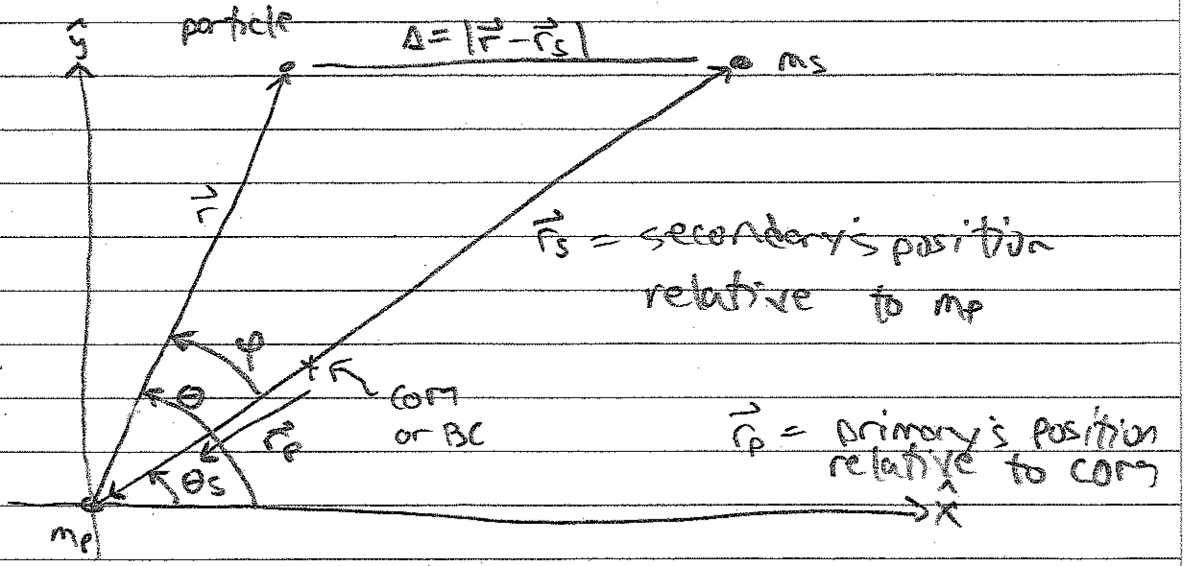
$\boxed{\text{5 Nov 2013}}$

Chapter 6: Lindblad Resonances

lets calculate the motion of a small particle orbiting near a secondary planet's Lindblad resonance



Recall the Fourier expansion of the secondary's orbit (pg 18 in previous lecture notes):

$$\Phi_2(r,\varphi) = -\frac{Gm_2}{\Delta} = \frac{1}{2}\phi_0(r) + \sum_{m=1}^{\infty} \phi_m(r)\cos(m\varphi)$$

m^{th} Fourier coefficients

Assume system is coplanar, $z=0$

solve for coefficients ϕ_m: How?

multiply above by $\cos m'\varphi$ and integrate over φ:

$\leftarrow$ m' = some integer to be specified later

$$\int_{-\pi}^{\pi} \Phi_2(r,\varphi)\cos m'\varphi \, d\varphi$$

$$= \int_{-\pi}^{\pi} d\varphi \cos m'\varphi \left[\tfrac{1}{2}\varphi_0(r) + \sum_{m=1}^{\infty} \varphi_m(r)\cos m\varphi \right]$$

if $m'=0$ the LHS $= \pi\varphi_0(r)$

for $m' \geq 1$ LHS $= \displaystyle\sum_{m=1}^{\infty} \varphi_m(r) \int_{-\pi}^{\pi} \underbrace{\cos(m'\varphi)\cos(m\varphi)}_{\frac{1}{2}\cos[(m'+m)\varphi] + \frac{1}{2}\cos[(m'-m)\varphi]} d\varphi$

what is the integral of this term? $\nearrow$ (to $\frac{1}{2}\cos[(m'+m)\varphi]$)

this? $\uparrow$ (to $\frac{1}{2}\cos[(m'-m)\varphi]$)

$$\text{LHS} = \sum_{m=1}^{\infty} \varphi_m(r) \int_{-\pi}^{\pi} \frac{1}{2}\delta_{mm'}$$

$$= \pi\varphi_{m'}(r)$$

so $\phi_m(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} \Phi_2(r,\varphi)\cos(m\varphi)\,d\varphi$

$= \frac{2}{\pi}\int_{0}^{\pi} \Phi_2(r,\varphi)\cos(m\varphi)\,d\varphi$

since $\Phi_2(-\varphi) = \Phi_2(\varphi)$
is even

The secondary's potential is

$$\Phi_2 = -\frac{Gm_s}{\Delta} = -\frac{Gm_s}{\sqrt{r^2+r_s^2-2rr_s\cos\varphi}}$$

so $\phi_m(r) = -\frac{Gm_s}{r_s}\underbrace{\frac{2}{\pi}\int_0^{\pi}\frac{\cos(m\varphi)\,d\varphi}{(1+\beta^2-2\beta\cos\varphi)^{1/2}}}_{\text{what is this?}}$

$\beta = \frac{r}{a_s}$

so $\phi_m(r) = -\frac{Gm_s}{r_s}\underset{\substack{\uparrow \\ \text{Laplace} \\ \text{coefficient}}}{b_{1/2}^{(m)}(\beta)}$

Now let's account for the indirect potential Φ_i:

Newton's 2^nd Law: $\ddot{\vec{r}}_{com} = -\nabla(\Phi_p + \Phi_s)$

in an inertial ref' frame, where

$\vec{r}_{com}$ = particle's position relative to com (ie BC)

$= \vec{r} + \vec{r}_p$

so $\ddot{\vec{r}} = \ddot{\vec{r}}_{com} - \ddot{\vec{r}}_p =$ where $\ddot{\vec{r}}_p = \frac{Gm_s}{r_s^3}\vec{r}_s$

= primary's acceleration due to m_s

Assignment #5

problem 6.1:

show that $\ddot{\vec{r}}_p = -\nabla\Phi_i = \frac{Gm_s}{r_s^3}\vec{r}_s$

where $\Phi_i = \frac{Gm_s}{r_s^3}\vec{r}\cdot\vec{r}_s$ = indirect potential

$$= \frac{Gm_s}{r_s^3} r r_s \cos\psi$$

$$= \frac{Gm_s}{r_s} r_s \cos\psi$$

so the particle's EOM in the non-inertial m_s-centered reference frame is

$$\ddot{\vec{r}} = -\nabla\left(\Phi_p + \Phi_s + \Phi_i\right)$$

where Φ_i is an effective potential that accounts for the additional fictitius forces that appear when you use a non-inertial coordianate system.

So the total potential on the particle due to all of the secondary's perturbations is

$$\Phi_s + \Phi_i = \frac{1}{2}\varphi_0 + \sum_{m=1}^{\infty} \varphi_m(r)\cos(m\varphi) + \frac{Gm_s}{r_s}\beta\cos\varphi$$

$$\text{where} \quad \varphi_m(r) = -\frac{Gm_s}{r_s} b^{(m)}_{1/2}(\beta)$$

The m's in the Fourier expansions are the perturbation's azimuthal wavenumber

ie the perturbing forces cycle m-times as you travel around 2π radians in longitude.

What is the azimuthal wavenumber for Φ_i?

