30 September 2013

Chapter 4: Restricted 3-Body Problem

Assignment #3, problems 2.11, 3.2, 3.5, 3.6, 4.1, 4.2, 4.3
due Thurs Oct 10

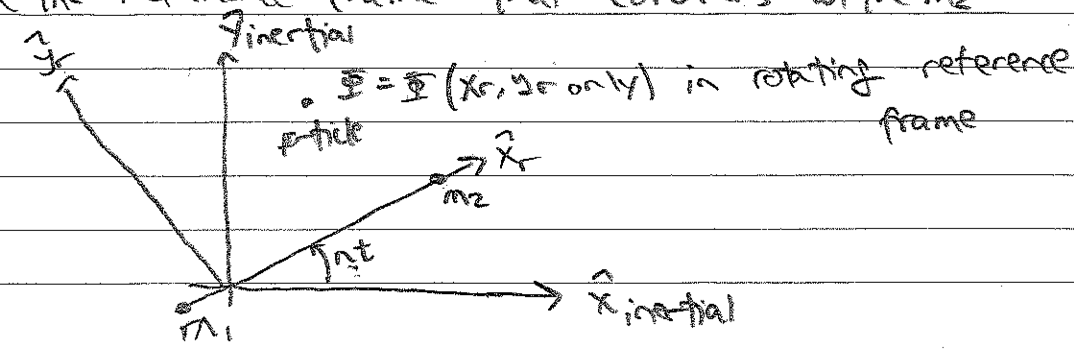
The circular restricted 3-body problem = study of the motion of massless particle orbiting a primary and perturbed by secondary in circular orbit... the next step in complexity beyond 2-body problem

circ' restr' 3-body problem (CR3BP) has a very useful integral of the motion:

Jacobi Integral

but lets consider an even more general problem: the motion of a particle in a gravitational potential $\Phi(\vec{r})$ that is stationary in a steadily rotating reference frame, ie $\Phi = \Phi(\vec{r})$ only and is independent of time in that rotating reference frame.

example: the CR3BP, since $\Phi = \Phi(\vec{r}$ only$)$ in the reference frame that corotates with m_2



The particle's equation of motion (EOM) in the rotating coordinate system is (see section 1.5 of text)

centrifugal

$$\ddot{\vec{r}}_r = -\nabla\Phi - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2\vec{\omega}\times\dot{\vec{r}}_r$$

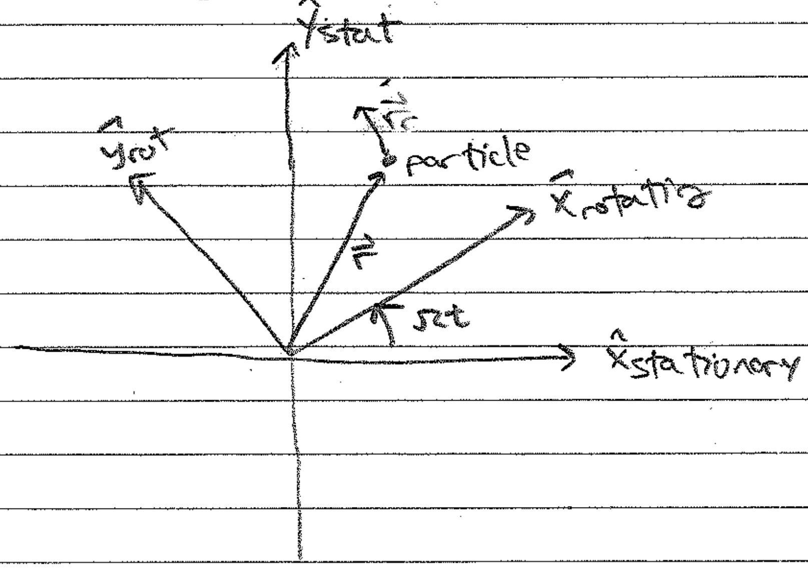
coriolis acceleration

where $\vec{r}$ = particle's position vector

$\dot{\vec{r}}_r$ = its velocity measured wrt rotating axes

$\vec{\omega} = \Omega\hat{z}$ = rotation axis

Ω = constant angular velocity



Use cylindrical coordinates: $\vec{r} = r\hat{r} + z\hat{z}$

centrifugal $-\vec{\omega}\times(\vec{\omega}\times\vec{r}) = -\Omega^2 r\,\underbrace{\hat{z}\times(\hat{z}\times\hat{r})}_{?}$

$= r\Omega^2\hat{r}$

$= \nabla\frac{1}{2}(r\Omega)^2$

$= \nabla\frac{1}{2}(\vec{\omega}\times r)^2$

recall relationship between velocity measured in rotating & stationary ref' frame:

$$\underset{\text{rot}}{\dot{\vec{r}}_r} = \underset{\text{stationary}}{\dot{\vec{r}}_s} - \vec{\omega} \times \vec{r} \qquad \text{(Section 1.5)}$$

so $$\ddot{\vec{r}}_r = -\nabla \Big[\underbrace{\Phi - \tfrac{1}{2}(\vec{\omega} \times \vec{r})^2}_{} \Big] - 2\vec{\omega} \times \dot{\vec{r}}_r$$

$U_{eff}(\vec{r})$ = gravity + centrifugal effective potential

recall that the 2-body energy integral was obtained by integrating $\dot{\vec{r}} \cdot \ddot{\vec{r}}$.

So lets consider

$$\dot{\vec{r}}_r \cdot (\ddot{\vec{r}}_r + \nabla U_{eff}) = -2\dot{\vec{r}}_r \cdot (\vec{\omega} \times \dot{\vec{r}}_r)$$

$$= -2\vec{\omega} \cdot (\dot{\vec{r}}_r \times \dot{\vec{r}}_r) = 0$$

also note that $\dot{\vec{r}}_r \cdot \nabla U_{eff} = \dfrac{dU_{eff}}{dt}$

where $U_{eff} = U_{eff}(x_r(t), y_r(t), z_r(t))$

U_{eff} has no explicit time dependance, but particle's trajectory $\vec{r}(t) = \vec{r}_s(t)$ implicit time dependance.

check: $\dfrac{dU_{eff}}{dt} = \dfrac{dU_{eff}}{dx_r}\dfrac{dx_r}{dt} + \dfrac{dU_{eff}}{dy_r}\dfrac{dy_r}{dt} + \dfrac{dU_{eff}}{dz_r}\dfrac{dz_r}{dt}$

$= \dot{\vec{r}}_r \cdot (\nabla U_{eff})$ since $x_r = x$ etc

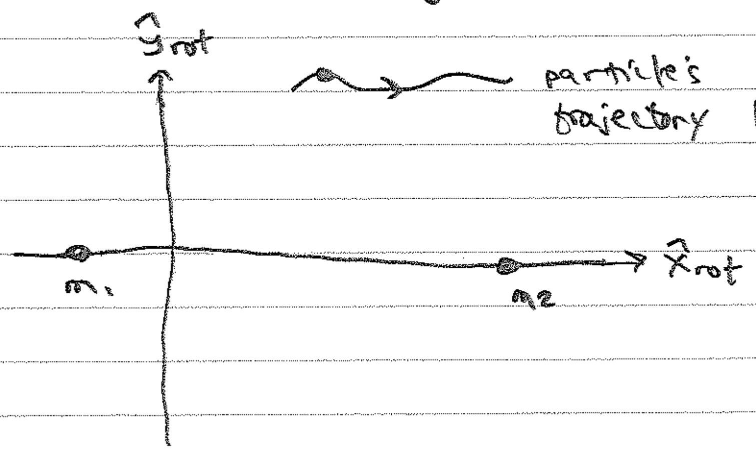
so $\dot{\vec{r}}_r \cdot (\ddot{\vec{r}}_r + \nabla U_{eff}) = \dfrac{d}{dt}\underbrace{\left(\tfrac{1}{2}\dot{\vec{r}}_r^{\,2} + U_{eff}\right)}_{J = \text{Jacobi integral,}} = 0$

where

$J = \frac{1}{2}\dot{\vec{r}}_r^{\,2} + U_{eff} = \frac{1}{2}\dot{\vec{r}}_r^{\,2} + \Phi - \frac{1}{2}(\vec{\omega}\times\vec{r})$

