Oct 17 2013

Chapter 5: Motion in an Axisymmetric Potential

A Keplerian system is one where the gravitational potential varies as $\Phi(r) \propto r^{-1}$

and a non-Keplerian system is one that does not.

It is the non-Keplerian systems that are now of interest here, and there are two types:

nearly Keplerian system: eg planetary systems, or satellites orbiting an oblate planet, etc.

and the very-non-Keplerian systems: such as stars orbiting in a disk (or spiral) galaxy.

The equations developed here apply to both kinds of non-Keplerian systems,

so I will occasionally discuss the motion of stars in a galaxy, since our results easily apply there

Let's consider motion in an axisymmetric potential where

$$\Phi = \Phi(r,z) \quad \text{with no } \theta\text{-dependance.}$$

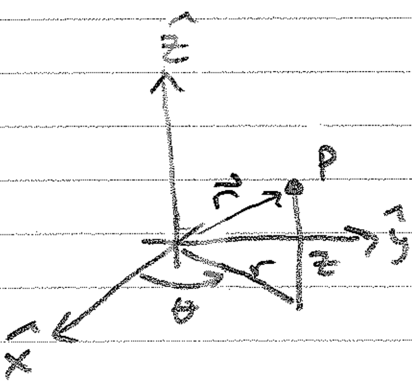
The equation of motion for a particle is

$$\ddot{\vec{r}} = -\nabla\Phi \quad \text{or}$$

so $$\ddot{r} - r\dot{\theta}^2 = -\frac{\partial\Phi}{\partial r}$$ ← radial component of $\ddot{\vec{r}} = -\nabla\Phi$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = 0$$ tangential

and $$\ddot{z} = -\frac{\partial\Phi}{\partial z}$$ vertical



(see Ch2 notes pg 8)

no tangential forces

so the z-component of the particle's specific angular momentum

$$h_z = r^2\dot{\theta} \quad \text{is conserved}$$

Remember: r = component of particle's position vector $\vec{r}$ in $\hat{x}$-$\hat{y}$ plane.

lets solve for the particle's zeroth-order motion: circular orbit in the $z=0$ plane:

which is trivial;

$$r(t) = r_0 = \text{constant}$$

$$\theta(t) = \theta_0 + \theta_1(t)$$

$$z(t) = 0$$

where r_0, θ_0 are constants

The radial EOM tells us that

$$\dot{\theta}_1^2 = \frac{1}{r_0}\frac{\partial \Phi}{\partial r}\bigg|_{\vec{r}_0} \equiv \Omega_0^2(r_0)$$

Note: this Ω is angular velocity, NOT the longitude of ascending node.

where $\frac{d\Phi}{dr}\Big|_{\vec{r}_0}$ means evaluate the derivative at $\vec{r} = \vec{r}_0$.
where $r=r_0$ $\theta=$any $z=0$

also Ω_0 is shorthand for Ω evaluated at $r=r_0$

The $\hat{\theta}$ EOM tells us that

$$r_0^2\dot{\theta}_1 = \text{constant}$$

so $\dot{\theta}_1 = \Omega_0$ is constant

and $\theta_1(t) = \Omega_0 t$

When the potential is Keplerian,

$$\Phi = -\frac{\mu}{r}$$

$$\text{and } \Omega^2 = \frac{1}{r}\frac{d\Phi}{dr} = \frac{\mu}{r^3} = n^2$$

so particle's angular velocity $\Omega(r) = n$ = mean motion, as expect

Now solve for the particle's ~~first order~~ motion;
the particle's motion is nearly circular and almost coplanar:

$$r(t) = r_0 + r_1(t)$$

$$\theta(t) = \theta_0 + \Omega_0 t + \theta_1(t)$$

$$z(t) = z_1(t)$$

where noncircular deviations are assumed small:

$$|r_1| \ll r_0$$

$$|\theta_1| \ll 1$$

$$|z_1| \ll r_0$$

the particle's EOM is

$$\ddot{r}_1 - (r_0+r_1)(\Omega_0+\dot{\theta}_1)^2 = -\frac{\partial \Phi}{\partial r} \qquad = \hat{r} \text{ EOM}$$

