Dec 2
2013

Tidal Evolution

Adopted from Chapter 4 of Murray & Dermott's Solar System Dynamics. (This lecture will eventually be Chap 9 in textbook, once written...)

Thus far we've treated all bodies as if they were point masses.

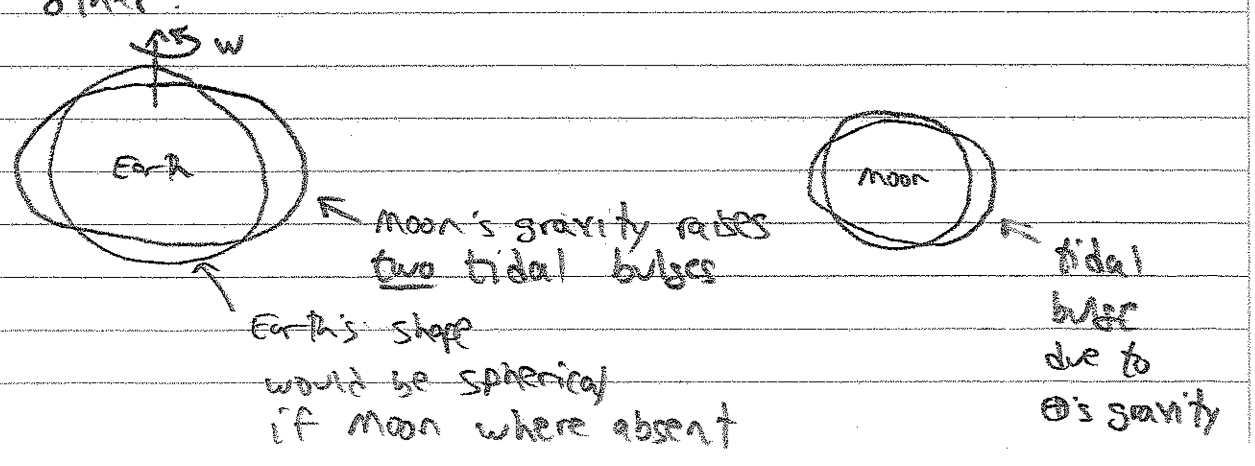
However real gravitating bodies have finite sizes, and when in close proximity, their r^{-2} gravitational results in differential (ie tidal) forces being exerted on each other

→ these tidal forces can alter the bodies shape, and cause their orbits and rotation rates to evolve over time. Examples: Earth's moon, Phobos & Deimos, all regular sattellites, Pluto-Charon, many asteroid sats, probably many hot Jupiters

The following will calculate the rate of orbit evolution due to tides.

Tidal bulges

we will soon see that
A planet (the primary) orbited by satellite (the secondary) will raise tidal bulges on each other:



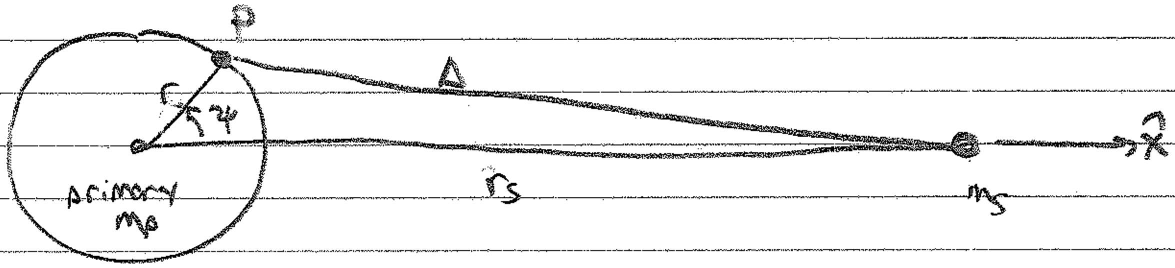
These tidal bulges are pointed to & away from the satellite

Earth is rotating, so someone watching the coast 24 hours would see two lunar high tides and two lunar low tides sweep by daily.

Plus another pair of high & low tides due to Sun which can also be regarded as a satellite of Earth $\oplus$.