= moving particles' constant Jacobi integral

example: CR3BP



J provides useful constraint on particle's $\vec{r}$ and $\dot{\vec{r}}_r$ as it roams system's potential $\Phi(\vec{r})$

Its also useful to express J in terms of particle's velocity $\dot{\vec{r}}_s = \dot{\vec{r}}_r + \vec{\omega} \times \vec{r}$ in stationary reference frame.

$$\text{so } \dot{\vec{r}}_r^{\,2} = \left(\dot{\vec{r}}_s - \vec{\omega} \times \vec{r} \right)^2$$

$$= \dot{\vec{r}}_s^{\,2} - 2\dot{\vec{r}}_s \cdot (\vec{\omega} \times \vec{r}) + (\vec{\omega} \times \vec{r})^2$$

$$= \dot{\vec{r}}_s^{\,2} - 2\vec{\omega} \cdot \vec{r} \times \dot{\vec{r}}_s + (\vec{\omega} \times \vec{r})^2$$

but $\vec{H} = \vec{r} \times \dot{\vec{r}}_s =$ particle's specific angular momentum

(not $\vec{r} = \vec{r}_r$ since origin is @ rotation axis)

$$\text{so } \dot{\vec{r}}_r^{\,2} = \dot{\vec{r}}_s^{\,2} - 2\vec{\omega} \cdot \vec{H} + (\vec{\omega} \times \vec{r})^2$$

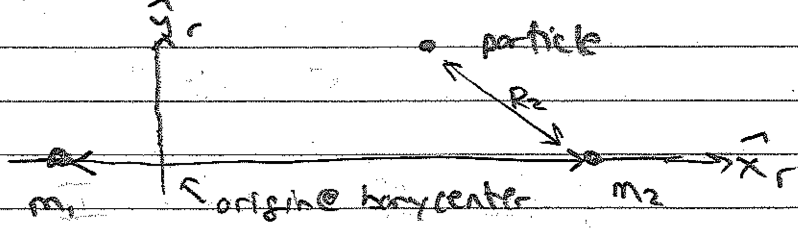
$$\text{and } J = \tfrac{1}{2}\dot{\vec{r}}_s^{\,2} + \Phi - \vec{\omega} \cdot \vec{H}$$

but $E = \frac{1}{2}\dot{\vec{r}}_s^{\,2} + \Phi =$ particle's specific energy in stationary frame

$$\Rightarrow J = E - \vec{\omega} \cdot \vec{H}$$

so the particle's energy and angular momentum are no conserved individually (as in 2-body problem), but the above combination of E and H are conserved.

problem 4.1 considers J for the CR3BP:



a_2 = secondary's sm axis

E_2 = system's specific energy in barycentric coordinate system

Prob 4.1: you will show that

$$\frac{J}{E_2} = J' = \text{dimensionless jacobi integral}$$

$$= \frac{a_2}{a} + 2\sqrt{\left(1+\frac{m_2}{m_1}\right)\frac{a}{a_2}(1-e^2)}\cos i + 2\frac{m_2}{m_1}\frac{a_2}{R_2}$$

where a, e, i = particle's orbit elements.

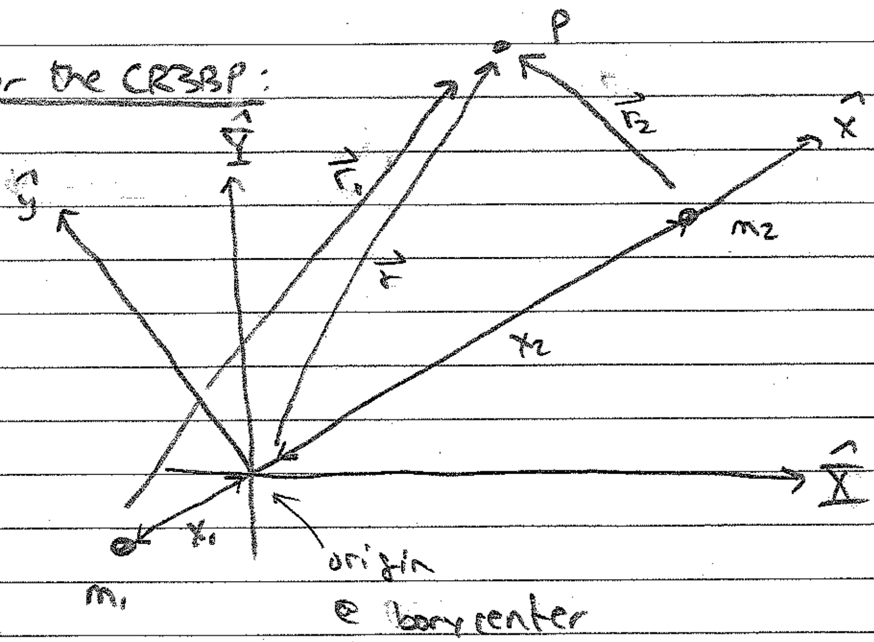
In most planetary systems, $m_2 \ll m_1$
(Jupiter has $m_2 \sim 0.001 m_1$)

and if the particle stays far enough away from m_2
(ie no gravitational scattering off m_2 or flybys)
such that $R_2 \gg \frac{2m_2}{m_1} a_2$

Then $J' \simeq \frac{a_2}{a} + 2\sqrt{\frac{a}{a_2}(1-e^2)}\cos i \equiv T$

also known as the Tisserand parameter

J for the CR3BP:



$\hat{X}, \hat{Y}$ are stationary axes

$\hat{x}, \hat{y}$ rotate about $\vec{\omega} = n\hat{z}$

$n = \sqrt{\frac{G(m_1+m_2)}{a^3}}$ = secondary's mean motion

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$
= particle P's location in rotating ref frame

V_{eff} = effective potential at particle P's location

$$= \Phi - \frac{1}{2}(\vec{\omega}\times\vec{r}) = -\frac{Gm_1}{r_1} - \frac{Gm_2}{r_2} - \frac{1}{2}n^2(x^2+y^2)$$

the particle's J is $J = \frac{1}{2}v_r^2 + V_{eff}$

the origin is at system's center of mass (com)

$$m_1 x_1 = m_2 x_2$$

also a = m_2's semimajor axis = $x_1 + x_2$

so $m_1 x_1 = m_2 (a - x_1) \Rightarrow x_1 = \dfrac{m_2 a}{m_1 + m_2}$

$$= \mu_2 a$$

where $\mu_i = \dfrac{m_i}{m_1 + m_2}$

Note that $\mu_1 + \mu_2 = 1$

and $x_2 = \mu_1 a$

Note a = system's natural unit of length

$a\Omega$ = unit of velocity

$(a\Omega)^2$ = unit of specific energy $= \dfrac{G(m_1+m_2)}{a}$

Form dimensionless quantities

$$v_r' = \frac{v_r}{a\Omega}$$

$$J' = -\frac{J}{(a\Omega)^2} = -\frac{1}{2}\left(\frac{v_r}{a\Omega}\right)^2 + \frac{G m_1 a}{r_1 G(m_1+m_2)}$$

$$+ \frac{G m_2 a}{r_2 G(m_1+m_2)} + \frac{1}{2}\left(\frac{x^2+y^2}{a^2}\right)$$

so $J' = -\frac{1}{2} v_r'^2 + \frac{\mu_1}{r_1'} + \frac{\mu_2}{r_2'} + \frac{1}{2}(x'^2 + y'^2)$

or $J' = -\frac{1}{2} v_r'^2 + U'$

where $U' = \frac{-V_{eff}}{(an)^2} = \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} + \frac{1}{2}(x'^2 + y'^2)$

= dimensionless V_{eff}

henceforth drop the primes.