$$h_z = (r_0+r_1)^2(\Omega_0+\dot{\theta}_1) \quad = \text{integration constant} \qquad \hat{\theta} \text{ EOM}$$

$$\ddot{z}_1 = -\frac{\partial \Phi}{\partial z} \qquad \hat{z}$$

The particle's deviation from circular motion is small, so linearize the EOM

on the RHS,

$$\frac{\partial \Phi}{\partial r} \simeq \left.\frac{\partial \Phi}{\partial r}\right|_{\vec{r}_0} + r_1 \left.\frac{\partial^2 \Phi}{\partial r^2}\right|_{\vec{r}_0}$$

$$= r_0 \Omega_0^2 + \left.\frac{\partial^2 \Phi}{\partial r^2}\right|_{r_0} r_1$$

similarly $\frac{\partial \Phi}{\partial z} \simeq \left.\frac{\partial \Phi}{\partial z}\right|_{\vec{r}_0} + \left.\frac{\partial^2 \Phi}{\partial z^2}\right|_{\vec{r}_0} z_1$

usually one orientates the coordinate system so that there is no vertical acceleration on particle in the z=0 plane, so this term is usually zero

ex: put z=plane in equitorial plane of an oblate planet so $\ddot{z}_1 = -\nu_0^2 z_1$

where $\nu_0^2 = \left.\frac{\partial^2 \Phi}{\partial z^2}\right|_{r_0}$ is a constant

from $\vec{\theta}$ EOM:

$$\Omega_0 + \dot{\theta}_1 = \frac{h_z}{r_0^2}\left(1+\frac{r_1}{r_0}\right)^{-2} = \frac{h_z}{r_0^2} - \frac{2h_z}{r_0^3} r_1$$

$$\text{so } \dot{\theta}_1 = \frac{h_z}{r_0^2} - \Omega_0 - \frac{2h_z}{r_0^3} r_1$$

we can set integration constant $h_z = r_0^2 \Omega_0$

$$\text{so } \dot{\theta}_1 = -\frac{2\Omega_0}{r_0} r_1$$

insert this into $\hat{r}$ EOM:

$$\ddot{r}_1 - (r_0+r_1)\,\Omega_0^2 \underbrace{\left(1-\frac{2r_1}{r_0}\right)^2}_{1-\frac{4r_1}{r_0}} = -r_0\Omega_0^2 - \left.\frac{\partial^2\Phi}{\partial r^2}\right|_{r_0} r_1$$

$$\text{so } \ddot{r}_1 - r_0\Omega_0^2 \underbrace{\left(1+\frac{r_1}{r_0}\right)\left(1-\frac{4r_1}{r_0}\right)}_{\approx 1-\frac{3r_1}{r_0}} = -r_0\Omega_0^2 - \left.\frac{\partial^2\Phi}{\partial r^2}\right|_{r_0} r_1$$

so

$$\ddot{r}_1 + \left(3\Omega_0^2 + \frac{\partial^2 \Phi}{\partial r^2}\Big|_{r_0}\right) r_1 = 0$$

or $\ddot{r}_1 + \kappa_0^2 r_1 = 0$ is the <u>linearized</u> $\vec{F}$ EOM

where $\kappa^2 = 3\Omega^2 + \frac{\partial^2 \Phi}{\partial r^2}$ = epicyclic frequency^2

$$= 3\Omega^2 + \frac{d}{dr}(r\Omega^2) = 4\Omega^2 + r\frac{d\Omega^2}{dr}$$

and $\kappa_0 = \kappa(r=r_0)$

This is the EOM for a SHO that is unforced, ie there is no driving term on the RHS (when we study orbit resonances, we will see this eqn again but with nonzero term on RHS)

The solution is $r_1(t) = -R\cos\kappa_0 t$

where R = particle's <u>epicyclic</u> amplitude

κ_0 = its epicyclic frequency

so $\dot{\theta}_1 = -\frac{2\Omega_0}{r_0} r_1 = +2\Omega_0 \frac{R}{r_0} \cos \kappa_0 t$

$$\theta_1(t) = \frac{2R}{r_0} \frac{\Omega_0}{\kappa_0} \sin(\kappa_0 t)$$

(The integration constant $= 0$ so $\theta(t=0) = \theta_0$)