Assess secondary's (m_s) tidal pull on primary M_p

Treat m_s as point mass, M_p as extended:



$\Phi(P) = -\frac{Gm_s}{\Delta}$ = gravitational potential due to m_s at point P on surface

$\vec{a} = -\nabla\Phi$ = acceleration at P

$\Delta^2 = r_s^2 + r^2 - 2rr_s\cos\psi$

r_s = m_s's distance from M_p

r would be M_p's radius if it were truly spherical (but it's not)

Note that if m_s orbited in primary's equatorial plane (not necessary tho) then ψ would be P's latitude on m_p; this is typical of regular satellites of giant planets.

so $\Phi(P) = -Gm_s\left(r_s^2 + r^2 - 2rr_s\cos\psi\right)^{-1/2}$

$$= \frac{-Gm_s}{r_s}\underbrace{\left(1 - 2\ell\cos\psi + \ell^2\right)^{-1/2}}_{\frac{r}{\Delta}}$$

where $\ell = \frac{r}{r_s}$

Many satellites orbit more that a few planetary radii away so $r \ll r_s$ and $\ell \ll 1$

so use binomial theorem:

$$(1+x)^n = 1 + nx + \tfrac{1}{2}n(n-1)x^2 + \cdots$$

$$\frac{r}{\Delta} = 1 + \ell\cos\psi - \frac{1}{2}\ell^2 + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-2\ell\cos\psi\right)^2 + O(\ell^3)$$

$$= 1 + \ell\cos\psi + \ell^2\left(-\frac{1}{2} + \frac{3}{2}\cos^2\psi\right)$$

so $$\frac{r}{\Delta} = 1 + \ell\cos\psi + \ell^2\,\frac{1}{2}\left(3\cos^2\psi - 1\right) + O(\ell^3)$$

compare to Legendre Polynomials, (A.26):

$$P_0(x = \cos\psi) = 1$$

$$P_1(x = \cos\psi) = x = \cos\psi$$

$$P_2(x = \cos\psi) = \tfrac{1}{2}(3x^2 - 1) = \tfrac{1}{2}(3\cos^2\psi - 1)$$

so $\frac{r}{\Delta} = 1 + \ell P_1(\cos\psi) + \ell^2 P_2(\cos\psi) + \cdots$

$$= \sum_{n=0}^{\infty} \left(\frac{r}{r_s}\right)^n P_n(\cos\psi)$$

rigorous proof is usually given in most E&M textbooks, valid for $r < r_s$

and $\Phi = -\frac{Gm_s}{\Delta} = \Phi_0 + \Phi_1 + \Phi_2 + \cdots$

where $\Phi_n = -\frac{Gm_s}{r_s}\left(\frac{r}{r_s}\right)^n P_n(\cos\psi)$

so $\Phi_0 = -\frac{Gm_s}{r_s} P_0 = \text{constant}$

$$\Phi_1 = -\frac{Gm_s r}{r_s^2} P_1(\cos\psi)$$

$$\Phi_2 = -\frac{Gm_s r^2}{r_s^3} P_2(\cos\psi)$$

set $\vec{a_0}$ = acceleration at m_p's center due to m_s

$$= \frac{G m_s}{r_s^2}\hat{x}$$

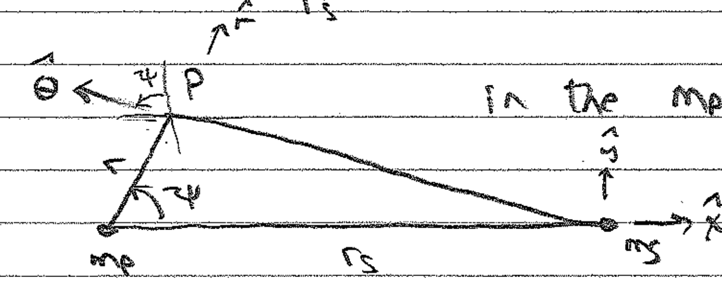
so $\vec{a}_{tide} = \vec{a}(P) - \vec{a_0}$ = tidal acceleration