The above can be written $U(x,y,z) = J + \frac{1}{2} v_r^2$

Zero velocity curves

since $v_r^2 > 0$,
the particles trajectory must satisfy

$$U(x,y,z) \geq J$$

if there are regions in 3D space that do not satisfy the above, then that region is Jacobi forbidden, that particle may not cross the surface that satisfies

$$U(x,y,z) = \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} + \frac{1}{2}(x^2+y^2) = J$$

where the dimensionless

$$r_1 = \sqrt{\left(x + \frac{x_1}{a}\right)^2 + y^2 + z^2}$$

$$= \sqrt{(x+\mu_2)^2 + y^2 + z^2}$$

and

$$r_2 = \sqrt{\left(x - \frac{x_2}{a}\right)^2 + y^2 + z^2}$$

$$= \sqrt{(x-\mu_1)^2 + y^2 + z^2}$$

The zero-velocity surface is the 3D surface that satisfies $U(x,y,z) = J$

The particle's J is conserved, and it is confined to the region that satisfies

$$U(x,y,z) \geq J$$

so the zero-velocity surface (where $U=J$) bounds a region where the particle is excluded

The zero-velocity curve is the slice through the Z-V surface in the $z=0$ plane.

the Z-V curve is NOT an orbit, it is a boundary that the particle may not cross

(unless for example the particle briefly fires its engines, which changes J and results in a new Z-V curve)

See Fig 4.2

the particle is allowed to roam where $U(x,y) \geq J$, white regions in Fig 4.2. Grey is off limits stop Oct 1

Lagrange Equilibrium Points:

The EOM for the particle in the rotating reference frame is

$$\ddot{\vec{r}}_r = -\nabla V_{eff} - 2\vec{\omega} \times \dot{\vec{r}}_r$$

an equilibrium site is where force $m\ddot{\vec{r}} = 0$, so equilibrium occurs where $\ddot{\vec{r}} = 0 = \dot{\vec{r}}_r$ and where

$$\nabla V_{eff}(x,y,z) = 0$$

where $$V_{eff} = -(an)^2\left[\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} + \frac{1}{2}(x^2+y^2)\right]$$

so $$\frac{dV_{eff}}{dx} = -(an)^2\left[-\frac{\mu_1}{r_1^2}\frac{dr_1}{dx} - \frac{\mu_2}{r_2^2}\frac{dr_2}{dx} + x\right] = 0$$

$$= (an)^2\left[\frac{\mu_1(x+\mu_2)}{r_1^3} + \frac{\mu_2(x-\mu_1)}{r_2^3} - x\right] = 0$$

and $$\frac{dV_{eff}}{dy} = (an)^2\left(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3} - 1\right)y = 0$$

and $$\frac{dV_{eff}}{dz} = (an)^2\left(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right)z = 0$$

The <u>Lagrange equilibrium points</u> (LEP) satisfy the above

So the LEP are in the $z=0$ plane

some LEP lie on $y=0$ axis (the $m_1 - m_2$ line)
$\Rightarrow$ these are the <u>collinear LEP</u>

where $r_1 = |x+\mu_2| = s_1(x+\mu_2)$

$s_1 = \text{sign}(x+\mu_2) = \pm 1$

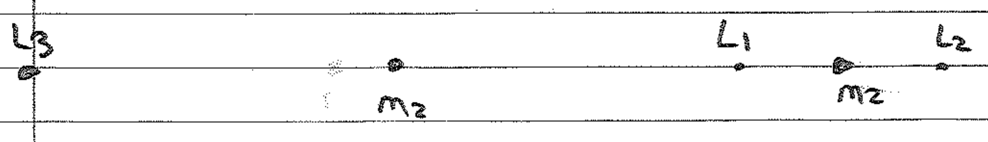
$r_2 = s_2(x-\mu_1)$

insert this into $\partial U/\partial x = 0$:

$$\frac{s_1\mu_1}{(x+\mu_2)^2} + \frac{s_2\mu_2}{(x-\mu_1)^2} = x$$

solve this numerically

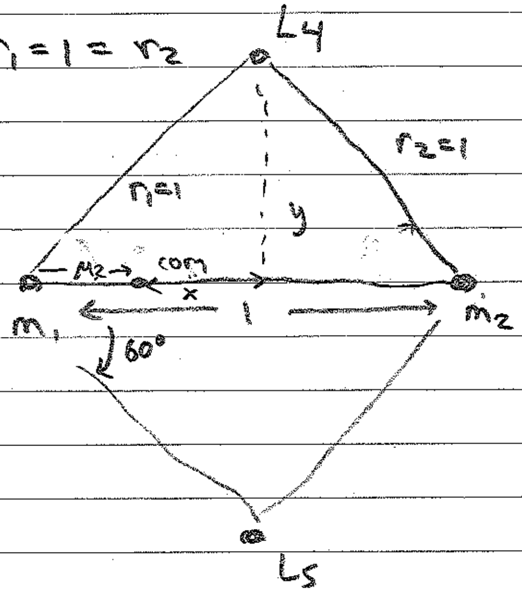
This equation has 3 roots for x,
those sites are the L_1, L_2, L_3 LEP:



two other LEP lie off the $|y|>0$ axis where

$$\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3} = 1 = \mu_1 + \mu_2$$

$\Rightarrow r_1 = 1 = r_2$



L_4, L_5 are the triangular equilibrium sites, they lead/trail the secondary by $\pm 60°$

$\mu_2 + x = \frac{1}{2}$

so $x = \frac{1}{2} - \mu_2$

and $\sin \pm 60° = \pm \frac{\sqrt{3}}{2} = y$

See Fig 4.3

The force on a particle at a LEP is zero, but nonzero just off the LEP.

If a miniscule perturbation nudges the particle away from the LEP,

Those forces will either:

i) cause the particle to oscillate about the LEP → that LEP is a stable equilibrium

or ii) drive the particle away from the unstable LEP

Fig 4.3 - which LEP sites are stable, unstable?

a zero-velocity curve that passes through the unstable LEP is called the separatrix that divides very distinct types of trajectories

so colinear $L_1 L_2 L_3$ sites are unstable,

the triangular $L_4 L_5$ sites are stable displace particle from L_1 or L_2 and it could go into orbit about m_1 or m_2.

displace a particle from L_3,
and it enters a <u>horseshoe orbit</u>,
blue curves in Fig 4.3

triangular L_4, L_5 points are stable
when $\mu_2 \lesssim 0.039$ (ie $m_2 < 40$ Jupiter mass orbiting sun)

displace particle from L_4, L_5 and it enters tadpole orbit (red curves)

Know any objects in horseshoe or tadpole orbits?

see Fig 4.5

Hill Sphere = volume of space where m_2's gravity dominates the particle's motion

R_H = radius of Hill sphere
= L_1 or L_2's distance from m_2

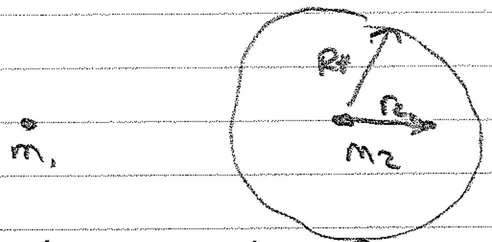
when $m_2 \ll m_1$,

$$R_H = \left(\frac{m_2}{3m_1}\right)^{1/3} a$$

which you derive from

$$\frac{s_1\mu_1}{(x+\mu_2)^2} + \frac{s_2\mu_2}{(x-\mu_1)^2} = x \quad \text{in this limit}$$

See problem 4.3



Satellites of planets live deep inside the Hill Sphere, ie

$$r_2 \ll R_H$$

in this case, the primary's gravity is only a weak perturbation, and the particle's motion is nearly 2-body motion about m_2

If however the satellite ever gets out to $r_2 \sim R_H$, the primary's gravity cannot be ignored, the motion is 3-body

Alternatively, if the particle is orbiting the primary, but happens to travel within $r_2 \sim R_H$ of secondary

The particle will recieve a large kick from m_2, This is gravitational scattering

Jupiter is the main scatterer of comets

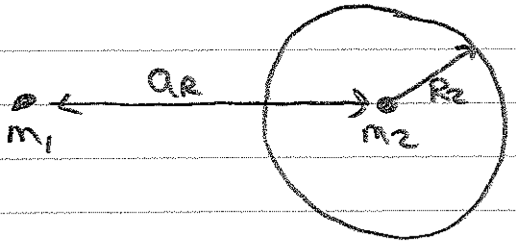
Roche limit: Note that $R_H \propto a_2$

$\uparrow$ secondary's sma

what if the secondary orbited too close to the primary, such that the secondary's physical radius $R_2 < R_H$?

$\Rightarrow$ tidal disruption

The secondary would shed mass... where would that go?



