2}\sqrt{1-e^2}ndt \tag{1.83}$$

$$= \sqrt{1-e^2}\left[1 - e\cos M + \mathcal{O}(e^2)\right]^{-2} dM \tag{1.84}$$

Now use the binomial expansion:

$$\sqrt{1-e^2} \simeq 1 - \frac{1}{2}e^2 \tag{1.85}$$

$$\text{and} \quad (1 - e\cos M)^{-2} \simeq 1 + 2e\cos M + \mathcal{O}(e^2) \tag{1.86}$$

$$\text{so} \quad df \simeq (1 + 2e\cos M)dM \tag{1.87}$$

$$\text{and} \quad f(t) \simeq M + 2e\sin M + \mathcal{O}(e^2) \tag{1.88}$$

when integrated.

Higher–order expressions for r/a and f are also given in Section 2.5 of M&D.

Assignment #1
due Thursday January 19
at the start of class

1. Repeat the above analysis to an accuracy of $\mathcal{O}(e^2)$ to show that

$$\frac{r}{a} \simeq 1 - e\cos(M) + \frac{1}{2}e^2(1-\cos 2M) + \mathcal{O}(e^3) \tag{1.89}$$

$$f \simeq M + 2e\sin M + \frac{5}{4}e^2\sin 2M + \mathcal{O}(e^3) \tag{1.90}$$

Of course, Section 2.5 of M&D derive the above by invoking Bessel functions. Your solution SHOULD NOT parrot M&D; use my method outlined above, which is conceptually much easier...

2. Using these same methods, show that

$$\cos E_c \simeq \cos M + \frac{1}{2}e(\cos 2M - 1) + \mathcal{O}(e^2) \tag{1.91}$$

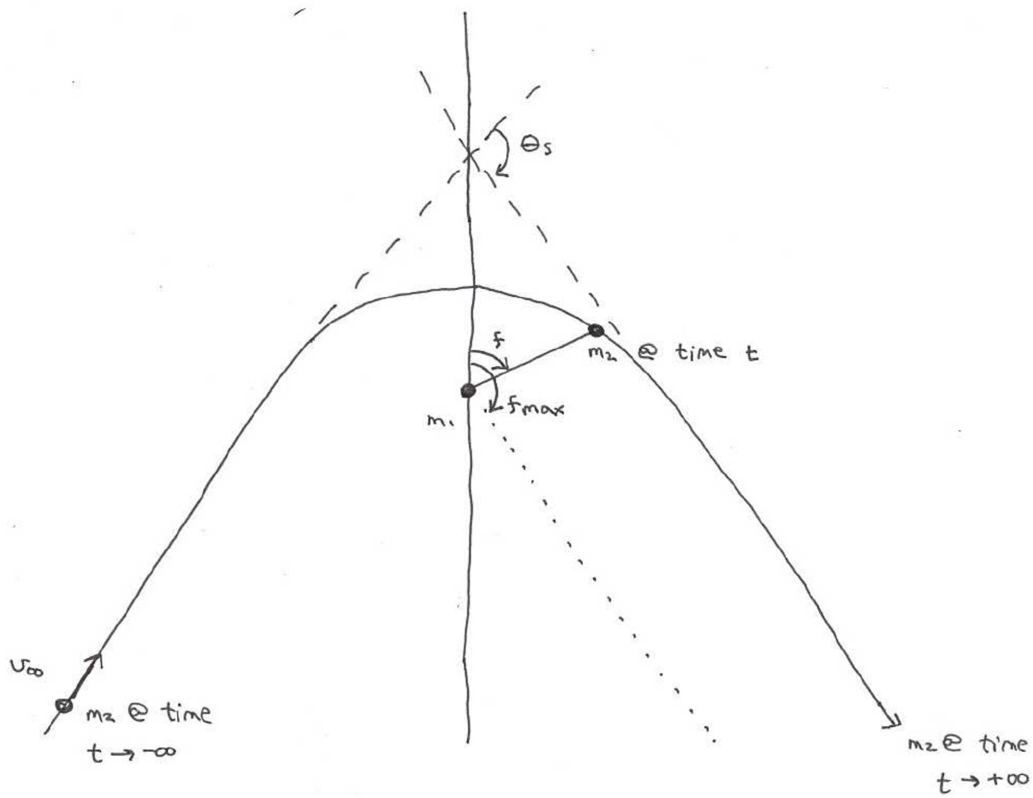
$$\sin f \simeq \sin M + e\sin 2M + \mathcal{O}(e^2) \tag{1.92}$$

(We will use the results of problems 1 & 2 later when we examine the effects of a planet's perturbations upon, say, an asteroid in a low–e orbit, or the motion of one satellite as it is perturbed by another.)

3. Consider two small satellites in close orbits with semimajor axes a and $a+\Delta a$, where $|\Delta a| \ll a$. Show that their synodic period, which is the time between successive encounters, is $T_{syn} \simeq 2Ta/3|\Delta a|$, where T is the orbit period of one of the satellites. We will use this result in our study of planetary rings.

more on next page ⇒

4. Consider two unbound stars m_1 and m_2 that encounter and gravitationally scatter each other; their motion is hyperbolic. Long before the encounter at time $t = -\infty$, the star's separation r is infinite, and star m_2 (the scattered star) has an initial speed v_∞ relative to m_1 (the scattering star) and impact paramete



a.) show that m_2's orbit has a semimajor axes a, specific angular momentum h, and eccentricity e that obey

$$a = -\frac{\mu}{v_\infty^2} \quad \text{where } \mu = G(m_1 + m_2) \tag{1.93}$$

$$h = bv_\infty \tag{1.94}$$

$$e^2 = 1 + \frac{b^2 v_\infty^4}{\mu^2} \tag{1.95}$$

b.) Let $f_{max} = m_2$'s maximum true anomaly, which is achieved at times $t = \pm\infty$. Also let $\theta_s =$ angle through which m_2 is scattered. Show that

$$\cos f_{max} = -\frac{1}{e} \tag{1.96}$$

$$\theta_s = 2\cos^{-1}\left(-\frac{1}{e}\right) - \pi. \tag{1.97}$$

We will use these results later when we discuss dynamical friction in star clusters.

The epicyclic approximation

Assume m_2 is in a nearly circular orbit, so

$$r(t) \simeq a - ae\cos nt \equiv a + x(t) \quad \text{where } x(t) = -ae\cos nt \tag{1.98}$$

$$f(t) \simeq nt + 2e\sin nt \equiv nt + \frac{y(t)}{a} \quad \text{where } y(t) = 2ae\sin nt \tag{1.99}$$

The text calls this the guiding center approximation,
but is also known as *epicyclic motion*.

The guiding center is the point G which travels about m_1
on a circular orbit of radius a with a constant angular rate n.

Meanwhile, m_2 is at the point x, y, which revolves around point G
in the opposite sense with a radial amplitude ae
(which is known as m_2's *epicyclic amplitude*),
and a tangential amplitude $2ae$.

Note that these formulas are only appropriate for the nearly Keplerian problem, namely, for the motion of m_2 as it orbits in a gravitational potential that varies as r^{-1}.

When we consider the motion of stars as they orbit in a galaxy's non–Keplerian potential, we will derive modified expressions for x and y.