$m=1$

So we can fold Φ_i into the $m=1$ term in the Fourier expansion:

$$\Phi_s + \Phi_i \rightarrow \Phi_i = \frac{1}{2}\phi_0 + \sum_{m=1}^{\infty} \phi_m(r)\cos(m\varphi)$$

where $$\phi_m = -\frac{Gm_s}{D}\left[b_{1/2}^{(m)}(\beta) - \delta_{m1}\beta \right]$$

$$\uparrow \qquad \uparrow$$

these two terms are usually comparable in magnitude

Had we ignored Φ_i, our results would have been in error by factor ~2, but only for a particle orbiting near the secondary's $m=1$ Lindblad resonance (also known as the 2:1 or 1:2 mean motion resonance)

motion of a particle near a Lindblad resonance (LR)

The particle is perturbed by an infinite number of terms in the secondary's Fourier-expand potential $\Phi_s = \Phi_0 + \Phi_1 + \Phi_2 + \Phi_3 + \cdots \Phi_m + \cdots$

but the following will show that if the particle is near an m^{th} LR, the particle behaves as if it whre only perturbed by the m^{th} term Φ_m

so write

$$\Phi_s(r,\varphi) \simeq \Phi_m = \phi_m(r)\cos(m\varphi)$$

and ignore the other terms

its not that the other terms are small and negligable (they are not)

its that the Φ_m term is <u>resonant</u>
ie the forcing frequency associated with Φ_m is nearly equal to the particle's natural oscillation frequeny

The resonant particle's response to all those m terms in Φ_s is

$$r = r_0 + r_1 + \overset{\swarrow \text{ due to } \Phi_2}{r_2} + \cdots \underset{\substack{\uparrow \\ \text{only this one is} \\ \text{large}}}{\overset{\swarrow \text{ due to } \Phi_m}{r_m}} + \cdots$$

So $r(t) = r_0 + r_m(t)$

which is why we can write

$$\Phi_s(r,\varphi) = \phi_m(r)\cos(m\varphi)$$

$$= \mathrm{Re}\left[\phi_m(r)\, e^{im(\theta-\theta_s)}\right]$$

when using complex notation,

and $\varphi = \theta - \theta_s$ = particles longitude relative to ms

and $e^{im\varphi} = \cos m\varphi + i\sin m\varphi$

The particle's Equation of Motion (EOM): in polar coordinates, see Ch 5 notes pg 2

$\hat{r}$: $$\ddot{r} - r\dot{\theta}^2 = -\frac{\partial}{\partial r}\left[\Phi_p + \phi_m e^{im(\theta-\theta_s)}\right]$$

$\hat{\theta}$: and $$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -\frac{1}{r}\frac{\partial}{\partial r}(\Phi_p + \Phi_s) = -\frac{im}{r}\phi_m e^{im(\theta-\theta_s)}$$

where I've dropped Re( ) = keep real part of ( )

⇒ only the real parts of these equations are to be preserved, their imaginary parts are to be ignored so the above eqn is

$$\frac{1}{r}\frac{d}{dr}\left(r^2\dot{\theta}\right) = \frac{m}{r}\Phi_m \sin\left[m(\theta-\theta_s)\right]$$

<u>the solution</u>:

as before, write $r(t) = r_0 + r_1(t)$

$\theta(t) = \theta_0 + \Omega_0 t + \theta_1(t)$

and assume
The noncircular
deviations
r_1, θ_1 are small: $|r_1| \ll r_0$ $|\theta_1| \ll 1$

which allows us to linearize the EOM

lets assume secodary's orbit is circular,
so $r_s(t) = a_s$ and $\theta_s(t) = \Omega_s t$

↖ this assumes
m_s is on $\hat{x}$ axis
when $t=0$

The acceleration from m_s is also small since $m_s << m_p$,

$$\text{so} \quad \left|\frac{d\phi_m}{dr}\right| \sim \theta\left(\frac{\phi_m}{r}\right) << \left|\frac{d\Phi_p}{dr}\right| = \frac{Gm_s}{r^2} \simeq \Omega^2 r$$

$$\text{so} \quad |\phi_m| << (\Omega r)^2$$

Now recall $h = r^2\dot{\theta}$ = specific angular momentum
so the $\hat{\theta}$ EOM is

$$\frac{dh}{dt} = -im\phi_m e^{\underbrace{im(\theta-\varpi_s)}_{\text{so we only need this to lowest order}}}$$

this is a small quantity

$$e^{im\varphi} = e^{im(\theta-\varpi_s)} = e^{im(\theta_0+\Omega_0 t+\underset{\text{small}}{\theta_1}-\Omega_s t)}$$

$$\simeq e^{i[m\theta_0 + m(\Omega-\Omega_s)t]}$$

set $\omega_m = m(\Omega-\Omega_s)$ = particle's Doppler-shifted forcing frequency

so $m(\theta-\varpi_s) \simeq m\theta_0 + \omega_m t$ to lowest order

and $\dfrac{dh}{dt} \simeq -im\phi_m e^{i(m\theta_0+\omega_m t)}$

which is integrable:

$$h(t) = \underset{\uparrow \text{ integration constant}}{h_0} - \frac{m\phi_m}{\omega_m} e^{i(m\theta_0 + \omega_m t)}$$

Note $h = r^2\dot{\theta} = (r_0 + r_1)^2(\Omega_0 + \dot{\theta}_1)$

$= r_0^2\Omega_0 + 2r_0\Omega_0 r_1 + r_0^2\dot{\theta}_1$

$=$ the above

so choose $h_0 = r_0^2\Omega_0$ = specific angular momentum of circular orbit of radius r_0

so $\dot{\theta}_1 = -\frac{2\Omega_0}{r_0} r_1 - \frac{m}{r_0^2\omega_m}\phi_m e^{\underbrace{i\varphi}_{\text{shorthand for } i(m\theta_0 + \omega_m t)}}$

This will be inserted into the $\hat{r}$ EOM:

$\hat{r}$ EOM: $r = r_0 + r_1$ where $|r_1| \ll r_0$

$$\ddot{r}_1 - (r_0 + r_1)(\Omega_0 + \dot{\theta}_1)^2 = -\frac{\partial}{\partial r}\left(\Phi_0 + \phi_m e^{im\varphi}\right)$$

we need the RHS to 1st order in the small quantities

evaluate derivatives at $r = r_0$

$$\frac{\partial \Phi_0}{\partial r} = \frac{\partial \Phi_0}{\partial r}\Big|_{r_0} + r_1 \frac{\partial^2 \Phi_0}{\partial r^2}\Big|_{r_0} + \mathcal{O}(r_1^2)$$

while $\frac{\partial \phi_m}{\partial r} e^{im\varphi}$ is 1st order small

so $$\ddot{r}_1 - (r_0 + r_1)(\Omega_0^2 + 2\Omega_0 \dot{\theta}_1) \simeq -\frac{\partial \Phi_0}{\partial r}\Big|_{r_0} - r_1 \frac{\partial^2 \Phi_0}{\partial r^2}\Big|_{r_0} - \frac{\partial \phi_m}{\partial r} e^{im\varphi}$$

insert $\dot{\theta}_1$

$$\ddot{r}_1 - r_0\Omega_0^2 - \Omega_0^2 r_1 - 2r_0\Omega_0\dot{\theta}_1 + \frac{\partial \Phi_0}{\partial r}\Big|_{r_0} + r_1 \frac{\partial^2 \Phi_0}{\partial r^2}\Big|_{r_0} = -\frac{\partial \phi_m}{\partial r} e^{im\varphi}$$

$$\ddot{r}_1 - \left(r\Omega^2 - \frac{\partial \Phi_0}{\partial r}\right)\Big|_{r_0} + \left(-\Omega^2 + \frac{\partial^2 \Phi_0}{\partial r^2} + 4\Omega^2\right)\Big|_{r_0} r_1 = -\left(\frac{\partial \phi_m}{\partial r} + \frac{2m\Omega_0}{r_0 \omega_m}\phi_m\right)\Big|_{r_0} e^{im\varphi}$$