R_2 = secondary's radius

solve for a_R = Roche limit
= semimajor axis of secondary that fills its Hill sphere,
$R_2 = R_H$

$$R_H = \left(\frac{m_2}{3m_1}\right)^{1/3} a_R = R_2$$

assume uniform density spheres ... is that a good approximation? $m_i = \frac{4\pi}{3}\rho_i R_i^3$

$$\left(\frac{\rho_2}{3\rho_1}\right)^{1/3} \frac{R_2}{R_1} a_R \approx R_2$$

$$\Rightarrow a_R \approx \left(\frac{3\rho_1}{\rho_2}\right)^{1/3} R_1$$

$$= 1.44 \left(\frac{\rho_1}{\rho_2}\right)^{1/3} R_1$$

This is only a ballpark estimate of a_R. For instance, a formal stability analysis assuming inviscid incompressible sphere yields coefficient $= 2.42$

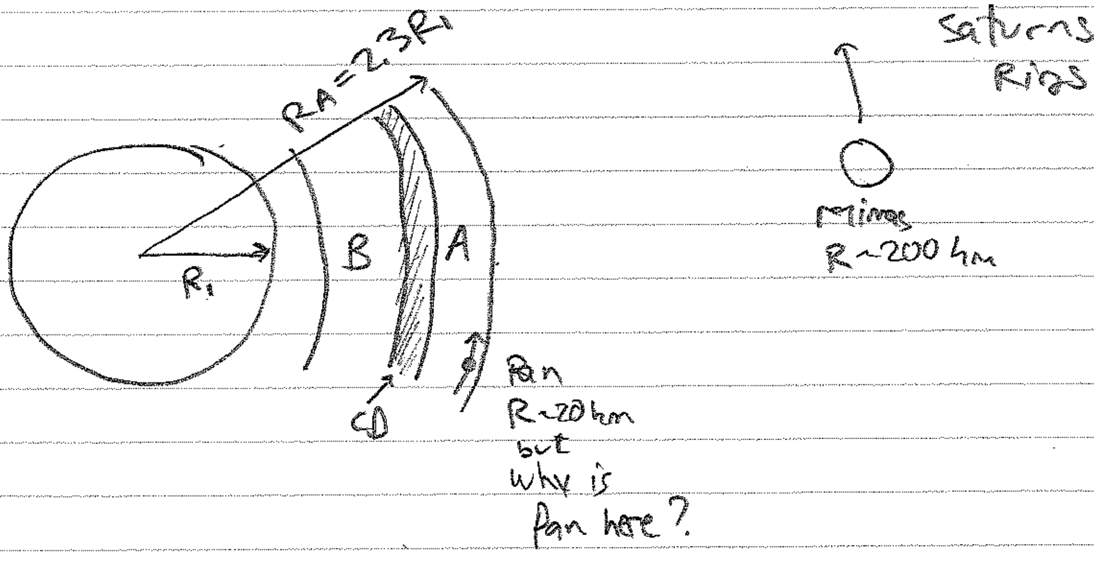
Point is, if you are in circular orbit smaller than $a_R \sim 2R_1$ ie 2 radii of primary, you are in danger of being tidally disrupted.

binary stars must orbit further than ~2 stellar radii to avoid exchanging mass (no surprise there)

But all major satellites orbit beyond ~2.4R_1

example: Saturn Rings

what does this suggest about Saturns Rings

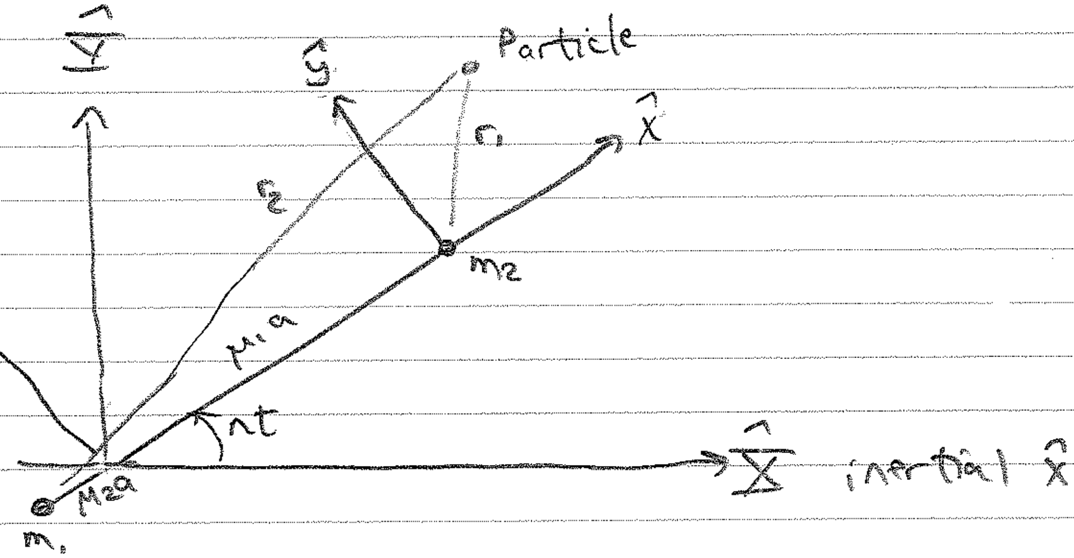


stop Oct 3

Hill's Equations = approx EOM, used to study motion of particles near secondary where $r_2 \ll r_1$

used to study: accretion of planetesimals (proto comets, asteroids, seeds of planets) from solar nebula (disk around young Sun from which planets formed)

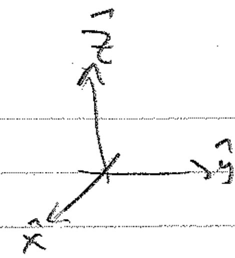
evolution of planetary rings, satellites embedded in rings (eg Pan)



The EOM in rotating ref frame is

$$\ddot{\vec{r}} = -\nabla U_{eff} - 2\vec{\omega} \times \dot{\vec{r}}$$ (dropping r subscript)

where $$U_{eff} = -(an)^2 \left[\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} + \frac{1}{2}(x^2+y^2) \right]$$



$$r_1 = \sqrt{(x+\mu_2)^2 + y^2 + z^2}$$

$$r_2 = \sqrt{(x-\mu_1)^2 + y^2 + z^2}$$

but keep in mind that all lengths are in units of $\underline{a}$

coriolis acceleration: $\vec{\omega} = n\hat{z}$

secondary's mean angular velocity

$$\text{so } -2n\hat{z}\times\left(\dot{x}\hat{x} + \dot{y}\hat{y}\right) = 2n\dot{y}\vec{x} - 2n\dot{x}\vec{y}$$

so set $a=1$

also assume system is coplanar $z=0$

$$\text{so } \ddot{x} = 2n\dot{y} + \frac{d}{dx}\left[\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} + \frac{1}{2}(x^2+y^2)\right](an)^2$$

$$= 2n\dot{y} + n^2\left[-\frac{\mu_1}{r_1^2}\frac{dr_1}{dx} - \frac{\mu_2}{r_2^2}\frac{dr_2}{dx} + x\right]$$

$$= 2n\dot{y} + n^2\left[-\frac{\mu_1}{2r_1^3}2(x+\mu_2) - \frac{\mu_2}{2r_2^3}2(x-\mu_1) + x\right]$$

$$\ddot{x} = 2n\dot{y} - n^2\left[\frac{\mu_1(x+\mu_2)}{r_1^3} + \frac{\mu_2(x-\mu_1)}{r_2^3} - x\right]$$

see page 12 notes

and $\ddot{y} = -2n\dot{x} + n^2\left[-\frac{\mu_1}{2r_1^2}\frac{\partial r_1}{\partial y} - \frac{\mu_2}{2r_2^2}\frac{\partial r_2}{\partial y} + y\right]$

$$= -2n\dot{x} - n^2\left(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3} - 1\right)y$$