Also $\ddot{z}_1 = -\nu_0 z_1 \Rightarrow z_1(t) = Z \sin(\nu_0 t + \phi_0)$

so $\nu_0 = \sqrt{\left.\frac{\partial^2 \Phi}{\partial z^2}\right|_{r_0}}$ = particle's vertical oscillation frequency

The particle's trajectory in cylindrical coordinates is:

$$r(t) = r_0 + r_1(t) = r_0 - e r_0 \cos(\kappa_0 t)$$

where $e = \frac{R}{r_0}$ resembles the eccentricity of the 2-body problem

$$\theta(t) = \theta_0 + \Omega_0 t + \theta_1(t)$$

$$= \theta_0 + \Omega_0 t + 2e \frac{\Omega_0}{\kappa_0} \sin(\kappa_0 t)$$

and set $Z = r_0 \sin i$ so $z(t) = r_0 \sin i \sin(\nu_0 t + \phi_0)$

So even if the particle is orbiting
in a non-Keplerian potential, its motion
resembles low-e Keplerian motion
in the Guiding-Center approximation.
(compare this solution to Ch 2 notes, pg 28)

so r_0 resembles 2-body semimajor axis

$$\frac{R}{r_0} \rightarrow \text{eccentricity } e$$

$$\frac{Z}{r_0} \rightarrow \sin(\text{inclination})$$

Note that the 2-body problem should
be recovered when the potential is
Keplerian, ie

$$\Phi(r) = -\frac{\mu_*}{r}$$

If so, then $\Omega_0, \kappa_0, \nu_0$ should all = mean motion n:

check: $\Omega^2 = \frac{1}{r}\frac{d\Phi}{dr} = \frac{\mu}{r^3} = n^2$ ✓

Assignment #5

problem 5.1: show $\kappa_0 = \Omega_0 = \nu_0 = n$
when Φ is Keplerian.

problem 5.2: show that phases θ_0 and ϕ_0
are related to ω and $\tilde{\omega}$

orbital Precession in a Non-Keplerian Potential

if the potential is keplerian, $\Phi \propto r^{-1}$, then

and $\kappa_0 = \nu_0 = \Omega_0 = n =$ mean motion

Then $r(t)$, $\theta(t)$, $z(t)$ all oscillate with period $T_{orb} = \frac{2\pi}{\Omega_0}$

and the orbit is closed, ie the motion repeats after time T_{orb}

If the particle is orbiting in a Non-keplerian potential then the frequencies $\kappa_0 \neq \nu_0 \neq \Omega_0$

and the orbit precesses

ie the eccentric orbital ellipse rotates over time

or the inclined orbit plane rotates.

Calculate particle's Precession Rates:

Since $r(t) = r_0 - R\cos(\kappa_0 t)$

$$\theta(t) = \theta_0 + \Omega_0 t + 2\frac{R}{r_0}\frac{\Omega_0}{\kappa}\sin \kappa_0 t$$

$$z(t) = Z\sin(\nu_0 t + \phi_0)$$

so $\Delta t = \frac{2\pi}{\kappa_0}$ = particle's epicyclic period

= time between two periapse passages

If time $t=0$ is time of one periapse passage,

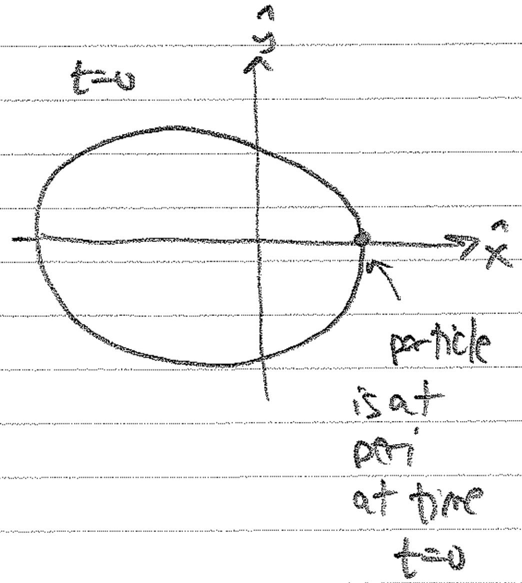
then $\Delta t = \frac{2\pi}{\kappa_0}$ = time of next periapse passage