= acceleration at point P **relative** to center's acceleration

$$= -\nabla \Phi(P) - a_0$$

$$= -\nabla\left(\underbrace{\Phi_0}_{\text{constant}} + \Phi_1 + \Phi_2 + \cdots\right) - a_0$$

but $\nabla \Phi_1 = -\nabla \frac{G m_s r}{r_s^2}\cos\psi$



in the m_p – P – m_s plane. (r,ψ) are polar coordinates

so $\nabla\, r\cos\psi = \frac{\partial}{\partial r}(r\cos\psi)\hat{r} + \frac{1}{r}\frac{\partial}{\partial\psi}(r\cos\psi)\hat{\theta}$

$$= \cos\psi\,\hat{r} - \sin\psi\,\hat{\theta}$$

$$= \cos\psi(\cos\psi\,\hat{x} + \sin\psi\,\hat{y}) - \sin\psi(-\sin\hat{x} + \cos\psi\,\hat{y})$$

$$= \hat{x}$$

so $-\nabla\Phi_1 = \frac{Gm_s}{r_s^2}\hat{x} = \vec{a}_0$

and $a_{tide}(r,\psi) \simeq -\nabla\Phi_2$ + other terms smaller by factors of $r/r_s \ll 1$

where $\Phi_2 = -\frac{Gm_s r^2}{r_s^3} P_2(\cos\psi)$

set $g_p = \frac{Gm_p}{R_p^2}$ = primary's surface gravity
where R_p = primary's mean radius

so $\frac{\Phi_2}{g_p} = -\frac{Gm_s r^2}{r_s^3}\frac{R_p^2}{Gm_p} P_2(\cos\psi)$

$$= -\underbrace{\left(\frac{m_s}{m_p}\right)\left(\frac{R_p}{r_s}\right)^3 R_p}_{\zeta\ =\ \text{tidal height parameter (constant}} \left(\frac{r}{R_p}\right)^2 P_2(\cos\psi)$$

so $\Phi_2 = \zeta g_p \left(\frac{r}{R_p}\right)^2 P_2(\cos\psi)$

measure's strength of tide due to various satellites

ζ on Earth due to Moon = 0.36 m
due to Sun = 0.16

$\Rightarrow$ solar tide on $\oplus$ is ~ ½ that due to moon.
Also note that $\zeta/R_p \ll 1$, will use that later

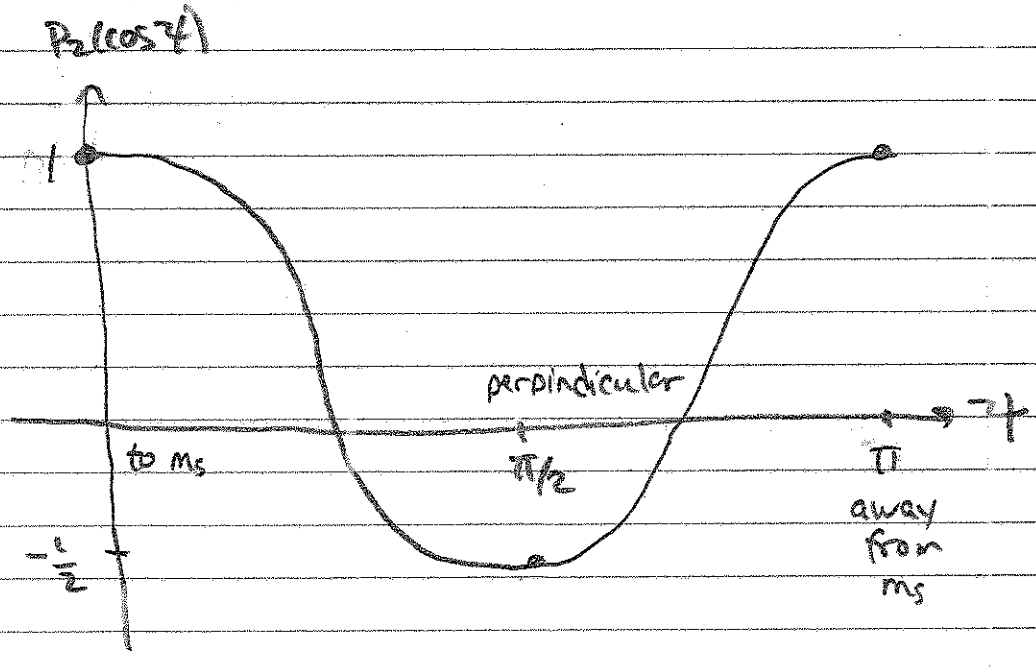
Tidal deformation of primary due to m_s:

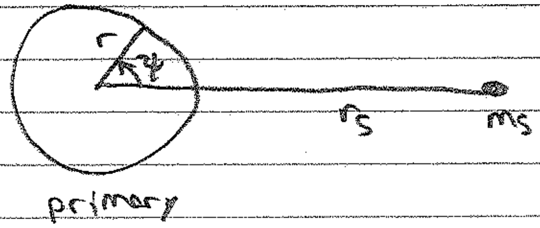
$$\Phi_2 = -\frac{G m_s r^2}{r_s^3} P_2(\cos\psi) = \text{tidal potential}$$

so $$a_r(r,\psi) = -\frac{\partial \Phi_2}{\partial r} = \frac{2 G m_s r}{r_s^3} P_2(\cos\psi)$$

= radial acceleration near primary's surface due to m_s

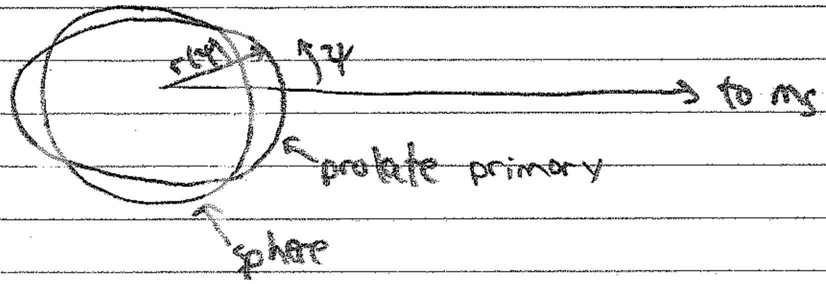
$$\Rightarrow -a_r \propto P_2(\cos\psi) = \tfrac{3}{2}\cos^2\psi - \tfrac{1}{2}$$





so the tide raise a bulge towards AND away from m_s, and flattens mp along perpendicular direction.

so the tide from m_s makes m_p prolate,
= elongated sphere



Anticipate that the prolate primary's surface will vary as

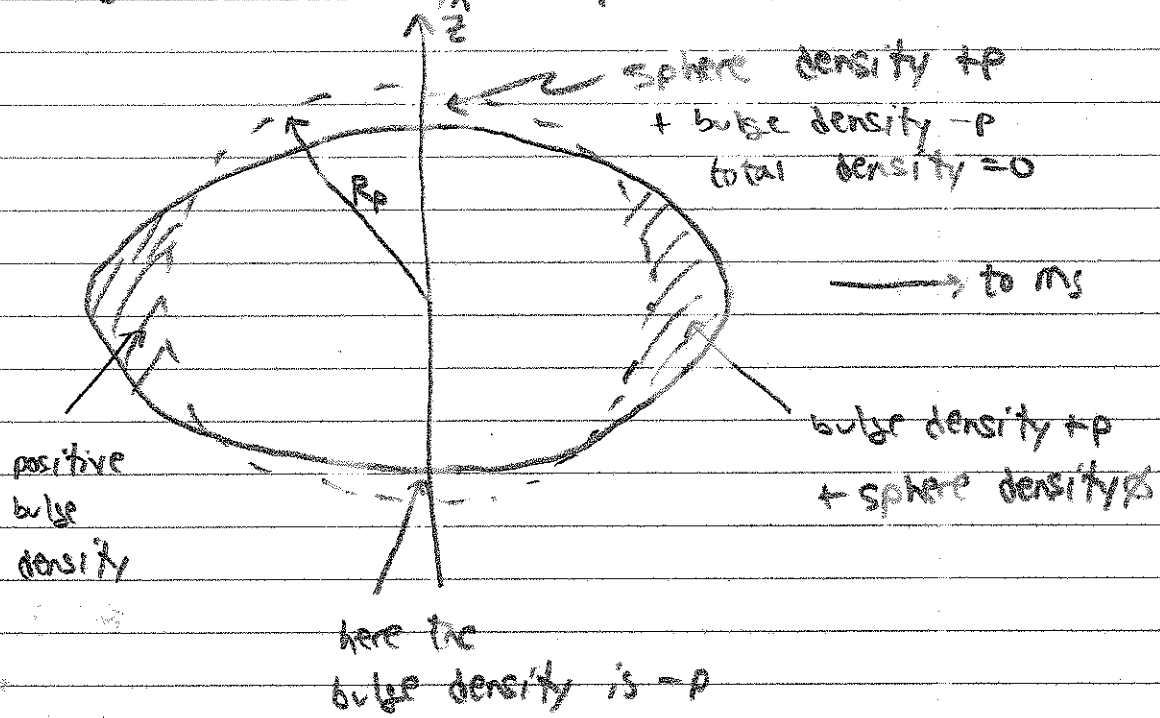
$$r(\psi) = \bar{R}_p \left[1 + \epsilon P_2(\cos\psi) \right]$$