(see pg 12)

where $\mu_i = \frac{m_i}{m_1+m_2}$

Next, move origin to m_2:

$$x = \mu_1 + x' \quad \text{and } y' = y$$

so $\hat{x}'$ points radially away
$\hat{y}'$ points along m_2's motion

assume $m_2 << m_1$ so $\mu_1 \simeq 1$
$\mu_2 \simeq \frac{m_2}{m_1} << 1$

Assume the moving particle is in vicinity of m_2
$\Rightarrow |x'|$ and $|y'| << 1$

so linearize the EOM — keep terms $O(x')$, drop $O(y'^2)$ and $O(x'y')$

$$r_1 = \sqrt{(1+x')^2 + y'^2} \simeq \sqrt{1+2x'} \simeq 1+x'$$

and $r_1^{-3} \simeq 1-3x'$

and $r_2 = \sqrt{x'^2 + y'^2}$ which I rename $r' = \Delta'$

so

$$\ddot{x}' = 2n\dot{y}' - n^2\left[(1+x')(1-3x') + \frac{\mu_2 x'}{\Delta'^3} - 1 - x'\right]$$

$$\simeq 2n\dot{y}' + n^2\left(3 - \frac{\mu_2}{\Delta'^3}\right)x'$$

Δ = particle's distance from m_2

and $$\ddot{y}' \simeq -2n\dot{x}' + n^2\left(3x' - \frac{\mu_2}{\Delta'^3}\right)y'$$

We are particularly interested in the motion of a particle in the vicinity of m_2's Hill sphere,

so $\Delta' \lesssim$ a few $\times \frac{R_H}{3} \sim$ few $\times \left(\frac{\mu_2}{3}\right)^{1/3}$

so Δ', x', y' are of order $\mu_2^{1/3} \ll 1$

while $\frac{\mu_2}{\Delta'^3} \sim O(1) \gg |x'|$

so EOM for particle in vicinity of secondary is

$$\left.\begin{aligned} \ddot{x} &= 2n\dot{y} + n^2\left(3 - \frac{\mu_2}{\Delta^3}\right)x \\ \ddot{y} &= -2n\dot{x} - n^2\frac{\mu_2}{\Delta^3}y \end{aligned}\right\}$$

Hill's Eqn's for George Hill, derived 1878

after dropping primes.

Hill's Eqn = EOM for particle in CR3BP

remember: origin on m_2

$\hat{x}$ point radially away from m_1-m_2

$\hat{y}$ points along m_2's motion about m_1

lengths are in units of m_2's sma a.

These eqn's are valid when

$\Delta \lesssim$ a few R_H/a

and $\mu_2 \ll 1$

They don't apply far from m_2,

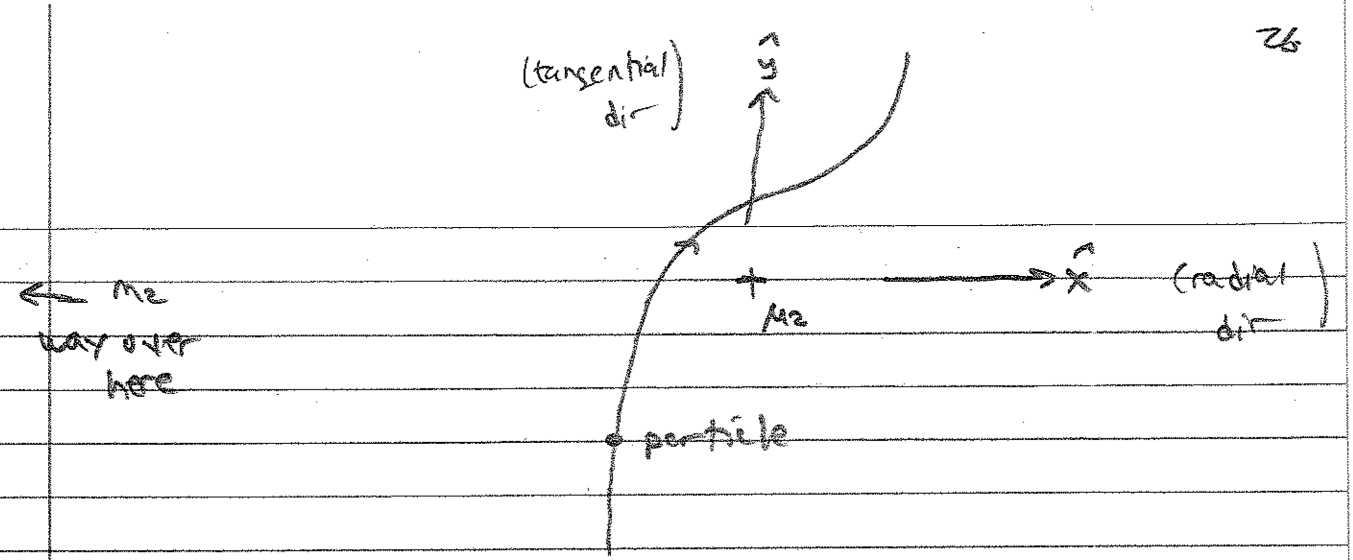
~~where $\Delta \gg R_H/a$,~~

and neglected nonlinear terms become important.

Scale Invariant Hill's Eqns:

lets change units so secondary's mass dissapears → will yield scale invariant eqns = eqns that have no physical parameters.

scale invariant eqns are very handy... if you solve them once, your solution applies to any secondary of any mass M_2.



We want scale-invariant EOM..

replace $\quad x \rightarrow L\, x_{\text{scale-invariant length}}$

↑ what is this systems natural unit of length L?

$t \rightarrow T\, t_{\text{scale-invariant time}}$

↑ natural unit of time?

try $\quad x_h = \dfrac{x}{r_H} \quad$ where $r_H = \dfrac{R_H}{a} = \left(\dfrac{\mu_2}{3}\right)^{1/3}$ = Hill radius in units of a.

= scale invariant x

so $\quad x = r_H x_h$

set $\quad \tau = nt \quad$ where n = secondary's mean motion $= \dfrac{2\pi}{T}$

= dimensionless time

$= 2\pi \dfrac{t}{T} \quad T = \mu_2$'s orbit period.

so τ increments by 2π every orbit of m_2

velocities are in units of $R_H \cdot \Omega$,
and time derivatives are wrt dimensionless
time $\tau = \Omega t$

see Fig 4.7

This is a Runge Kutta integration of Hills Eqns,
particles approach secondary while on
initially circular orbits having
impact parameter b = initial radial
separation x

so these particles initial positions are

$$x_0 = b$$

$$y_0 = \pm 200$$

what are their initial velocities?

$$\dot{x}_0 = ?$$

$$\dot{y}_0 = ?$$

which way are particles moving in
lower portion of figure? upper?

whats happening to trajectories
having $|x| \lesssim 1$?

Note the straight lines... are real trajectories straight after traveling out to very large longitudes, ie out to $|y| \sim 9$?

Hills Eqns are the EOM that result from 'straightening out' the curvature that trajectories exhibit at large longitudes at $\Delta\theta \sim \frac{y}{R_H} \sim 1$

so solutions to HEqn's ignore the curvature that occurs in real orbits

see mng. video

trajectories having which impact parameters are in danger of striking secondary?

see Fig 4.8 $\quad 2.0 \lesssim |b| \lesssim 2.4$.

what happens to those particles with $|b| < 2.0$? why?

Note also wavy edges, These are <u>wakes</u>, also seen at edge of Enke gap, Fig 4.9

Fig 4.9: which side of gap is pre-encounter w/ Pan, which is post?

what is the sense of orbital motion, up or down?

where is Pan?

whats that in center of gap? what kind of trajectories is that in?