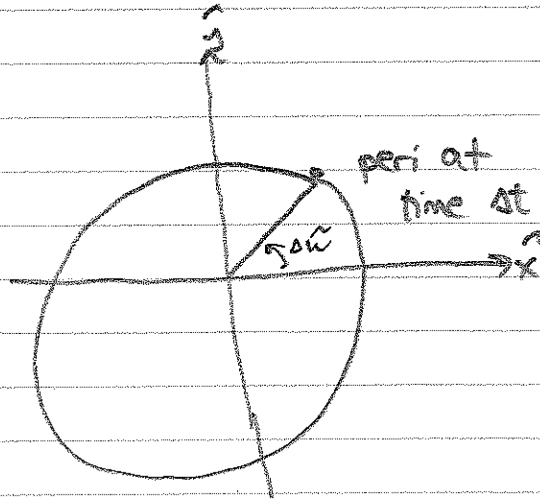
But the particle's longitude will have advanced

by $\Delta\theta = \Omega_0 \Delta t$ = new longitude of periapse passage

so $\Delta\tilde{\omega} = \Delta\theta - 2\pi$ = angle $\tilde{\omega}$ advanced

during time Δt (modulo 2π)





so $\frac{\Delta\tilde{\omega}}{\Delta t} = \dot{\tilde{\omega}} = \Omega_0 - \frac{2\pi}{2\pi/\kappa_0} = \Omega_0 - \kappa_0$

= particle's periapse precession rate

Likewise, the period for the particle's vertical oscillations is $\Delta t = \frac{2\pi}{\nu_0}$

The <u>longitude of the ascending node</u> Ω_{node} is the value of $\theta(t)$ when $z(t)=0$

suppose $t=0$ = time of first passage thru the $z=0$ plane

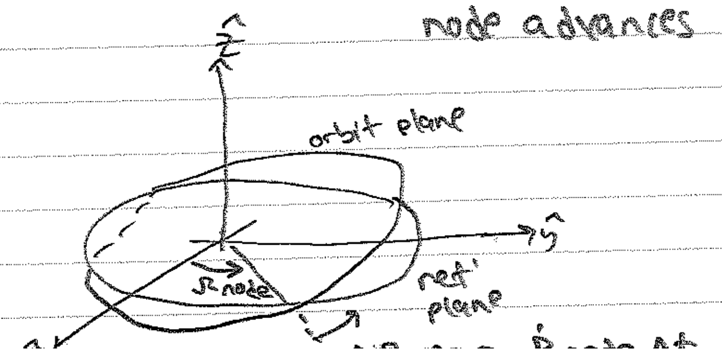
so $\Delta t = \frac{2\pi}{\nu_0}$ = time of next passage thru $z=0$

and $\theta(\Delta t)$ will have advanced by

$\Delta\theta = \Omega_0 \Delta t - 2\pi$

so $\dot{\Omega}_{node} = \frac{\Delta\theta}{\Delta t} = \Omega_0 - \nu_0$

= rate at which particle's node advances



most planetary environments have $\nu_0 > \Omega_0 > \kappa_0$

so $\dot{\tilde{\omega}} > 0$ (usually) and longitude of periapse precesses (usually)

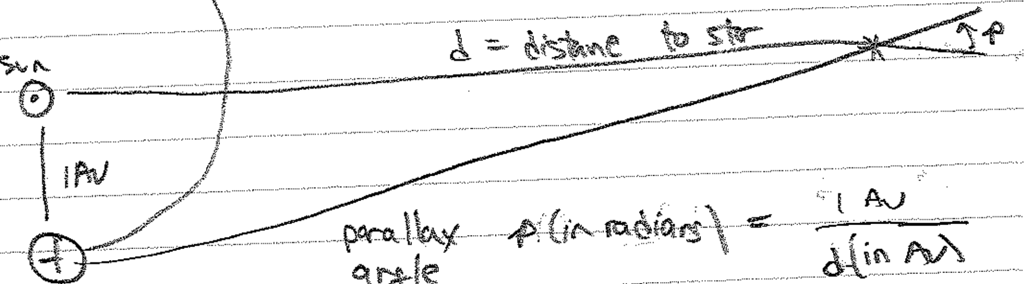
and $\dot{\Omega}_{node} < 0$ (usually) so the particle's node regresses (usually)