$\bar{R}_p$ = primary's radius when m_s is absent

ϵ = dimensionless measure of primary's tidal distortion due to m_s

next: solve the EOM to relate ϵ to the primary's physical properties, ρ_p, R_p etc.

treat the prolate primary as the sum of
sphere of constant density ρ
+ bulge of density $\pm\rho$:



The gravitational potential due to oblate primary is

$$\Phi = \Phi_{sphere} + \Phi_{bulge}$$

easy   next calculation

to calculate Φ_{bulge}, we need to a model
that describes the primary's interior,
namely, an equation of state (EOS)

lets first consider the simplest possible scenario:
the primary planet is entirely water
$\Rightarrow$ incompressible so $\rho =$ constant
and flows freely (no friction)

ie the EOS is incompressible and inviscid

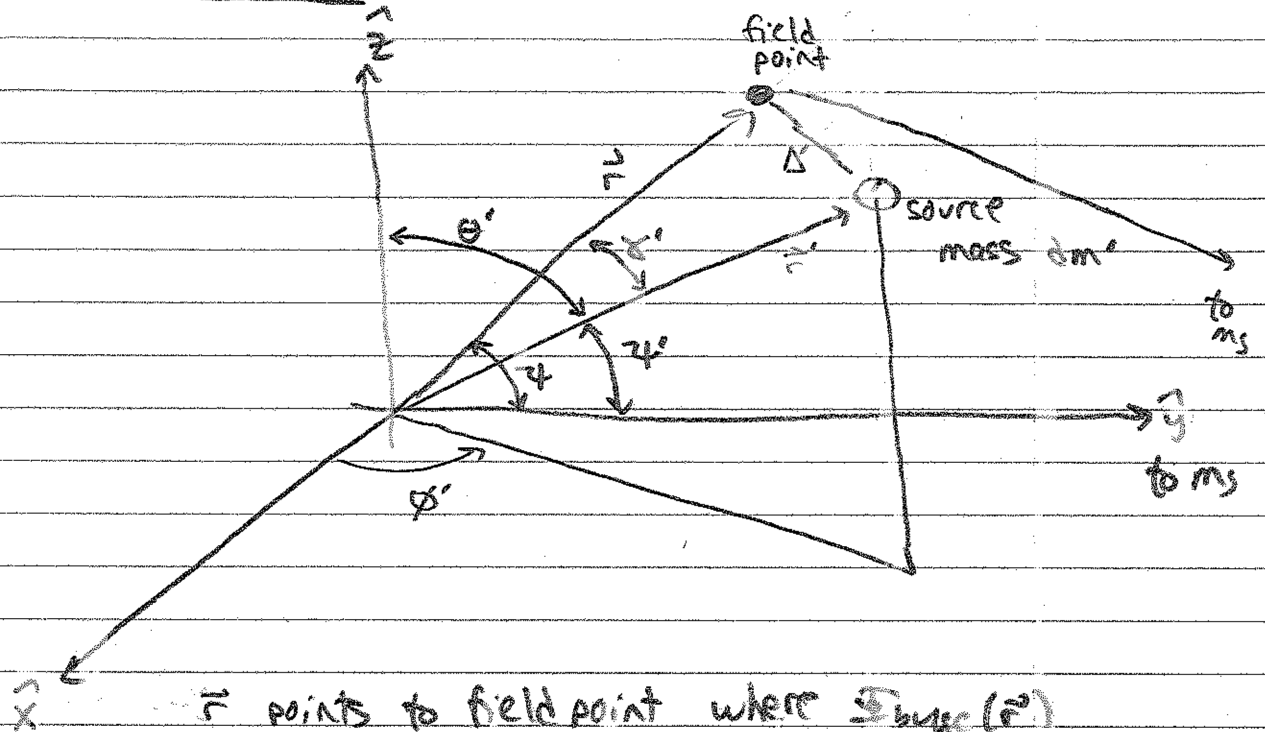
Note that real tidally evolved planet-satellite systems are usually gaseous (and compressible)
(Jupiter - Io, star - hot Jupiter)

or rocky (ie viscous incompressible fluid)
like Earth - moon

or icy like Pluto - Charon.