so $$\frac{dx}{dt} = \frac{\Omega_H}{n^{-1}} \frac{dx_h}{d\tau} = n\Omega_H \frac{dx_h}{d\tau}$$

and $$\frac{dx^2}{dt^2} = n^2 \Omega_H \frac{d^2 x_h}{d\tau^2}$$

$$\Delta_h = \frac{\Delta}{\Omega_H} = \text{p's distance from } \mu_2 \text{ in units of } \Omega_H$$

$$\mu_2 = 3\Omega_H^3$$

so $$n^2 \Omega_H \frac{d^2 x_h}{d\tau^2} = 2n^2 \Omega_H \frac{dy_h}{d\tau} + n^2 \left(3 - \frac{3\Omega_H^3}{\Delta_h^3 \Omega_H^3}\right) \Omega_H x_h$$

$$\Rightarrow \frac{d^2 x_h}{d\tau^2} = 2\frac{dy_h}{d\tau} + 3\left(1 - \frac{1}{\Delta_h^3}\right) x_h$$

and $$\frac{d^2 y_h}{d\tau^2} = -2\frac{dx_h}{d\tau} - \frac{3y_h}{\Delta_h^3}$$

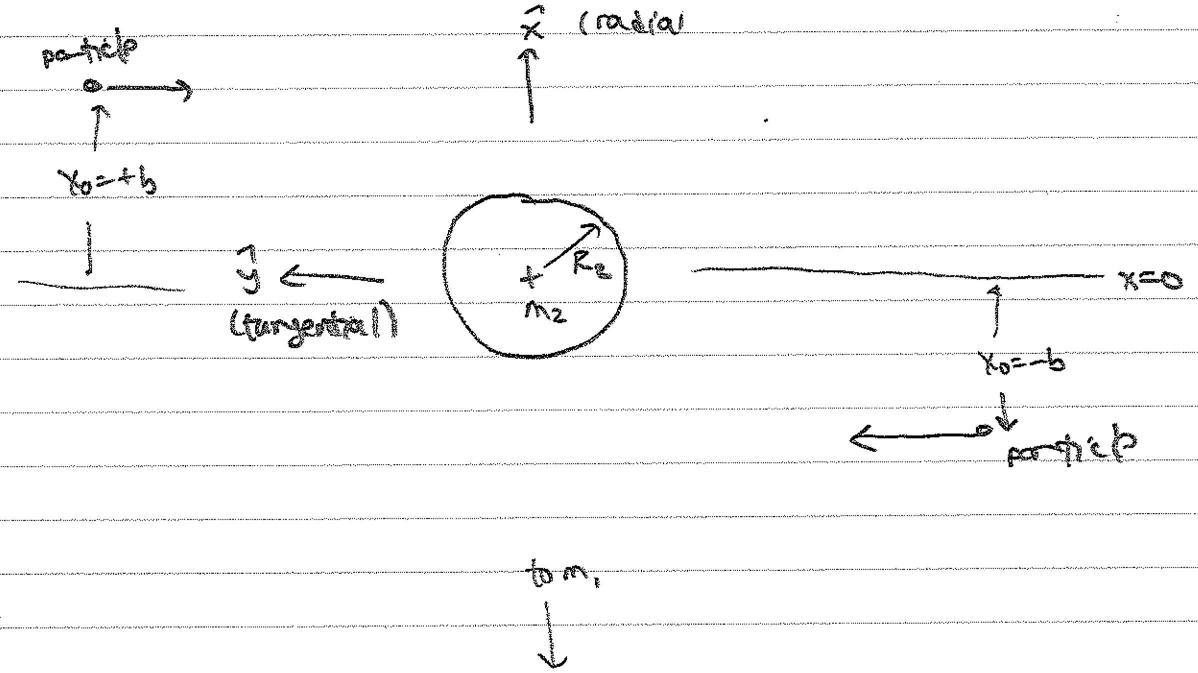
lets write $$\left.\begin{aligned} A_x &= -3x/\Delta^3 \\ A_y &= -3y/\Delta^3 \end{aligned}\right\}$$ secondary's acceleration of particle

so $$\ddot{x} - 2\dot{y} - 3x = A_x$$
$$\ddot{y} + 2\dot{x} = A_y$$

after dropping h subscript. These are scale-invariant Hills eqns, all lengths are in units of Ω_H

stopped
Oct 8

Approximate solution to Hills Eqns for particle advancing on secondary from initially circular orbit



This solution is relevant to: Pan's maintenance of Encke gap

wavy gap edges

accretion of planetesimals by growing protoplanet.

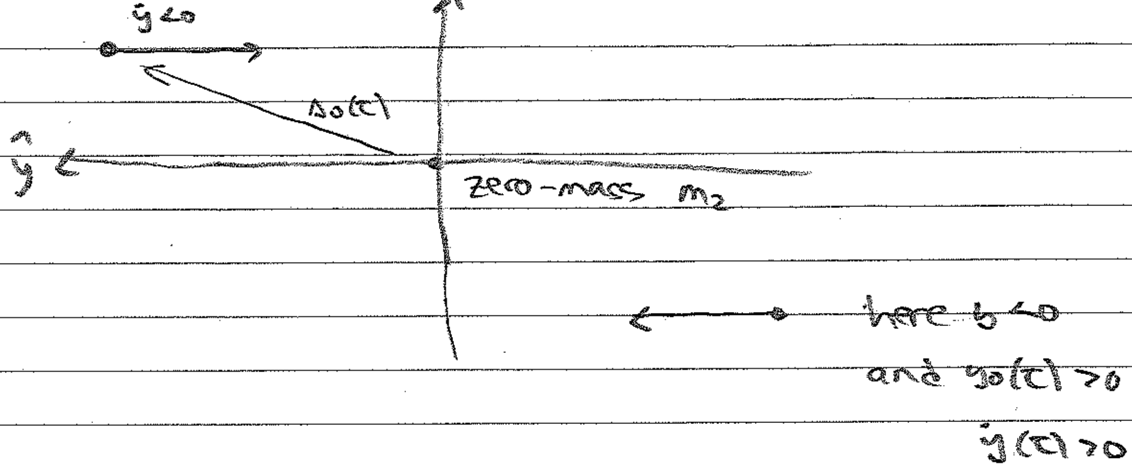
consider particle in circular orbit that is approaching zero-mass secondary so $A_x = 0 = A_y$

The orbit is circular so $x(\tau) = b$ = constant

and $\dot{y}(\tau)$ = constant

so $\ddot{x} = \dot{x} = \ddot{y} = 0 \Rightarrow \dot{y} = -\frac{3}{2}b$ = p's azimuthal velocity

and $y_0(\tau) = -\frac{3}{2}b\tau$ = particle's tangential coordinate



$\Delta_0(\tau)$ = unperturbed particle's distance from secondary

$$= \sqrt{b^2 + y_0^2(\tau)} = |b|\sqrt{1 + (3\tau/2)^2}$$

This is the particle's unperturbed motion,

sometimes called the <u>zeroth order solution</u>

Now derive the first-order solution for a particle having a distant encounter with a secondary whose mass is non-zero

set $x(t) = b + x_1(t)$ $\quad$ $y(t) = y_0(t) + y_1(t)$

$= 0^{th} + 1^{st}$ order solution $\quad$ $= 0^{th} + 1^{st}$ order

insert this into the EOM: $\dot{x} = \dot{x}_1$ $\quad$ $\dot{y} = -\frac{3}{2}b + \dot{y}_1$

$\ddot{x} = \ddot{x}_1$ $\quad$ $\ddot{y} = \ddot{y}_1$

so $\quad \ddot{x}_1 + 3b - 2\dot{y}_1 - 3b - 3x_1 = A_x$

or $\quad \ddot{x}_1 - 2\dot{y}_1 - 3x_1 = A_x = -\dfrac{3(b+x_1)}{\Delta^3}$

and $\quad \ddot{y}_1 + 2\dot{x}_1 = A_y = \dfrac{9b\tau/2 - 3y_1}{\Delta^3}$

where $\Delta(\tau) = \sqrt{(b+x_1)^2 + (-3b\tau/2 + y_1)^2}$

note $\tau = 0$ is time of closest approach

how do you solve this coupled 2^{nd} order DEQ?

what approximations can you make, to simplify and hopefully obtain equations you can solve?

assume the encounter is distant so that the particle's deviations from noncircular motion is small ie

$$|x_1| \ll |b| \quad \text{and} \quad |y_1| \ll |3b\tau/2|$$

so $\Delta(t) = \Delta_0(\tau)$

and the RHS of EOM are $A_x = -\dfrac{3b}{\Delta_0^3}$

$$A_y = \frac{9b\tau}{2\Delta_0^3}$$

then calculate $\frac{d}{dt}$ (upper EOM):

$$\dddot{x}_1 - 2\left(-2\dot{x}_1 + A_y(t)\right) - 3\dot{x}_1 = \frac{dA_x}{dt}$$

so $\dddot{x}_1 + \dot{x}_1 = \dfrac{dA_x}{dt} + 2A_y(\tau)$

integrate: $\ddot{x}_1 + x_1 = \underbrace{A_x(\tau) + 2\int_{-\infty}^{\tau} A_y(\tau')d\tau'}_{= g(\tau)}$

this is the EOM for a simple harmonic oscillator that is driven by a time-dependant force $g(\tau)$