However the Sun's somewhat circular orbit through the Galaxy has $\nu_0 > \kappa_0 > \Omega_0$

$$\left.\begin{array}{l} \text{so } \dot{\tilde{\omega}} < 0 \\ \dot{\Omega}_{node} < 0 \end{array}\right\} \text{Sun's orbit regresses.}$$

Ex 5.1: Treat the Milky Way Galaxy as a uniform slab of matter of density $\rho = 0.16\ M_\odot/pc^3$. Calculate the Sun's vertical oscillation frequency

what is a parsec? pc = parsec = distance to a star whose parallax = 1 arc second



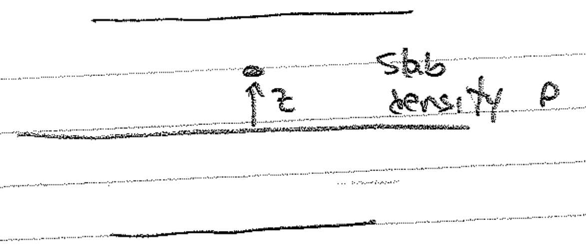
parallax angle $p(\text{in radians}) = \dfrac{1\text{ AU}}{d(\text{in AU})}$

$1\text{ arcsec} = \dfrac{1\text{ degree}}{60\text{ min} \times 60\text{ sec}}$ or $p(\text{in arcsec}) = \dfrac{1}{d(\text{in parsecs})}$

anyway, $1\text{ pc} = 3.26\text{ ly} = 3.09 \times 10^{18}\text{ cm}$

$M_\odot = 2 \times 10^{33}\text{ gm} = \text{solar mass}$

recall from problem I.3:



$$g(z) = -\frac{\partial \Phi}{\partial z} = -4\pi G \rho z$$

using Gauss law

so $\frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \rho = \nu_0^2$ $G = 6.67 \times 10^{-8}\ cm^3/gm\,sec^2$

so $\nu_0 = \sqrt{4\pi G \rho}$ where $\rho_0 = 0.16\ M_\odot/pc^3$

$= 1.1 \times 10^{-23}\ gm/cm^3$

$= 3 \times 10^{-15}\ sec^{-1}$

so $T_z = \frac{2\pi}{\nu_0} = 2 \times 10^{15}\ sec$

$\simeq 70$ myrs

$=$ Sun's vertical oscillation period

Section 5.3 of text shows how to estimate the Sun's orbit period $T_{orb} = \frac{2\pi}{\Omega_0} \simeq 230$ myrs

so $T_z \simeq \frac{T_{orb}}{3.5}$ ie Sun will executed 3.5 vertical oscillation for every longitudinal orbit about the Sun.

that section also shows that Sun's precession period is

$$T_{\tilde{\omega}} = \left|\frac{2\pi}{\dot{\tilde{\omega}}}\right| \simeq 2.9\ T_{orb}$$

So Sun's $\tilde{\omega}$ rotates once every ~3 orbits

⇒ in disk galaxies, precession is relatively fast

ie pression periods are comparable to a star's orbit period

Also keep in mind that the universe is ~15 Gyrs old, so the Sun has orbited the Galaxy only

$$N \sim \frac{15\,Gyr}{230\,Myr} \sim 50 \text{ times}$$

⇒ galaxies (which are physically ancient) are also dynamically young... stars have only existed long enough to make a few tens of orbits about the Galaxy

whereas the Earth has orbited the Sun ~4.5×10^9 times

⇒ planetary systems (which are younger than a galaxy) are dynamically ancient

The orbits of Solar System planets have had lots of time (ie orbits) to settle down to their current see also section 5.3.1 which derives the so-called Oort Constants:

which are actually the functions

$$A(r) = -\frac{1}{2}\left(r\frac{d\Omega}{dr}\right)$$

$$\text{and } B(r) = A(r) - \Omega(r)$$

Galactic astronomers can measure $A(r)$ and $B(r)$, which are related to Galaxy's rotation curve

$v_\theta(r) = r\Omega(r)$

which is governed by the Galaxy's matter distribution... more matter in the galactic disk, the faster the stars orbit

