14 September 2013

Chapter 3: Evolution of the 2-body orbit

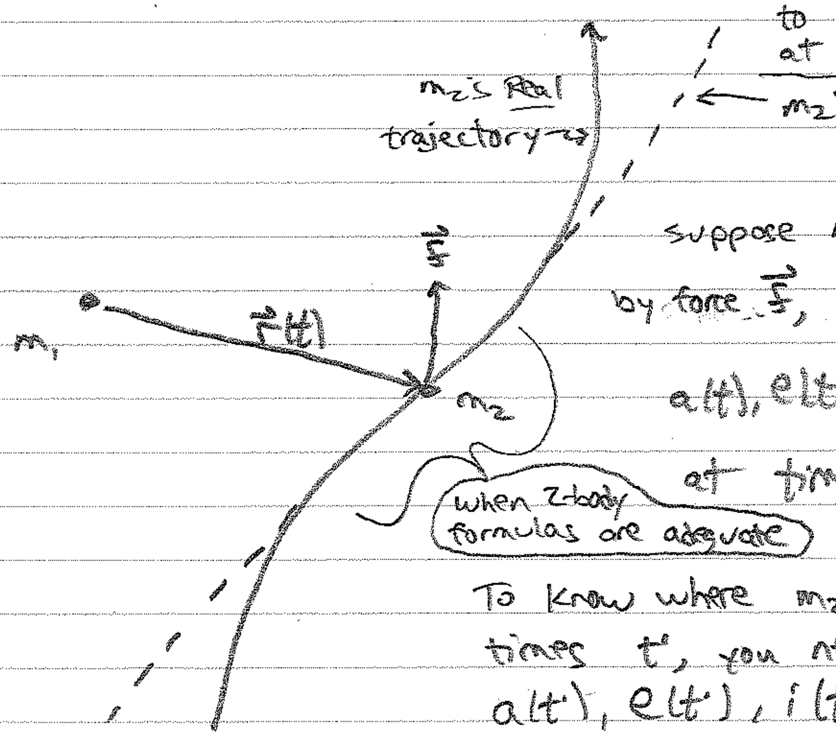
In the 2-body problem, the secondary's motion about primary m_1 is described by six <u>constant</u> orbit elements: $a, e, i, \omega, \Omega, \tau$

But if m_2 is perturbed by other forces, gravity from another planet, or a drag force, then the orbit elements vary with time t.

So $\underbrace{a(t), e(t), \text{etc}}$ are NOT constants

these are m_2's <u>osculating</u> orbit elements

↑ Latin: to kiss (ie, m_2's osculating orbit is tangent to real orbit at time t)



suppose m_2 is perturbed by force $\vec{F}$, and m_2's osc. elements $a(t), e(t), i(t)$ are known at time t.

To know where m_2 is at different times t', you need to know $a(t'), e(t'), i(t')$ etc

which we compute from <u>Gauss' planetary Eqns.</u>

However, you can still use m_2's osc' $a(t), e(t)$ etc to estimate where at nearby times $t + \Delta t$

using 2-body formula (eg, solve Kepler's eqn for E_c at time $t+\Delta t$ then use $r(t+\Delta t) = a(1 - e\cos E_c)$ to get $\vec{r}(t+\Delta t)$)

which is adequate provided Δt is sufficiently small.

For example, one often uses 2-body formula + osculating elements to estimate where an asteroid will be a few years hence.

But asteroids are subject to weak perturbing forces [resonances with Jupiter and Mars + Yarkovsky effect (YE) (force due to asymmetric re-radiation of incident sunlight)] which might cause large changes in a, e etc over longer periods of time ($\sim 10^3$ or 10^6 yrs?)

So if you are an astronomer that wants to observe this asteroid 6 months from now, use osculating orbit elements to easily calculate $\vec{r}(t + 6\text{ months})$, which will tell you where to point your telescope.

but if you are a planetary dynamicist, and you want to know if the YE is going to cause that asteroid to drift into resonance w/Jup say 10^6 yrs from now (which is how most asteroids get ejected from asteroid belt), then you would use Gauss Eqn's

to estimate the orbit drift time.

same Gauss of Gauss' Law.

Should I add YE to syllabus?