but we will first tackle tides on a (simple) water planet,
then adapt these results to other systems.

calculate $\Phi_{bulge}(\vec{r})$



$\vec{r}$ points to field point where $\Phi_{bulge}(\vec{r})$ is being calculated

$$d\Phi_{bulge}(\vec{r}) = -\frac{G\,dm'}{\Delta'}$$ = potential at $\vec{r}$ due to small bulge mass $dm' = \rho\, dV'$

so $\Phi_{bulge} = -\int \frac{G\rho\, dV'}{\Delta'}$

use spherical coordinates (for once!)
so $dV' = r'^2 \sin\theta' \, dr' \, d\theta' \, d\phi'$ = volume of small source dm'

Note: ψ = angle between $\vec{r}$ and m_s

ψ' = angle btwn $\vec{r}'$ and m_s

γ' = angle btwn $\vec{r}$ and $\vec{r}'$

treat bulge as a thin shell of radial width dr'

where $dV' = r'^2 \sin\theta' \underbrace{dr'}_{\text{bulge radial width}} d\theta' d\phi'$

since $r' = \bar{R}_p \left[1 + \epsilon P_2(\cos\gamma') \right]$

= radius of prolate surface

$\Rightarrow dr' = \epsilon \bar{R}_p' P_2(\cos\gamma')$ = radial thickness of dm'

so $dV' = \bar{R}_p^2 \sin\theta' \, \epsilon R_p' P_2(\cos\gamma') \, d\theta' d\phi'$

but $d\Omega' = \sin\theta' d\theta' d\phi'$ = differential solid angle that dV' subtends

so $dV' = \epsilon \bar{R}_p^3 P_2(\cos\gamma') \, d\Omega'$
= volume of dm'

and $\Phi_{bulge}(\vec{r}) = -\epsilon G \rho \bar{R}_p^3 \int \frac{P_2(\cos\gamma') \, d\Omega'}{\Delta'}$

$$= -\epsilon G \rho \bar{R}_p^3 \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \, \frac{P_2[\cos\gamma'(\theta', \phi')]}{\Delta'}$$

Now recall our result from page 4 for $\Delta' = |\vec{r} - \vec{r}'|$

$$\frac{1}{\Delta'} = \frac{1}{r}\sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\gamma')$$

where $\Delta' = |\vec{r} - \vec{r}'|$ = separation between $\vec{r}, \vec{r}'$
and $r \geq r'$
and note $r' \simeq \overline{R}_p + O(\epsilon \overline{R}_p)$

so $$\frac{1}{\Delta'} \simeq \frac{1}{r}\sum_{n=0}^{\infty} \left(\frac{\overline{R}_p}{r}\right)^n P_n(\cos\gamma') + O(\epsilon)$$

and $$\Phi_{bulge}(\vec{r}) = -\epsilon G\rho \overline{R}_p^2 \sum_{n=0}^{\infty} \left(\frac{\overline{R}_p}{r}\right)^{n+1} \int P_2(\cos\psi') P_n(\cos\gamma')\, d\Omega'$$

= grav. potential of bulge on m_p raised by m_s, in regions where $r \geq \overline{R}_p$

ie beyond the planet's surface

The Legendre polynomials are orthogonal, so only the $n=2$ term in the above has nonzero integral, so

$$\Phi_{bulge}(\vec{r}) = -\epsilon G\rho \overline{R}_p^2 \left(\frac{\overline{R}_p}{r}\right)^3 I_2(\psi)$$

where $$I_2(\psi) = \int P_2(\cos\psi') P_2(\cos\gamma')\, d\Omega'$$

It is straightforward (but tedious) to show that:

$$I_n(\psi) = \frac{4\pi}{2n+1} P_n(\cos\psi)$$ ← see orthogonality relations for spherical harmonics, ie E&M text

so $$\Phi_{bulge}(\vec{r}) = -\frac{4\pi}{5}\epsilon G\rho \bar{R}_p^2 \left(\frac{\bar{R}_p}{r}\right)^3 P_2(\cos\psi)$$

when $r \geq \bar{R}_p$ ie exterior to primary

(you get a somewhat different expression when $r < \bar{R}_p$, see M&D section 4.3)