Anyway, some papers on planetary dynamics will sometimes make use of $A(r)$ and $B(r)$ instead of $\Omega(r)$ and $\kappa(r)$

but these are all related via

$$r\frac{d\Omega}{dr} = -2A$$

$$\text{so } \kappa^2 = 4\Omega^2 + r\frac{d\Omega^2}{dr} = 4\Omega^2 + 2\Omega\left(r\frac{d\Omega}{dr}\right)$$

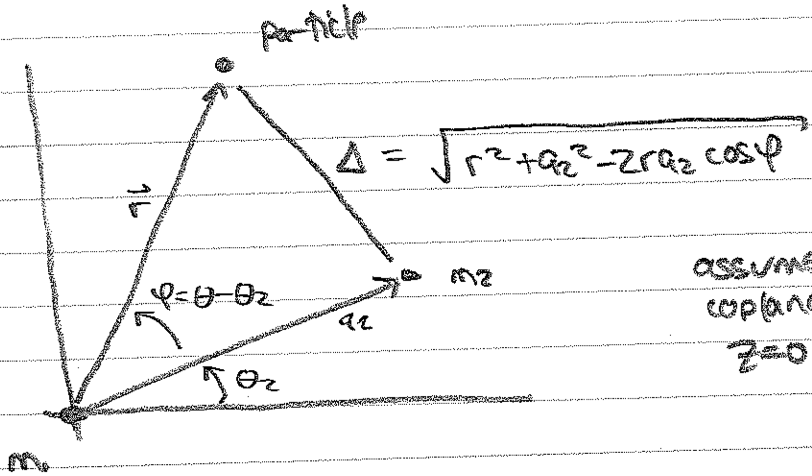
$$= 4\Omega^2 - 4\Omega A = 4\Omega^2 - 4\Omega(B+\Omega)$$

$$\text{so } \kappa^2 = -4B\Omega$$

so if you are reading a paper on planetary dynamics, and it invokes Oort's A, B 'constant's', you can use the above to replace $A, B \to \kappa, \Omega$

Precession in a planetary system:

Suppose the particle is perturbed by secondary m_2 that is in circular orbit about m_1:



assume coplanar, $z = 0$

The system's total gravitation potential is

$$\Phi = \Phi_1 + \Phi_2 = -\frac{Gm_1}{r} - \frac{Gm_2}{\Delta}$$

note that Φ_2 is periodic in φ = particle's relative longitude lets Fouir-expand Φ_2:

$$\Phi_2(r,\varphi) = -\frac{Gm_2}{\Delta} = \frac{1}{2}\varphi_0(r) + \sum_{m=1}^{\infty}\left[\varphi_m(r)\cos(m\varphi) + \eta_m(r)\sin(m\varphi)\right]$$

$\varphi_m(r)$ and $\eta_m(r)$ = m^{th} Fourier coefficients in the Fourier expansion of Φ_2

is Φ_2 even or odd in φ?

what are the η_m then?

Note that $\varphi_0(r)$ is the <u>axisymmetric</u> part of m_2's potential, This is the part of Φ_2 that will cause the particle's orbit to precess.

Solve for φ_0 by averaging the above over all relative longitudes φ:

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{Gm_2}{\Delta}\,d\varphi' = \frac{1}{2\pi}\int_{-\pi}^{\pi}d\varphi\left[\frac{1}{2}\varphi_0(r) + \sum_{m=1}^{\infty}\varphi_m(r)\cos(m\varphi')\right]$$

$$= ?$$

so $\frac{1}{2}\Phi_0(r) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{Gm_2 \, d\varphi'}{\sqrt{r^2 + a_2^2 - 2ra_2\cos\varphi'}}$

$$= -\frac{Gm_2}{\pi a_2}\int_0^{\pi} \frac{d\varphi'}{\sqrt{1+\beta^2-2\beta\cos\varphi}}$$

where $\beta = \frac{r}{a_2}$

The above integral is related to the
<u>Laplace coefficient</u>:

$$b_s^{(m)}(\beta) = \frac{2}{\pi}\int_0^{\pi} \frac{\cos(m\phi)\, d\phi}{(1+\beta^2-2\beta\cos\phi)^s}$$

so $\frac{1}{2}\Phi_0(r) = -\frac{Gm_2}{2a_2} b_{1/2}^{(0)}(\beta)$

so the system's total gravitational potential is

$\Phi(r,\varphi) = \Phi_1 + \Phi_2$

$$= -\frac{Gm_1}{r} - \underbrace{\frac{Gm_2}{2a_2} b_{1/2}^{(0)}(\beta)}_{\text{axisymmetric part}} + \text{secondary's non-axisymmetric terms}$$