Gauss' Planetary Eqn's (PE)

secondary m_2 orbits primary m_1, and m_2 is subject to perturbing acceleration

$$\vec{a}_p = a_r \hat{r} + a_\theta \hat{\theta} + a_n \hat{n}$$

in cylindrical coordinates, where $\hat{n}$ is normal to m_2's orbit

so we are using orbit-plane coordinate system.

perturbation $\vec{a}_p$ causes m_2's a, e, i etc to vary over time

derive rates $\dot{a} = \frac{da}{dt}$, $\dot{e}$, $\frac{di}{dt}$ etc

= Gauss' PE

<u>calculate $\dot{q}$</u>

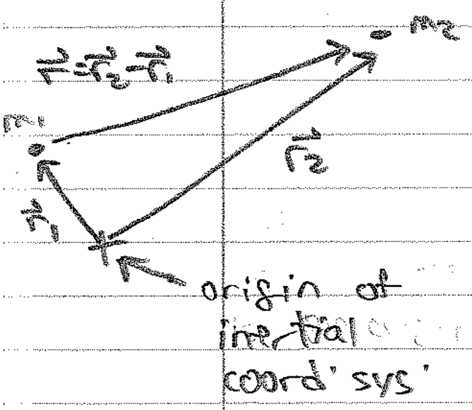
$$W = m_2 \int_{\vec{r}_1}^{\vec{r}_2} \vec{a}_p \cdot d\vec{r} \;=\; \text{work done on } m_2 \text{ by acceleration } \vec{a}_p$$

so $\Delta W = m_2 \vec{a}_p \cdot \Delta\vec{r}$ = small work done on m_2 during small displacement $\Delta\vec{r}$

$$\dot{W} = \frac{\Delta W}{\Delta t} = m_2 \vec{a}_p \cdot \dot{\vec{r}}_2$$

= rate at which work is done on m_2

= rate at which m_2's energy changes in <u>inertial</u> coordinate system



you will show in problem 2.4 shows

$$\dot{\vec{r}}_2 = \frac{\mu_r}{m_2} \dot{\vec{r}}$$

where $\mu_r = \dfrac{m_1 m_2}{m_1 + m_2}$ = reduced mass

so $\dot{W} = \mu_r \vec{a}_p \cdot \dot{\vec{r}}$ = rate m_2's energy E changes

= rate at which system <u>total</u> energy E changes

Prob 2.4: the system's total energy

$$E = \mu_r \mathcal{E} \quad \text{where } \mathcal{E} = -\frac{\mu}{2a}$$

= 2body energy integral

so $\dot{W} = \mu_r \vec{a}_p \cdot \dot{\vec{r}} = \mu_r \dot{\mathcal{E}} = \frac{\mu_r \mu \dot{a}}{2a^2}$

so $\dot{a} = \frac{2a^2 \vec{a}_p \cdot \dot{\vec{r}}}{\mu} = \frac{2 \vec{a}_p \cdot \dot{\vec{r}}}{n^2 a}$

$$= \frac{2}{n^2 a}\left(a_r \dot{r} + a_\theta r\dot{\theta} + a_n \dot{z} \right)$$

what is m_2's $\dot{z}$ measured in the orbit-plane coordinate system?

recall $\dot{r} = \frac{ean \sin f}{\sqrt{1-e^2}}$ $\quad r\dot{\theta} = r\dot{f} = \frac{an}{\sqrt{1-e^2}}(1+e\cos f)$

so $\dot{a} = \frac{2}{n\sqrt{1-e^2}}\left[a_r e \sin f + a_\theta (1+e\cos f) \right]$

usually one uses Gauss' eqn's when doing analytic work, which usuall requires $e \ll 1$, so for nearly circular orbits

$$\dot{a} \simeq \frac{2a_\theta}{n}$$

(for example, orbit decay due to drag has $\dot{e} < 0$ so e usually small)

$\Rightarrow$ the tangential or 'along-track' part of the perturbing acceleration $\vec{a}_p$ will cause semimajor axis a to grow or shrink.

to get di/dt:

set $\vec{h} = h\hat{n}$ = m_2's angular momentum integral

so $\vec{T} = \frac{d\vec{h}}{dt}$ = specific torque on m_2

use right-hand fingers to evaluate cross products, A.18

$$= \vec{r} \times \vec{a}_p = r\hat{r} \times \left(a_r \hat{r} + a_\theta \hat{\theta} + a_n \hat{n} \right)$$

$$= r a_\theta \hat{n} - r a_n \hat{\theta}$$

$$= \dot{h}\hat{n} + h\frac{d\hat{n}}{dt}$$

so $\dot{h} = r a_\theta$

and $$\frac{d\hat{n}}{dt} = -\frac{r a_n}{h}\hat{\theta} = \frac{r a_n}{h}\underbrace{\hat{r} \times \hat{n}}_{-\hat{\theta}}$$

so a_θ alters the magnitude of $\vec{h}$,

and a_n alters the direction $\vec{h}$ points towards

ie only normal acceleration a_n
can tip m_2's orbit plane

since $h = \sqrt{\mu a (1-e^2)}$

$$\dot{h} = \frac{1}{2}\sqrt{\frac{\mu(1-e^2)}{a}}\,\dot{a} + \frac{1}{2}\sqrt{\frac{\mu a}{1-e^2}}\,(-2e\dot{e}) = r a_\theta$$

so $$\dot{e} = \frac{1}{e}\sqrt{\frac{1-e^2}{\mu a}}\left(\frac{1}{2}\sqrt{\frac{\mu(1-e^2)}{a}}\,\dot{a} - r a_\theta\right)$$