The RHS involves all perturbations due to ms, They are all oscillatory.

what does that tell us about the first parentheses?

$$\Rightarrow \Omega^2 = \frac{1}{r}\frac{d\Phi_0}{dr}$$

This is the familiar formula for angular velocity of a circular orbit

The next ( ) is

$$3\Omega^2 + \frac{d^2\Phi_0}{dr^2} = ? = \kappa^2 = 4\Omega^2 + r\frac{d\Omega^2}{dr}$$

↑ epicyclic frequency ²

$$\text{Lets set } \Psi_m(r) = -\frac{d\phi_m}{dr} - \frac{2m\Omega}{r\omega_m}\phi_m$$

= secondary's <u>forcing function</u>

so the $\hat{r}$ EOM is

$$\ddot{r}_1 + \kappa_0^2 r_1 = -\underbrace{\Psi_m(r_0)}_{\text{is real since } \phi_m \text{ is}} \underbrace{e^{i(m\theta_0 + \omega_m t)}}_{\cos(m\theta_0 + \omega_m t)}$$

what kind of DEQ is this?

forced simple harmonic oscillator

The solution has two parts:

$$r_1(t) = \underbrace{r_{free}(t)}_{\text{The solution to FEOM when RHS = 0}} + \underbrace{r_{forced}(t)}_{\text{part of solution entirely due to } m_s}$$

in your DEQ class, these are the homogeneous solution + particular solution

phase

what is $r_{free}(t)$? $= -R\cos(K_0 t + \varphi_0)$

$= -Re^{i(K_0 t + \varphi_0)}$

and $r_{forced}(t)$ will resemble

$r_{forced}(t) = -Re^{i(m\theta_0 + \omega_m t)}$

↑ insert into EOM to solve for the amplitude of particle's forced motion

$$-(i\omega_m)^2 R e^{im\varphi} - \kappa^2 R e^{im\varphi} = -\Phi_m e^{im\varphi}$$

$$\text{so } -(\kappa^2 - \omega_m^2) R = -\Phi_m$$

set $D(r) = \kappa^2(r) - \omega_m^2(r)$

$$\text{so } R(r) = \frac{-\Phi_m(r)}{D(r)}$$

= amplitude of the particle's radial motion excited by ms

The particle's <u>forced eccentricity</u> is

$$e = \left|\frac{R}{r}\right| = \left|\frac{\Phi_m}{rD}\right|$$

<u>resonance location</u>

where is the resonance?

where $D(r) = 0$

$\uparrow$ <u>particle's distance from resonance</u> in frequency^2 units let r_r = resonance radius

$$D(r_r) = \kappa^2(r_r) - w_m^2(r_r) = 0$$

$$\Rightarrow \kappa(r_r) = \underset{\uparrow}{\epsilon}\, w_m(r_r)$$

$\epsilon = +1$ or -1

recall $w_m(r_r) = m\left[\Omega(r_r) - \Omega_s\right]$ = Doppler shifted forc' freq

angular velocity of m_s

so $\kappa(r_r) = \epsilon m\left[\Omega(r_r) - \Omega_s\right]$

is the equation for radius of resonance

Suppose the system is Keplerian
(ie planet orbits a star)

Then $\kappa = \Omega$ and $\Omega \propto r^{-3/2}$

and $(\epsilon m - 1)\Omega = \epsilon m \Omega_s$

$$\text{or} \quad \frac{\Omega}{\Omega_s} = \frac{\epsilon m}{\epsilon m - 1} = \frac{m}{m - \epsilon}$$

$$\left(\frac{a_s}{r_r}\right)^{3/2} = \frac{m}{m - \epsilon}$$

$$\text{or} \quad \frac{r_r}{a_s} = \left(\frac{m - \epsilon}{m}\right)^{2/3} = \left(1 - \frac{\epsilon}{m}\right)^{2/3}$$

$$r_r = \left(1 - \frac{\epsilon}{m}\right)^{2/3} a_s$$

(LR)

= radius of the m^{th} <u>Lindblad</u> resonance

resonance with $\epsilon=+1$ are inner LR (ILR)
since $r_r < a_s$

and $\epsilon=-1$ is an outer LR (OLR)
since $r_r > a_s$

see Fig 6.2

Note that the $m>>1$ resonances tend to pile up at $r_r \simeq a_s$ at secondary's orbit, which is also known as the corotation circle (cc) since a particle there would corotate with the secondary

Note that when m is large, the resonances are very close to each other. In that case, our assumption that a particle responds only to a single resonance breaks down.

These LR resonances are also known as mean motion resonances that satisfy:

$$\frac{\Omega}{\Omega_s} \rightarrow \frac{n}{n_s} = \frac{m}{m-\epsilon}$$

or commensurability resonances since

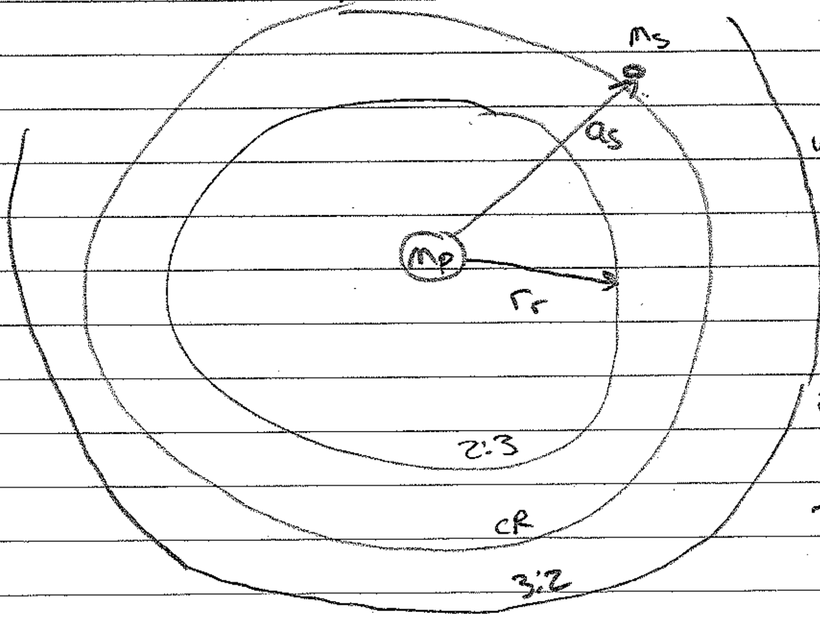
$$\frac{\Omega}{\Omega_s} \rightarrow \frac{P_s}{P} = \frac{m}{m-\epsilon}$$

the orbit periods are ratios of integers

ex. The $m=2$, $\epsilon=+1$ ILR

has $\frac{P}{P_s} = \frac{m-\epsilon}{m} = \frac{2}{3}$

so the particle's orbit period is $\frac{2}{3}$ x secondary's



where

$$r_r = \left(1 - \frac{\epsilon}{m}\right)^{2/3} a_s$$

$$= 0.76\, a_s$$

is sometimes called the 2:3 mean motion resonance

while the $m=2$ $\epsilon=-1$ OLR is at $r_r = 1.21 a_s$

which has $\frac{P}{P_s} = \frac{3}{2}$ ie the 3:2

The particle's <u>forced</u> eccentricity due to the secondary's m^{th} LR:

$$R(r) = -\frac{\Phi_m(r)}{D(r)}$$

where $D(r_0) = \kappa^2 - \omega_m^2$

if the particle is <u>near</u> exact resonance wh where $r_0 = a_r$, then write

$$D(r_0) \simeq (r_0 - r_r)\frac{dD}{dr}\bigg|_{r=r_0} + \text{small terms}$$ (typo in text)

then set $x = \frac{r_0 - r_r}{r_r}$ = particle's <u>fractional distance from exact resonance</u>

so $D(r_0) \simeq x\mathcal{D}$ where

where $\mathcal{D} = r\frac{dD}{dr}\Big|_{r=r_r}$

In Assignment #5 problem 6.4
you will show that $\mathcal{D} = 3\epsilon(m-\epsilon)\Omega_0^2$