The secondary's non-axisymmetric terms do not contribute much to particle's precession rate... rather, those terms give rise to mean-motion or Lindblad resonances, which will be examined in greater detail in Chapter 6.

So to calculate the particle's $\dot{\tilde{\omega}}$, we can use

BTW, this term is responsible for a secular resonance

$$\Phi = -\frac{Gm_1}{r} - \frac{Gm_2}{2a_2} b^{(0)}_{1/2}(\beta) \quad \text{for } \beta = \frac{r}{a_2}$$

~~where~~

so $$\Omega^2 = \frac{1}{r}\frac{d\Phi}{dr} = \frac{Gm_1}{r^3} - \frac{Gm_2}{2a_2 r}\frac{db^{(0)}_{1/2}}{dr}$$

use chain rule

$$\frac{db^{(0)}_{1/2}}{dr} = \frac{db^{(0)}_{1/2}}{d\beta}\frac{d\beta}{dr} = \frac{1}{a_2}\frac{db^{(0)}_{1/2}}{d\beta}$$

Also set $n_0^2 = \frac{Gm_1}{r^3}$ = mean motion^2

so $$\Omega^2 = n_0^2 - \frac{Gm_2}{2a_2^2 r}\frac{db^{(0)}_{1/2}}{d\beta}$$

$$= n_0^2\left(1 - \frac{1}{2}\frac{m_2}{m_1}\beta^2\frac{db^{(0)}_{1/2}}{d\beta}\right)$$

set $\mu_2 = \frac{m_2}{m_1}$

Note $\mu_2 \ll 1$ for planetary systems so $\quad \Omega \simeq n_0\left(1 - \frac{1}{4}\mu_2 \beta^2 \frac{\partial b^{(0)}_{1/2}}{\partial \beta}\right)$

Inspect Fig 5.6 ... does m_2's gravity slow down or speed up the particle's mean angular velocity?

The particle's precession rate is $\dot{\tilde{\omega}} = \Omega - \kappa$
so we also need κ:

$$\kappa^2 = 3\Omega^2 + \frac{\partial^2 \Phi}{\partial r^2} = \underbrace{4\Omega^2 + r\frac{\partial \Omega^2}{\partial r}}_{\text{will use this formula}}$$

$r = \beta a_2$
so $r\frac{\partial}{\partial r} \to \beta\frac{\partial}{\partial \beta}$

$$\text{so } \kappa^2 = 4n_0^2\left(1 - \frac{1}{2}\mu_2\beta^2\frac{\partial b_{1/2}^{(0)}}{\partial \beta}\right)$$

$n_0^2 = \frac{Gm_1}{r^3}$

$$+ \underbrace{r\frac{\partial}{\partial r}\left[n_0^2\left(1 - \frac{1}{2}\mu_2\beta^2\frac{\partial b_{1/2}^0}{\partial \beta}\right)\right]}_{-3n_0^2\left(1 - \frac{1}{2}\mu_2\cdots\right) - n_0^2\,\beta\frac{\partial}{\partial \beta}\left(\frac{1}{2}\mu_2\beta^2\frac{\partial b_{1/2}^0}{\partial \beta}\right)}$$

$$= n_0^2 - \frac{1}{2}\mu_2\beta^2\frac{\partial b_{1/2}^{(0)}}{\partial \beta}n_0^2 - \frac{1}{2}\mu_2\left(2\beta^2\frac{\partial b_{1/2}^0}{\partial \beta} + \beta^3\frac{\partial^2 b_{1/2}^0}{\partial \beta^2}\right)n_0^2$$