Assignment # 4 : problem 4.10

show that $g(\tau) = -\frac{4}{b\,s_0} - \frac{3b}{s_0^3}$

So the solution ~ (time-varying amplitude) × sinusoids:

$$x_1(\tau) = \sin\tau \int_{-\infty}^{\tau} g(\tau')\cos\tau'\, d\tau'$$

$$-\cos\tau \int_{-\infty}^{\tau} g(\tau')\sin\tau'\, d\tau'$$

confirm by inserting $\dot{x}_1$ and $\ddot{x}_1$
and showing that $x_1(\tau)$ satisfies the EOM:

$$\dot{x}_1 = c(\tau)\int g\, c(\tau')\, d\tau' + g(\tau)\sin\tau\cos\tau$$

$$+ s(\tau)\int g\, s(\tau')\, d\tau - g\cos\tau\sin\tau$$

$$\text{so } \dot{x}_1 = \cos(\tau)\int^{\tau} g(\tau')\cos(\tau')\, d\tau' + \sin\tau \int^{\tau} g(\tau')s(\tau')\, d\tau'$$

$$\text{and } \ddot{x}_1 = -\sin(\tau)\int^{\tau} g(\tau')\cos(\tau')\, d\tau' + \cos(\tau)\int^{\tau} g(\tau')s(\tau')\, d\tau'$$

$$+ g(\tau)$$

$$= -x_1 + g(\tau)$$

which is the EOM ⇒ solution is confirmed.

If we wanted to study, say, wavy edges at the edge of Encke gap, we'd be interested in the solution when $\tau \gg 1$. long after particle's encounter with Pan.

So set $\tau \to \infty$ in upper integration limits:

$$x_1(\tau) = \sin\tau \int_{-\infty}^{\infty} g(\tau')\cos\tau'\,d\tau'$$

$$-\cos\tau \underbrace{\int_{-\infty}^{\infty} g(\tau')\sin\tau'\,d\tau'}_{\text{what is this intgral?}}$$

Assignment #4 prob 4.12

The other integral is proportional to

$$k = 2K_0(2/3) + K_1(2/3) \doteq 2.52$$

$\uparrow$ modified Bessel functions

and $x_1(\tau) \doteq -\text{sign}(b)\,\frac{8k}{3b^2}\sin\tau$

much simpler expression

lets check our assumptions.

Recall $|x_1| \ll |b|$ ie particle's radial excursion due to m_2's kick is small

so $\frac{8h}{3b^2} \ll |b|$

so particle's impact parameter must satisfy $|b| \gg \left(\frac{8h}{3}\right)^{1/3} \simeq 2$

units?

lets convert these scale invarient results into physical units, ie $\tau \to nt$, and recall that all distances are in units of $R_H = (\mu_2/3)^{1/3} a$

so $$\frac{x_1}{R_H} = -\text{sgn}(b) \frac{8h}{3(b/R_H)^2} \sin(nt)$$

or $$x_1(t) = -\text{sgn}(b) \frac{8k\mu_2}{9} \left(\frac{a}{b}\right)^2 \sin(nt)\, a$$

here x_1 has physical length.

The particle's distance from the primary is

$$r(t) = a + x = a + b + x_1(t)$$

which has the same form as epicyclic motion, $r(t) = a + x(t) = a - ae\cos(nt)$

$\Rightarrow$ particle's semimajor axis $= \underset{\text{secondary's sma}}{a} + \underset{\text{particle's impact parameter}}{b}$

so pre & post encounter sma is unchanged, in this solution to the linearized EOM... but see pg 40!

and its eccentricity after getting kicked by μ_2:

$$e = \frac{|x_1|}{a} = \frac{8k\mu_2}{9}\left(\frac{a}{b}\right)^2$$

Next: use these results to study <u>shepherding</u>

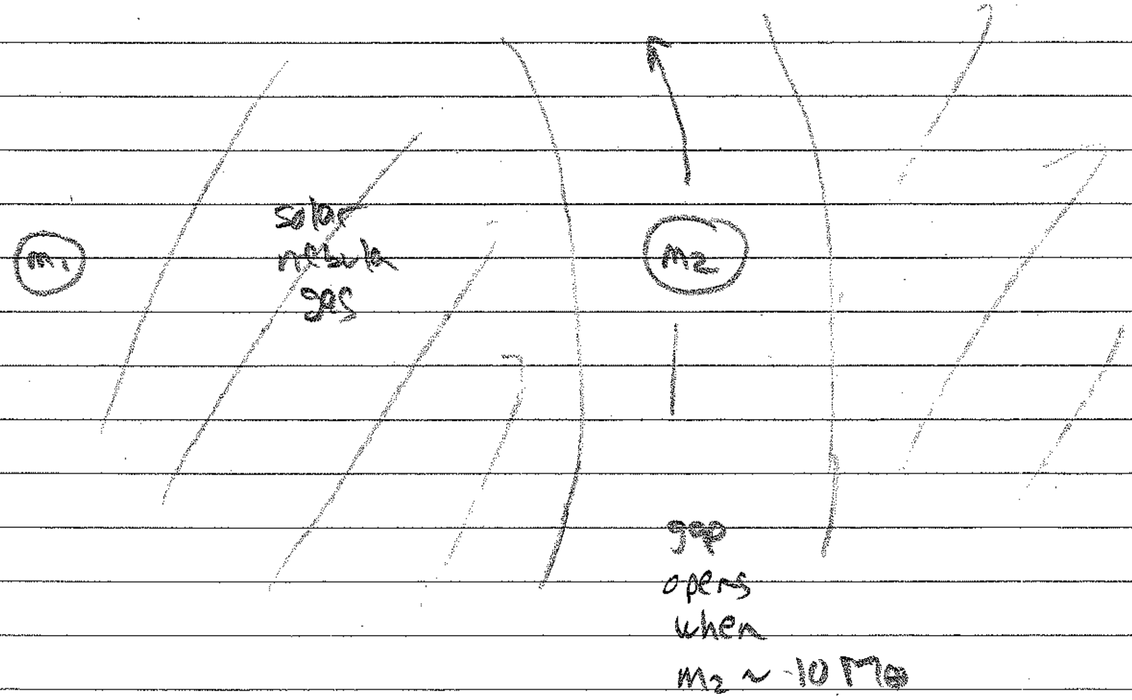
= maintenance of gap in a ring (or disk) by an embedded perturber (satellite in a ring or a protoplanet in a disk)

Shepherding: process by which a gravitating ~~perturber~~ orbiting in a disk of matter tends to nudge disk matter radially away

counterintuitive, since gravity is attractive

but this is the process by which Pan keeps Encke gap open in Saturn's A ring,

and is how protoplanets in solar nebula tend to open gap in circumstellar gas disk



so how did Jupiter acquire its $\sim 300 M_\oplus$ atmosphere?

Start w/ dimensionless J integral for a particle that is perturbed by low-mass secondary $m_2 << m_1$

$$J' = \frac{a}{a+b} + 2\sqrt{\frac{a+b}{a}(1-e^2)} + 2\frac{m_2}{m_1}\frac{a}{\Delta}$$

$\cos i \simeq 1$ assume coplanar disk

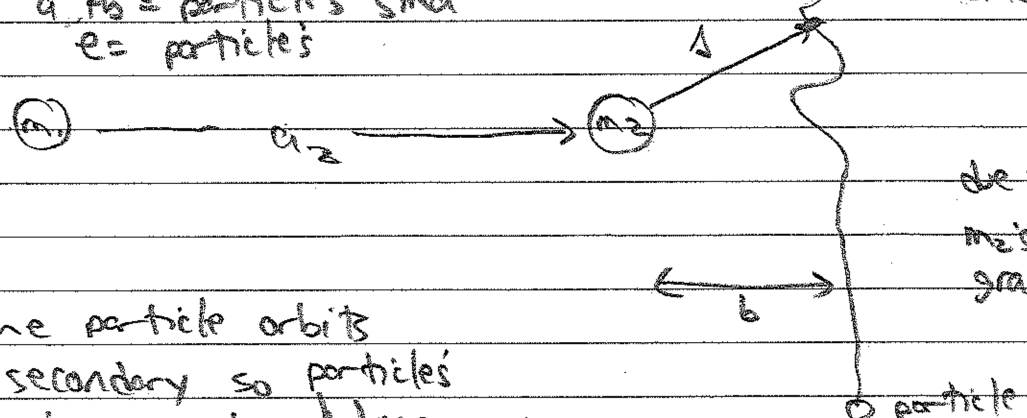
due to m_2's gravity

see pg 6 with $a \to a+b$ and $a_2 \to a$

$a = m_2$'s sma

$a+b =$ particle's sma

$e =$ particle's



Assume particle orbits near secondary so particle's impact parameter $|b| << a$

before encounter, the particle is far away so $2\frac{m_2}{m_1}\frac{a}{\Delta} << 1$ and in circular orbit

$$\text{so } J' \simeq \left(\frac{a+b}{a}\right)^{-1} + 2\sqrt{1+\frac{b}{a}}$$

$$\simeq 1 - \frac{b}{a} + \frac{1}{2}(-1)(-2)\left(\frac{b}{a}\right)^2 + 2\left[1 + \frac{1}{2}\frac{b}{a} + \frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{b}{a}\right)^2\right]$$

$$\simeq 3 + \frac{3}{4}\left(\frac{b}{a}\right)^2 + \mathcal{O}\left(\frac{b}{a}\right)^3 = J'_{initial}$$