Assign #5
problems 5.1, 5.7, 6.1, 6.4, 6.5, 6.10
due Tues Nov 19

and in problem 6.5 you show that

$$\Psi_m(r_0) = -\frac{d\phi_m}{dr} - \frac{2m\Omega_0}{r_0\omega_m}\phi_m$$

where $\phi_m = -\mu_s\Big(b^m_{1/2}(\beta) - \delta_{m1}\beta)\Big)(\Omega r)^2$

$\uparrow$ $\mu_s = \frac{m_s}{m_p}$

$$\Rightarrow \Psi_m = \epsilon f^\epsilon_m \mu_s r_0 \Omega_0^2$$

[f^ϵ_m = Eqn 6.27

= combination of Laplace coefficients
evaluated at resonance,
Tabulated in 6.1

for $1 \lesssim m \lesssim 10$
$1 \lesssim f^\epsilon_m \lesssim 10$

so the particle's forced eccentricity

$$e(x) = \left|\frac{\Psi_m}{D}\right| = \frac{f^\epsilon_m \mu_s r\Omega^2}{x\,3(m-\epsilon)r\Omega^2}$$

so $e(x) = \frac{\mu_s f^\epsilon_m}{3x(m-\epsilon)}$ = eccentricity
versus
fractional distance x
from resonance

example : consider an asteroid orbiting
0.1 AU away from Jupiter's
$m=3$ $\epsilon=+1$ ILR

what is e excited by this resonance?

Jupiter has $\mu_s = 10^{-3}$ and $a_s = 5\,AU$

so $r_r = \left(1 - \frac{\epsilon}{m}\right)^{2/3} a_s = 3.8\,AU$

= resonance radius

so $x = \frac{0.1\,AU}{r_r} = 0.026$

Table 6.1: $f_m^\epsilon = 3$

so $e(x) = \frac{\mu_s f_m^\epsilon}{3(m-\epsilon)x} \simeq 0.02$

what if asteroid was 0.01 AU away from resonance?

$e(x) = 0.2$

$\Rightarrow$ very large e's can get excited when close to a LR

Forced eccentricity:

recall $r_1(t) = r_{free}(t) + r_{forced}$

= particle's displacement from circular orbit

where $r_{free}(t) = -R\cos(\kappa_0 t + \varphi)$

p's semimajor axis

$$= -e_{free}\, r_0 \underbrace{\cos(\kappa_0 t + \varphi)}$$

eccentricity associated w/ the particle's free motion $= \frac{R}{r_0}$

corresponds to elliptic motion in the Guiding center approx when $e_{free} \ll 1$

see ps 10 notes

while $r_{forced}(t) = -R\cos(m\theta_0 + \omega_m t) = -R\cos m(\theta - \theta_s)$

$$= -e_{forced} \underbrace{\cos(m\theta_0 + \omega_m t)}$$

Note this is NOT elliptic motion

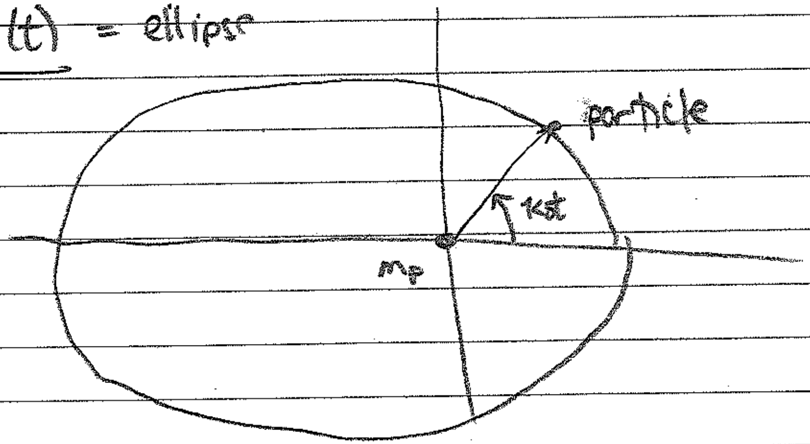
we call this the p's 'forced' eccentricity even tho the p's forced motion is NOT a Keplerian ellipse

actually, $e_{forced} = \frac{R}{r_0}$ = p's fractional radial displacement due to m_s

Compare the particle's free & forced motions:

assume $\kappa_0 = \Omega_0 =$ Keplerian mean motion and set $\varphi = 0$

$r_{free}(t)$ = Keplerian ellipse

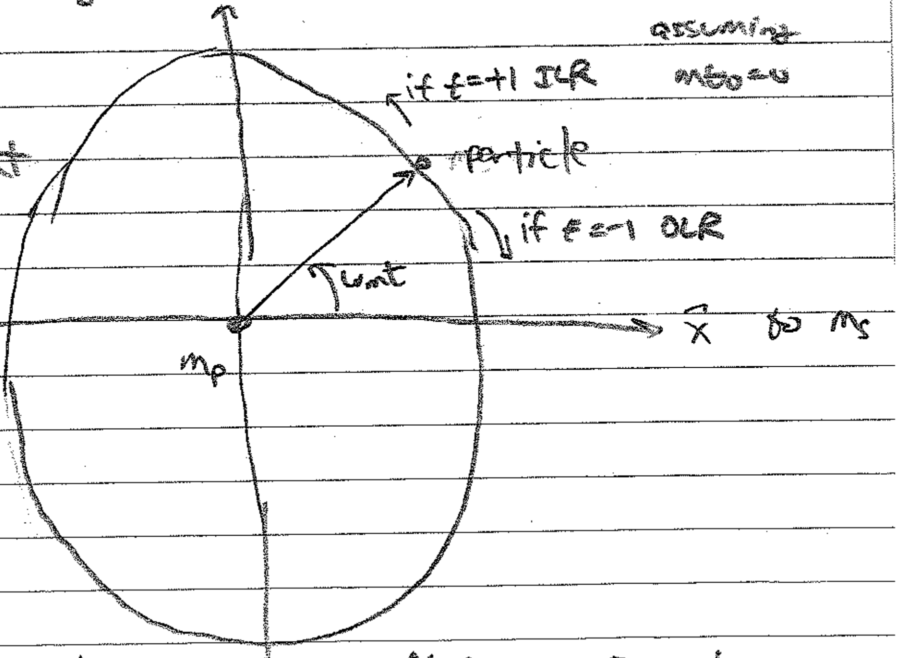


$r_{forced}(t)$ assuming particle orbits at $m=2$

$r_{forced} = -R\cos m(\theta-\theta_s)$

choose coordinate system to rotate w/ m_s

so $\theta_s = 0$



Then longitude $m\theta = \omega_m t$

but $\omega_m = \epsilon\kappa t$ at resonance

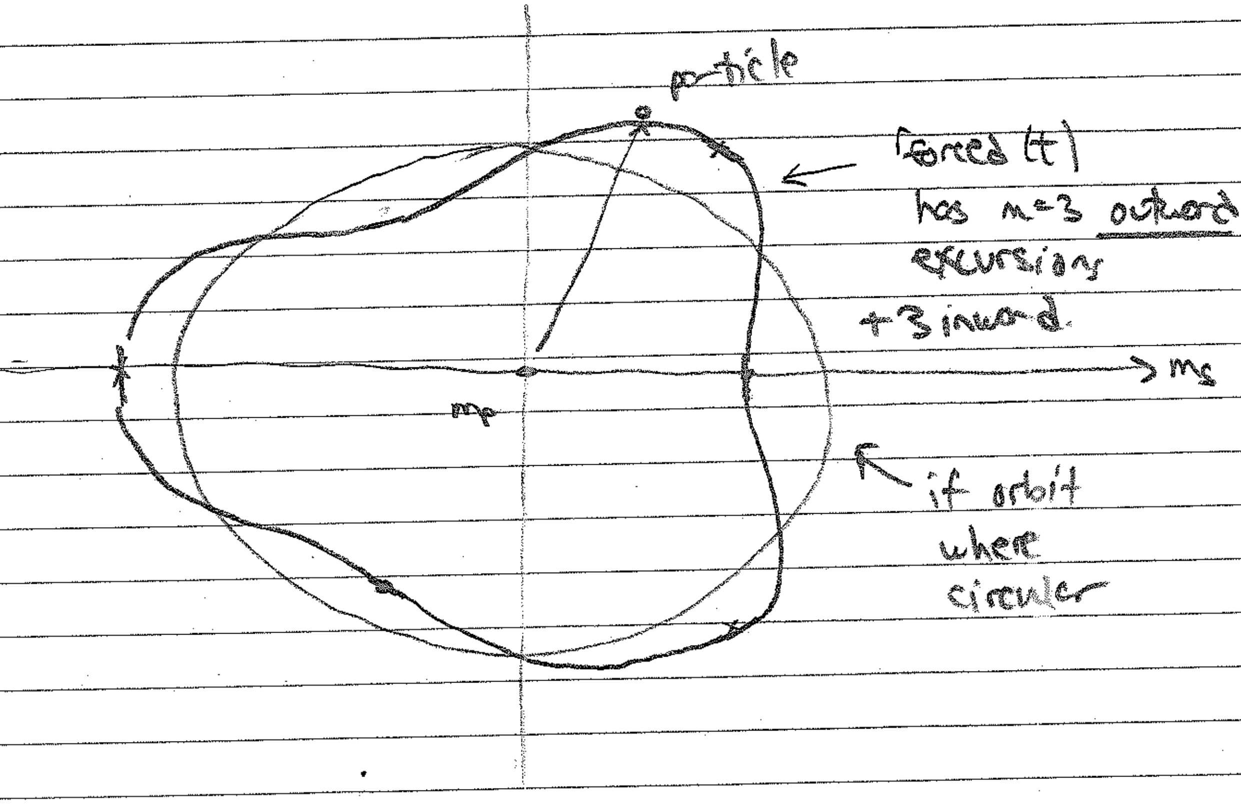
so $\theta = \frac{\epsilon\kappa t}{m}$

where $\epsilon = +1(-1)$

at inner (outer) LR ⇒ In the corotating ref' frame, the p's longitude increases (decreases) if at inner (outer) LR

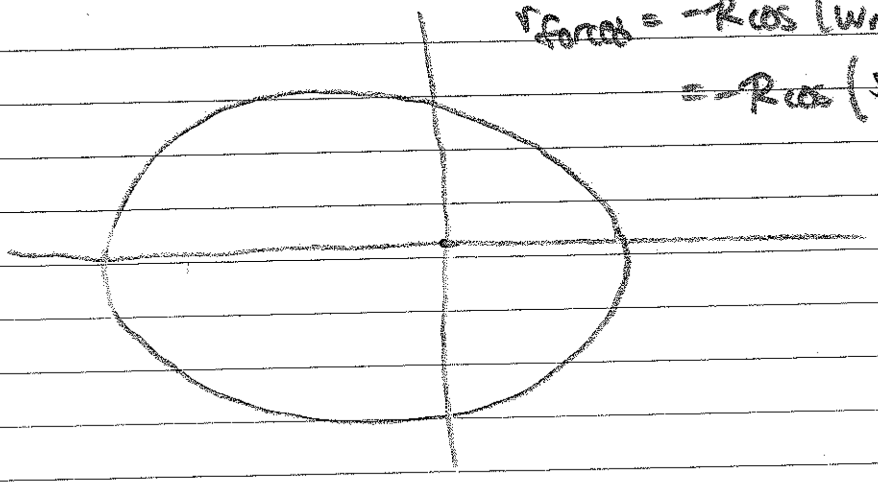
Note that the p's $m=2$ forced motion is a planet-centered ellipse.

if at $m=3$ LR:



$\Rightarrow$ only the $m=1$ OLR results in a forced trajectory that is elliptic

$$r_{\rm forced} = -R\cos(\omega_m t)$$
$$= -R\cos(\Omega - \Omega_s)t$$



This is why the B ring's outer edge
(at Mimas' m=2 ILR)
has an m=2 Saturn-centered shape (mostly)

and why the A ring's outer edge
has m=7 shape, due to m=7 ILR
with Janus & Epimetheus.

Other resonances:

recall $r_1(t) \propto R$

m_s's
= forcing fn

where $R \propto \Psi_m = -\frac{\partial \phi_m}{\partial r} - \frac{2m\Omega}{r\omega_m}\phi_m$

$\omega_m = m(\Omega - \Omega_s)$ = Doppler shifted forcing freq

so R gets large when $\Omega = \Omega_s$

This is a corotation resonance (CR resonance)

ie on CR circle

This particular CR lies on the secondary's orbit,
at $r_0 = a_s$

and is not particularly important (why?)

however there are other weaker Lindblad & CR resonances that do not lie on the CR circ

and thus occur in nature (ie ring-satellite systems, asteroid belt + Jupiter Mars etc)

They are associated with the secondary's eccentric motion, which results in additional radial forcing on the particle that is periodic and thus can result in resonant excitation.

The strength of these higher-order LR and CR resonances are weaker than the zeroth-order LR studied here, by factor of $e_s^{|k|}$ where $|k| = 1$ (or more)

This is assessed in Section 6.2, which shows where these higher order LR and CR resonances are.

see Fig 6.3.

See section 6.22 for derivation of vertical resonances... if particle & m_s are not coplanar, the particle experiences periodic vertical forces from m_s, which can excite the particle's forced inclination

Resonance Trapping

many planetary bodies are in Lindlad resonances:

Pluto is in a 3:2 resonance w/Neptune
(ie at Neptune's m=2 OLR)

3 of Jupiter's 4 Galilean sat's
(Io, Europa, Ganymede + Callisto)
are in Laplace resonance, ie in 2:1
resonances

(The 4 that Galileo could see w/his telescope)

Saturn's A, B rings w/ Mimas, Janus & Epimetheus

Mimas is in 2:3 resonance w/ Enceladus

Uranian sats are resonant:
Umbriel – Titania – Oberon
1:2 2:3

HST discovered 3 new sats at Pluto,
all seem to be resonant w/ Charon

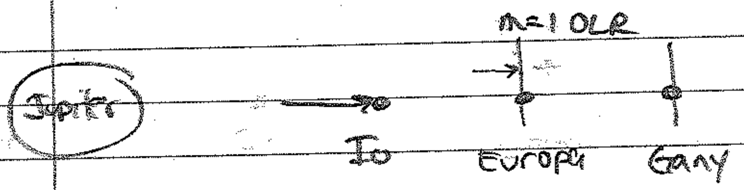
Exoplanet systems
~~GJ 876 has 2 Jupiter mass planets in 2:1~~