So the total gravitational potential at field point $\vec{r}$ where $r \geq \bar{R}_p$ is

$$\Phi(\vec{r}) = \Phi_{sphere} + \Phi_{bulge} + \Phi_{tide}$$

$$= -\frac{Gm_p}{r} - \frac{4\pi}{5}\epsilon G\rho \bar{R}_p^2 \left(\frac{\bar{R}_p}{r}\right)^3 P_2(\cos\psi)$$

$$- \zeta g_p \left(\frac{r}{\bar{R}_p}\right)^2 P_2(\cos\psi)$$

Now evaluate Φ at the primary's <u>surface</u> where

$$r = \bar{R}_p\left[1+\epsilon P_2(\cos\psi)\right]$$

$$\text{so } \Phi \simeq -\frac{Gm_p}{\bar{R}_p}\left(1-\epsilon P_2\right) - \frac{4\pi}{5}\epsilon G\rho\bar{R}_p^2 P_2 - \zeta g_p P_2$$

where $m_p = \frac{4\pi}{3}\rho\bar{R}_p^3$

$$\text{so } \Phi(\psi) \simeq -\frac{Gm_p}{\bar{R}_p} + \epsilon G\rho\bar{R}_p^2 P_2(\cos\psi)\underbrace{\left(\frac{4\pi}{3}-\frac{4\pi}{5}\right)}_{8\pi/15} - \zeta g_p P_2$$

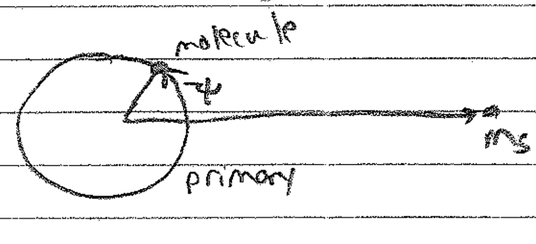
$$= -\frac{Gm_p}{\bar{R}_p} + \left(\frac{8\pi}{15}\epsilon G\rho\bar{R}_p^2 - \zeta g_p\right) P_2(\cos\psi)$$

= gravitational potential on water planet's liquid surface

consider a water molecule on that surface:

the acceleration on that molecule is

$$\vec{a} = -\nabla\Phi(\psi)$$



what is $\vec{a}$?

what does that tell us about the P_2 term in Φ?

so $\quad \epsilon = \dfrac{15\,\xi\,g_p}{8\pi G \rho \bar{R}_p^2}$

$$= \frac{15}{8\pi G\rho \bar{R}_p^2}\left[\left(\frac{m_s}{m_p}\right)\left(\frac{\bar{R}_p}{r_s}\right)^3 \bar{R}_p\right]\left(\frac{G m_p}{\bar{R}_p^2}\right)$$

$$= \frac{15}{8\pi}\frac{4\pi}{3}\left(\frac{\bar{R}_p}{r_s}\right)^3$$

$$= \frac{5}{2}\left(\frac{\bar{R}_p}{r_s}\right)^3$$

alt: $\quad \epsilon = \dfrac{15\,\xi}{8\pi G\rho \bar{R}_p^2}\,\dfrac{G m_p}{\bar{R}_p^2} = \dfrac{15\,\xi}{8\pi\rho \bar{R}_p^4}\,\dfrac{4\pi\rho \bar{R}_p^3}{3}$

$\epsilon = \dfrac{5}{2}\dfrac{\xi}{\bar{R}_p}$ = height of tidal bulge on primary, in units of primary's radius $\bar{R}_p$

if Earth where entirely water, then

$$\epsilon \sim \frac{5}{2}\,\frac{0.36\,m}{6400\,km} \sim 10^{-7}$$

this confirms our earlier assumption, that ϵ as well as $\xi/\bar{R}_p$ are both $\ll 1$