The encounter with m_2 also kick's the particles orbit elements so

$$sma \to a + b + \Delta a = a + b'$$

where $b' = b + \Delta a$ = p's new impact parameter.

Also the particles post-encounter $e > 0$, so

$$J' = \left(\frac{a+b'}{a}\right)^{-1} + 2\sqrt{\left(1+\frac{b'}{a}\right)(1-e^2)}$$

$$\simeq 1 - \frac{b'}{a} + \left(\frac{b'}{a}\right)^2 + 2\left[1 + \frac{1}{2}\frac{b'}{a} - \frac{1}{8}\left(\frac{b'}{a}\right)^2\right]\left(1 - \frac{1}{2}e^2\right)$$

$$= 3 + \frac{3}{4}\left(\frac{b'}{a}\right)^2 - e^2$$

$$= 3 + \frac{3}{4}\left(\frac{b+\Delta a}{a}\right)^2 - e^2$$

$$= 3 + \frac{3}{4}\left(\frac{b}{a}\right)^2 + \frac{3}{2}\frac{b\Delta a}{a^2} - e^2 = J'_{final}$$

so $\quad J'_{final} = J'_{inial} + \frac{3}{2}\frac{b}{a}\frac{\Delta a}{a} - e^2$

what is the relationship between $J'_{initial}$ J'_{final}?

J' is conserved, $J'_{init} = J'_{final}$

and $\Delta a = \frac{2}{3}\frac{a^2}{b}e^2$

$= \mathrm{sgn}(b)\,\frac{2}{3}\left|\frac{a}{b}\right| e^2 a$

since $b > 0$ when particle orbits exterior to m_2
and $b < 0$ when p is interior to m_2

what happens to sma of p's in exterior orbits?
interior

if the particle is one of many in the disk,
what happens to the p's eccentricity?

This is shepherding: m_2 pump's up p's e
while J conservation
nudges sma away from m_2.

this is how Pan prevents ring particles
from diffusing into the Encke gap,

and is how protoplanets open a gap
about their orbit in the gas disk.

calculate the particle's synodic period = time until particle encounters m_2 again

$$n = \sqrt{\frac{Gm_1}{a^3}} = m_2\text{'s angular velocity about } m_1$$

$$n_p = \sqrt{\frac{Gm_1}{(a+b)^3}} = n\left(1+\frac{b}{a}\right)^{-3/2} \simeq n\left(1-\frac{3b}{2a}\right)$$

so $\Delta n = n - n_p = \frac{3b}{2a}n$ = particle's angular speed relative to m_2

so $\Delta t = \left|\frac{2\pi}{\Delta n}\right| = \frac{4\pi}{3n}\left|\frac{a}{b}\right|$ = particle's synodic period

= time for particle to lap (or be lapped by) m_2

so repeated encounter's with m_2

cause particle's sma to evolve away at the rate

$$\dot{a} = \frac{\Delta a}{\Delta t} = \mathrm{sgn}(b)\,\frac{2}{3}\left|\frac{a}{b}\right| a\,\frac{64h^2\mu_2^2}{81}\left(\frac{a}{b}\right)^4 \frac{3n}{4\pi}\left|\frac{b}{a}\right|$$

$$\dot{a} = \mathrm{sgn}(b)\,\frac{32h^2\mu_2^2}{81\pi}\left|\frac{a}{b}\right|^4 an$$ = rate at which particle is shepherded away from m_2

If the particle has mass m, then its total angular momentum is

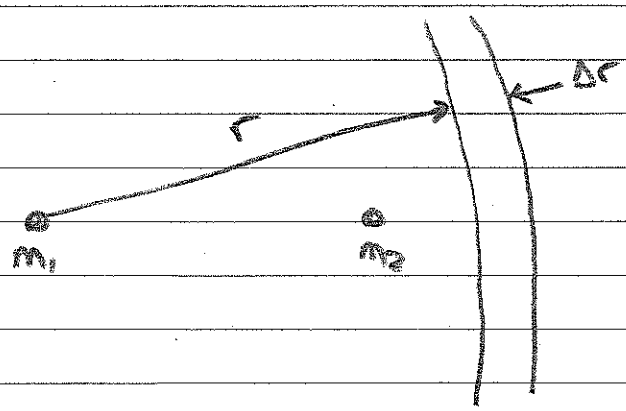
$$L = mh \simeq m\sqrt{Gm_1 a} \quad \text{assuming } e \ll 1$$

since a is evolving over time, the secondary is exerting a torque on the particle,

$$T = \frac{dL}{dt} = \frac{m}{2}\sqrt{\frac{Gm_1}{a}}\,\dot{a} = \frac{1}{2} m n a \dot{a}$$

= shepherding torque on the particle

Now suppose there are <u>many</u> particles in similar orbits, with semimajor axes $r = a + b$, and they inhabit a narrow annulus in disk of radial width Δr



If annulus has surface density $\sigma = \frac{\Delta m}{\Delta A}$

where Δm = mass in annulus

$\Delta A = 2\pi r\,\Delta r$ = area of annulus then $\Delta T = \frac{1}{2}\overbrace{\left(2\pi\sigma r\Delta r\right)}^{\Delta m} n a \dot{a}$

$r=a$

so $\frac{\Delta T}{\Delta r} = \pi\sigma a^2 n \,\mathrm{sgn}(b)\, \frac{32k^2\mu_2^2}{81\pi}\left|\frac{a}{b}\right|^4 an$

$b = r-a$

also set $\mu_d = \frac{\pi\sigma a^2}{m_1}$ = so-called normalized disk mass ~ roughly the mass of the disk in units of m_1

so $\frac{dT}{dr} = \frac{\Delta T}{\Delta r} = \mathrm{sgn}(r-a)\, \frac{32k^2}{81}\left(\frac{a}{r-a}\right)^4 \mu_d \mu_2^2 m_1 a n^2$

= radial torque density

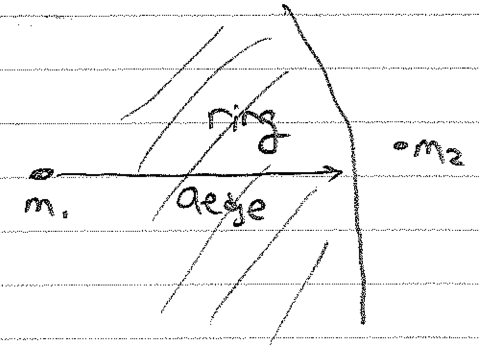
strip radial width

Note that $\frac{dT}{dr}dr$ = torque that m_2 exerts on a narrow strip of disk matter

suppose satellite orbits exterior to a planetary ring.

then $T = \int_0^{a_{edge}} \frac{dT}{dr}\,dr$

= total torque that m_2 exerts on ring.



what is sign of T?

does the ring exert torque on m_2?

what is the sign of that torque?

how is m_2's orbit going to respond?

Use these results to answer problems 4.14, 4.15 in Assignment #4

Assign #4

text problems 4.9, 4.10, 4.12, 4.14, 4.15

due Thurs Oct 24

Midterm Tues Oct 29

take-home exam, 2 or 3 hours

pickup exam in class (no lecture)

turn it in 2pm next day, location TBA