+ many others

Probably due to N's outward migration
+ resonance capture

to solve for this evolution,
requires a nonlinear theory for particle's
motion at resonance
(libration, level curves, conservation of 'action')

likely explains how Laplace resonance was
established



our <u>linearized</u>
treatment is
not adequate

Alternatively, weak drag force can also
cause a small body's orbit to decay inwards
into a planet's resonance

if planet's perturbation can deposit
enough E and L with the particle
to offset losses due to drag,
the particle gets trapped at the resonance:

ex: PR drag can cause dust to spiral
in and get trapped at LR's

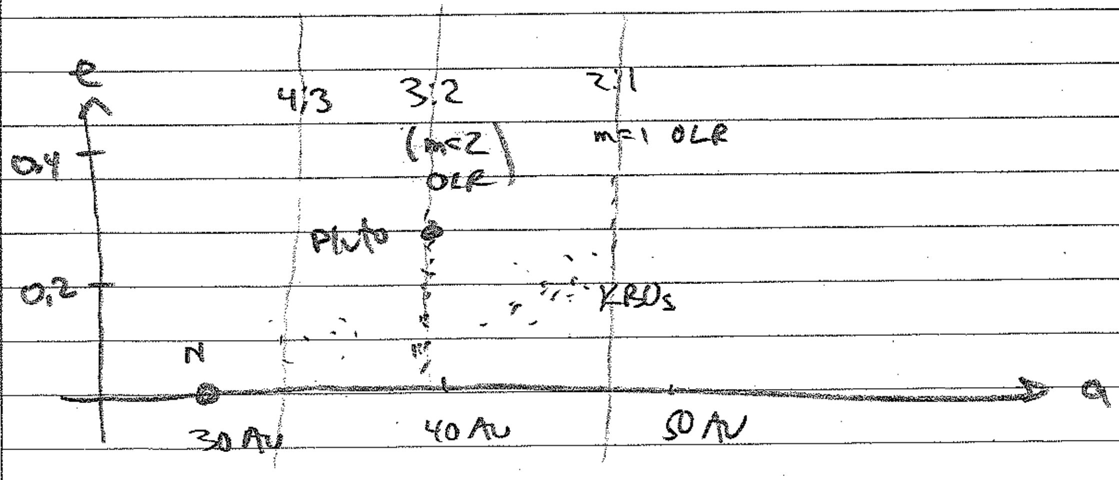
(astronomers are looking for this in
circumstellar debris disks, but no
unambiguous detections yet)

How do bodies get into resonance?
do they form in resonance?

Plutos e=0.25. Did it form in this orbit

2 easy ways to establish resonance:
resonance sweeping: the secondary's orbit migrates radially, its LR resonances also migrate, or "sweep". This can cause small bodies to get trapped at Lindblad resonances

Neptune, Pluto, & The Kuiper Belt:

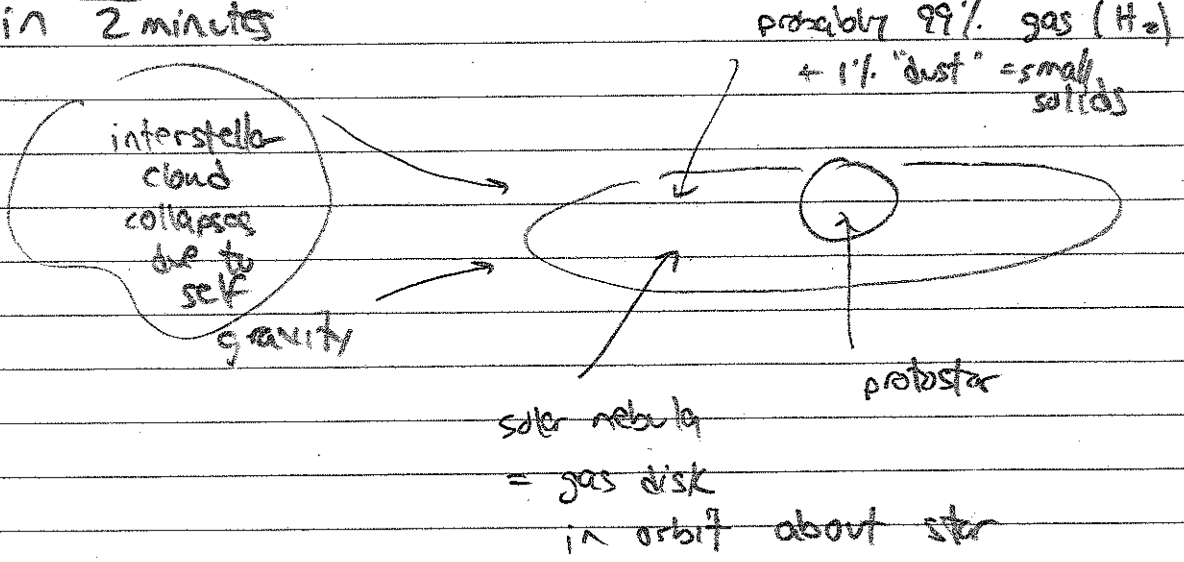


How did this system come to be?

also: gas drag on planetesimals due to solar nebula gas drag can deliver planetesimals to resonances.

This kind of resonant trapping can be solved using linear theory.

The solar nebula & origin of planets, in 2 minutes



dust assemble into planetesimals → protoplanets

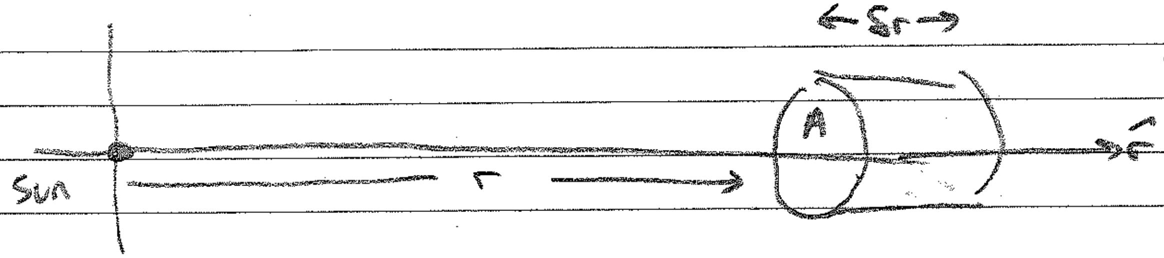
$\sim 10 M_\oplus$ protoplanets accrete nebula gas → giant planets

smaller protoplanets → terrestrial planets

remaining planetesimals get accreted or scattered by planets, or they survive and persist in asteroid, Kuiper Belts

The solar nebula's sub-Keplerian rotation about The Sun. From Section 3.2.2

consider a parcel of nebula gas of mass δm in cylinder of size $A \times \delta r$:



$\delta m = \rho A \delta r$ = mass of gas parcel

↖ nebula density

The $\hat{r}$ EOM is $\delta m(\ddot{r} - r\dot{\theta}^2) = -\frac{Gm_1 \delta m}{r^2} + A\delta p$

↗ sun's gravity on δm

where δp = gas pressure difference across cylinder

$= p(r) - p(r+\delta r) \simeq -\frac{dp}{dr}\delta r$

so $\ddot{r} - r\dot{\theta}^2 = \frac{-Gm_1}{r^2} - \frac{1}{\rho}\frac{dp}{dr}$

The nebula gas will have settled into circular orbits so $\ddot{r} = 0$ and $\dot{\theta} = \Omega$

so $$\Omega^2 = \frac{Gm_1}{r^3} + \underbrace{\frac{1}{\rho r}\frac{\partial p}{\partial r}}_{\text{negative since } p(r) \text{ likely decreases with } r}$$

set $\Omega_0^2 = \frac{Gm_1}{r^3}$ = Keplerian angular velocity^2

so $$\Omega^2(r) = (1-2\eta)\Omega_0^2$$

where $$-2\eta = \frac{\partial p/\partial r}{\rho r \Omega_0^2}$$

or $$\eta = -\frac{\partial p/\partial r}{2\rho r \Omega_0^2}$$

so $\Omega(r) = (1-\eta)\Omega_0$ = gas' angular velocity

so $$\vec{v}_{gas} = r\Omega\hat{\theta} = (1-\eta) r\Omega_0\hat{\theta}$$

↑ when $\eta > 0$, the gas motion is subkeplerian so planetesimals orbiting in solar nebula experience "headwind" which exerts aerodynamic drag, causing p'mal to spiral inwards and into a protoplanet's LR