$$\text{so } \kappa^2 = n_0^2\left[1 - \frac{1}{2}\mu_2\beta^2\left(3\frac{\partial b_{1/2}^{(0)}}{\partial \beta} + \beta\frac{\partial^2 b_{1/2}^{(0)}}{\partial \beta^2}\right)\right]$$

$$\text{and } \kappa \simeq n_0\left[1 - \frac{1}{4}\mu_2\beta^2\left(3\frac{\partial b_{1/2}^{(0)}}{\partial \beta} + \beta\frac{\partial^2 b_{1/2}^{(0)}}{\partial \beta^2}\right)\right]$$

so the particle's precession rate $\tilde{\omega} = \Omega - \kappa$ is

$$\tilde{\omega} = \frac{1}{2}\mu_2 \beta^2 \left(\frac{\partial b_{1/2}^{(0)}}{\partial \beta} + \frac{1}{2}\beta \frac{\partial^2 b_{1/2}^{(0)}}{\partial \beta} \right) \Omega_0$$

which is kinda complicated-looking, but can be simplified further via Laplace coefficient identities:

$$\frac{\partial b_s^{(m)}}{\partial \beta} = \frac{\partial}{\partial \beta} \frac{2}{\pi} \int_0^\pi \frac{\cos(m\phi)\, d\phi}{(1+\beta^2-2\beta\cos\phi)^s}$$

$$= \frac{2}{\pi} \int_0^\pi \frac{d\phi \cos(m\phi)(-s)(2\beta - 2\cos\phi)}{(1+\beta^2-2\beta\cos\phi)^{s+1}}$$

$$= (-2s)\frac{2}{\pi}\int_0^\pi d\phi \frac{\cos(m\phi) - \cos m\phi \cos\phi}{(1+\beta^2-2\beta\cos\phi)^{s+1}}$$

A.6

$$\frac{1}{2}\cos(m+1)\phi + \frac{1}{2}\cos(m-1)\phi$$

so $$\frac{\partial b_s^{(m)}}{\partial \beta} = -2s\left(-\frac{1}{2} b_{s+1}^{m+1} + b_{s+1}^{m} - \frac{1}{2} b_{s+1}^{m-1} \right)$$

$$= s\left(b_{s+1}^{(m-1)} - 2b_{s+1}^{(m)} + b_{s+1}^{(m+1)} \right)$$

and $$\frac{\partial b_{1/2}^{(0)}}{\partial \beta} = \frac{1}{2}\left(b_{3/2}^{(-1)} - 2b_{3/2}^{(0)} + b_{3/2}^{(1)} \right)$$

$$= b_{3/2}^{(1)} - b_{3/2}^{(0)}$$

Another useful identity:

$$2\frac{\partial b_{1/2}^{(0)}}{\partial \beta} + \beta\frac{\partial^2 b_{1/2}^{(0)}}{\partial \beta^2} = b_{3/2}^{(1)}(\beta)$$

extra credit, +10 points on Assignment #5

find an <u>easy</u> proof for the above
"easy enough" for textbook

so $\dot{\tilde{\omega}} = \frac{1}{2}\mu_2\beta^2\left(\frac{\partial b_{1/2}^{(0)}}{\partial\beta} + \frac{1}{2}b_{3/2}^{(1)} - \frac{\partial b_{1/2}^{(0)}}{\partial\beta}\right)n_0$

$\Rightarrow \dot{\tilde{\omega}} = \frac{1}{4}\mu_2\beta^2 b_{3/2}^{(1)}(\beta)\, n_0$

which is always positive.

So μ_2's gravity causes particle's $\tilde{\omega}$
to advance over time,
at a rate that is faster nearer m_2's orbit
where $\beta \to 1$ and $b_{3/2}^{(1)}(\beta)$ blows up.

If there where multiple planets in the system, then

$$\Phi \rightarrow -\frac{Gm_c}{r} - \sum_{j=2}^{N} \frac{Gm_j}{2a_j}\, b^{(0)}_{1/2}(\beta_j)$$

and $$\dot{\tilde{\varpi}} \rightarrow \frac{1}{4} \sum_{j}^{\text{planets}} \mu_j \beta_j^2\, b^{(1)}_{3/2}(\beta_j)\, n_0$$

$\beta_j = \frac{r}{a_j}$

a_j = planet j's sma.