<u>Calculate η</u> for planetesimal orbiting at $r = 1AU$:

ideal gas law: $p = \frac{\rho k_B T}{m_g}$

ρ = gas volume density
k_B = Boltzman constant
T = gas temp
m_g = mass of gas molecule, H_2

assume ρ and T are power-laws in r:

$\rho(r) = \rho_1 \left(\frac{r}{r_1}\right)^{-k}$ $T(r) = T_1 \left(\frac{r}{r_1}\right)^{-\ell}$

$r_1 = 1AU$ ρ_1, T_1 = values at $r = r_1$
and $k, \ell > 0$

so $p(r) \propto r^{-(k+\ell)}$

$\frac{dp}{dr} = -(k+\ell)p/r$

so $T \sim 280K$ at $r = 1AU$

The gas sound speed is

$$c = \sqrt{\frac{3k_BT}{m_g}} \qquad m_g = 3.34 \times 10^{-24} gm$$

$$k_B = 1.38 \times 10^{-16} \ erg/K$$

$$\simeq 3 \ km/sec$$

while the circular Keplerian speed at $r = 1AU$ is

$$r\Omega = \sqrt{\frac{GM_\odot}{r}} \simeq 30 km/sec$$

a typical solar nebula model has $k \simeq 3$, $\ell \sim 1$

(see reference at end of chapter 3)

so $\eta \sim 0.005$

so the nebula gas' orbital velocity is sub-Keplerian by $\sim 0.5\%$, due to gas radial pressure support which offsets (slightly)

$\Rightarrow$ orbiting planetesimals feel headwind, which exerts gas-drag force on p'mal and $\eta = \frac{(k+\ell)p}{2\rho(r\Omega)^2}$

for ideal gas,
p is also related to the nebula's sound speed c
via $p = \frac{1}{3}\rho c^2 = \rho k_B T / m_H$

so $\eta = \frac{1}{6}(k+\ell)\left(\frac{c}{r\Omega}\right)^2$

To evaluate η, we need gas temp T.
The nebula is mostly H_2 gas (transparent)
+ ~1% dust by mass, so assume Sun
warms the dust which in turn warms H_2 gas
(naive assumption, but good enough for now)

The dust grains are blackbody radiators, so

$$\underbrace{\left(\frac{L_\odot}{4\pi r^2}\right)}_{\text{incident solar flux}} \underbrace{\pi R^2}_{\text{dust grain cross section \& area}} = \underbrace{4\pi R^2}\sigma T^4 = \text{rate at which blackbody radiates}$$

$L_\odot = 4 \times 10^{33}$ ergs/sec = solar luminosity

so $T = \left(\frac{L_\odot}{16\pi r^2 \sigma}\right)^{1/4}$

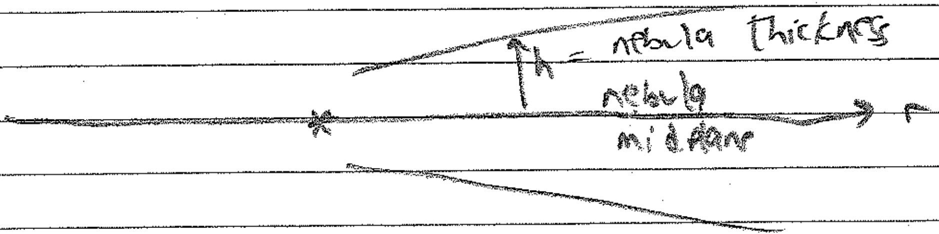
$\sigma = 5.67 \times 10^{-5}$ ergs/cm²/s/K asside: how thick (vertically) is The solar nebula at r = 1 AU? ?

The gas molecules are in nearly Keplerian motion about sun. The vertical velocity of a Keplerian orbit is

$$\dot{z} = a\Omega\sin I\cos(\omega+f) \qquad \text{(Eqn 2.51)}$$

so $c \sim a\Omega\sin I$

$\uparrow$ inclination of gas molecule



since $h = a\sin I$

Then $c \simeq h\Omega$ $\qquad \Omega = \sqrt{\frac{Gm_\odot}{r^3}} \simeq 2\times10^{-7}$

$h = \frac{c}{\Omega} \simeq 0.1$ AU

so $\frac{h}{r} \sim 0.1$ = nebula's fractional thickness drag acceleration on planetesimal due to aerodynamic drag is

$$\vec{a}_d = -\frac{\overbrace{|\vec{u}|}^{\text{magnitude of } \vec{u}}\vec{u}}{\lambda_d}$$

$\lambda_d \leftarrow$ drag damping length

where $\vec{u} = \dot{\vec{r}} - \vec{v}_{gas}$ = p'mal's velocity relative to gas

$$= \dot{r}\hat{r} + \left[r\dot{\theta} - (1-\eta)r\Omega\right]\hat{\theta}$$

assume low-e motion: $\dot{r} \simeq ean\sin f$

$r\dot{\theta} \simeq an(1+e\cos f)$

↑ note mean motion $n \simeq \Omega$ $r \simeq a$

so $\vec{u} \simeq ean\sin f\,\hat{r} + \left[an + ean\cos f - an + \eta an\right]\hat{\theta}$

$$\vec{u} = an\left[e\sin f\,\hat{r} + (\eta + e\cos f)\hat{\theta}\right]$$

= p'mals speed relative to gas

The relative speed is from

$$|u|^2 = (an)^2\left[e^2\sin^2 f + \eta^2 + 2e\eta\cos M + e^2\cos^2 f\right]$$

$$= an\left(e^2+\eta^2 + 2e\eta\cos f \right)$$

$$= an(e^2+\eta^2)\left(1+ \frac{2e\eta\cos f}{e^2+\eta^2} \right)$$

so $|u| = an\sqrt{e^2+\eta^2}\; g(e/\eta, f)$

where $g(e/\eta, f) = \sqrt{1+\dfrac{(2e/\eta)\cos f}{1+(e/\eta)^2}}$

$\simeq 1$ for almost any value of e/η

(except that can take $0 \lesssim g \lesssim \sqrt{2}$ when $e \simeq \eta$)

assume $g \simeq 1$ so $\vec{a}_d \simeq -\dfrac{\sqrt{e^2+\eta^2}\, an\, \vec{u}}{\lambda_d}$

where $\lambda_d = \dfrac{8}{3C_d}\left(\dfrac{\rho_p}{\rho_g}\right) R_p$

planetesimal volume density

planetesimal radius

drag coefficient $C_d \simeq 1$

gas density

λ_d is roughly the distance the planetesimal must travel to sweep out its own mass, in gas

So $$\vec{a_d} = -\frac{|\vec{u}|\vec{u}}{\lambda_d}$$

$$= -\frac{\sqrt{e^2+\eta^2}\,|an|^2}{\lambda}\left[e\sin f\,\hat{r} + (\eta+e\cos f)\,\hat{\theta}\right]$$

set $$= -\underset{\uparrow}{\alpha_d}\, a n^2\left[e\sin f\,\hat{r} + (\eta+e\cos f)\,\hat{\theta}\right]$$

dimensionless drag constant

$$\alpha_d = \sqrt{e^2+\eta^2}\,\frac{a}{\lambda}$$

Note that when $\eta=0$, this drag force resembles problem #3 from midterm

what did that drag do?

what happens when $\eta > 0$?

Aerodynamic drag on planetesimal is

$$\vec{u} = \vec{v} - \vec{v}_{gas}$$

$$\vec{a}_d = -\frac{|\vec{u}|\vec{u}}{\lambda_d}$$

where $\vec{u} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} - \overbrace{(1-\eta)\, r\Omega_0(r)}$

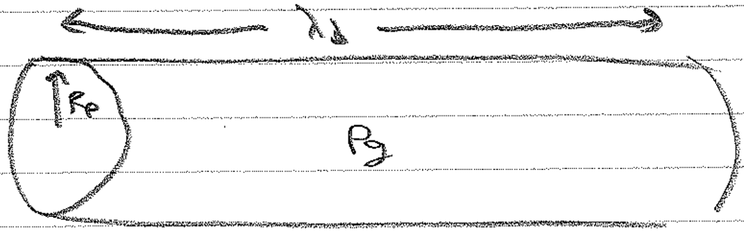
= p'mal's velocity relative to $\vec{v}_{gas}$

and $|\vec{u}| = \underset{\simeq \text{ constant}}{r\Omega_0\sqrt{e^2+\eta^2}}\ \underbrace{g(e/\eta, f)}_{\text{almost always } 1}$

$\lambda_d = \frac{8}{3C_D}\left(\frac{\rho_p}{\rho_g}\right)R_p$ ~ distance the p'mal must travel to sweep a mass of gas equivalent to its own

$C_D \sim 1$

consider $R_p = 1$ km planetesimal



$\rho_g \sim 10^{-9}$ gm/cm³ (see Section 3.2.2.3)

$\rho_p \sim ?$

so $\lambda_d \sim 10^{15}$ cm ~ 50 AU

if this planetesimal orbits at r = 1 AU, then $\lambda_d \sim \frac{50}{2\pi} \sim 10$ orbits set $\vec{a}_d = \frac{-|\vec{u}|\vec{u}}{\lambda_d} \equiv -k_d \Omega_0 \vec{u}$

where $k_d = \frac{|\vec{u}|}{\lambda_d \Omega_0}$ = dimensionless drag coefficient

Note: $k_d \ll$ when drag is weak

so $\vec{a}_d = -k_d \Omega_0 \left[\dot{r}\hat{r} + r(\dot{\theta} - \Omega_0 + \eta\Omega_0)\hat{\theta} \right]$

$= -k_d \Omega_0 \left[(\dot{r}_1 + \dot{r}_d)\hat{r} + r(\Omega_0 + \dot{\theta}_1 + \dot{\theta}_d - \Omega_0 + \eta\Omega_0)\hat{\theta} \right]$

let's assume forcing by m_s >> that due to drag

ie $|\dot{r}_d| \ll |\dot{r}_1|$ and $|\dot{\theta}_d| \ll |\dot{\theta}_1|$

(check this assumption later)

so $\vec{a}_d \simeq -k_d \Omega_0 \dot{r}_1 \hat{r} - k_d r_0 \Omega_0 (\dot{\theta}_1 + \eta\Omega_0)\hat{\theta}$

is the linearized acceleration of particle due to gas drag

ie we also drop small nonlinear $r_1\dot{\theta}_1$ terms we already wrote the EOM for a particle perturbed by ms (see notes pg 8), so we just need to add $\vec{a}_d$ to RHS:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\partial}{\partial r}\left[\Phi_p + \phi_m e^{im(\theta-\Omega t)}\right] - k_d \Omega_0 \dot{r}_1$$

and $$\frac{dh}{dt} = -im\phi_m e^{i(m\theta_0+\omega_m t)} - k_d r_0^2 \Omega_0 (\dot{\theta}_1 + \eta\Omega_0)$$

↳ the $\hat{\theta}$ EOM × r_0

where $h = r^2\dot{\theta}$ = particle's specific angular momentum

but $$\frac{dh}{dt} = \dot{h}_1 + \dot{h}_d \leftarrow$$ secular (steady) ang. mom loss due to drag

↑ due to ms, is periodic

The oscillatory parts and secular parts of this EOM must be satisfied <u>separately</u> so

$$\dot{h}_1 = -im\phi_m e^{i(m\theta_0+\omega_m t)} - k_d r_0^2 \Omega_0 \dot{\theta}_1$$

and $$\dot{h}_d = -k_d \eta (r_0\Omega_0)^2$$ = specific torque that gas drag exerts on particle.

can integrate $\dot{h}_1$:

$$h_1(t) = -\frac{m\phi_m}{\omega_m} e^{i(m\theta_0+\omega_m t)} - k_d r_0^2 \Omega_0 \theta_1$$

Also $h = r^2\dot{\theta} = (r_0 + r_1 + r_2)^2(\Omega_0 + \dot{\theta}_1 + \dot{\theta}_2)$

$$\simeq (r_0^2 + 2r_0r_1 + 2r_0r_2)(\Omega_0 + \dot{\theta}_1 + \dot{\theta}_2)$$

$$= (r_0^2\Omega_0 + 2r_0\Omega_0r_2 + r_0^2\dot{\theta}_2) + \underbrace{(2r_0\Omega_0r_1 + r_0^2\dot{\theta}_1)}_{\text{periodic term}}$$ when linearized

these are secular (non-oscillatory) terms

so $h = h_0(t) + h_1(t)$

where $h_0 = r_0^2\Omega_0 + 2r_0\Omega_0r_2 + r_0^2\dot{\theta}_2$

$$h_1(t) = 2r_0\Omega_0r_1 + r_0^2\dot{\theta}_1$$

$$= -\frac{m}{\omega_m}\phi_m e^{i(m\theta_0 + \omega_m t)} - k_2 r_0^2\Omega_0\theta_1$$

The oscillatory part of the particles motion will have the form

$$r_1(t) = -\mathrm{Re}\left[R e^{i(m\theta_0 + \omega_m t)}\right]$$

$$\theta_1(t) = \mathrm{Re}\left[\Theta e^{i(m\theta_0 + \omega_m t)}\right]$$

so $\dot{\theta}_1 = i\omega_m\theta_1$

$$2 r_0 \Omega_0 r_1 + i \omega_m r_0^2 \theta_1 = -\frac{m}{\omega_m} \phi_m e^{i(m\theta_0 + \omega_m t)} - k\delta r_0^2 \Omega_0 \theta_1$$

$$(i \omega_m r_0^2 + k\delta r_0^2 \Omega_0)\, \theta_1 = -\left(\frac{m}{\omega_m} \phi_m e^{i()} + 2 r_0 \Omega_0 r_1 \right)$$

$$\theta_1 = -\frac{1}{i \omega_m r_0^2 (1 + k\delta \Omega_0 / i\omega_m)} \left(\frac{m}{\omega_m} \phi_m e^{i()} + 2 r_0 \Omega_0 r_1 \right)$$

$$= \frac{i}{\omega_m r_0^2} \left(1 - \frac{i k\delta \Omega_0}{\omega_m} \right)^{-1} \left(\frac{m}{\omega_m} \phi_m e^{i()} + 2 r_0 \Omega_0 r_1 \right)$$

↑ small since $k\delta \ll 1$

$$\text{so } \theta_1 \simeq \frac{i}{r_0 \omega_m^2} \left(1 + \frac{i k\delta \Omega_0}{\omega_m} \right) \left[\frac{m}{\omega_m} \phi_m e^{i(m\theta_0 + \omega_m t)} + 2 r_0 \Omega_0 r_1 \right]$$

Next, solve for $r_1(t)$ using the $\hat{r}$ part of the EOM