$\mu = n^2 a^3$

$$\dot{e} = \frac{\sqrt{1-e^2}}{e a^2 n}\left\{\frac{a n}{2}\,\frac{\sqrt{1-e^2}\,2}{n\sqrt{1-e^2}}\left[a_r e \sin f + a_\theta(1+e\cos f)\right] - r a_\theta\right\}$$

so $$\dot{e} = \frac{\sqrt{1-e^2}}{e a n}\left[a_r e \sin f + a_\theta\left(1 + e\cos f - \frac{r}{a}\right)\right]$$

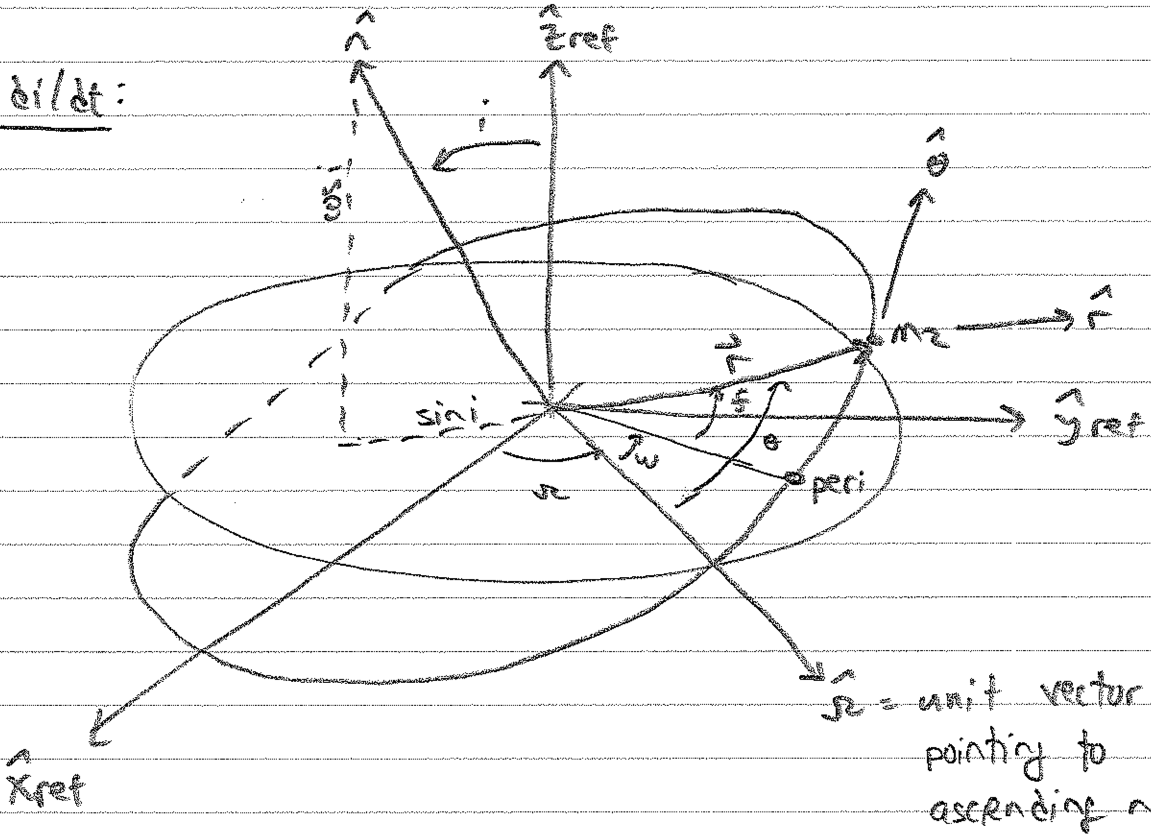
if we use $r = a(1 - e\cos E_c)$

then the e in the denominator goes away for $e \ll 1$ since $f \simeq E_c + O(e)$

$$\dot{e} \simeq \frac{1}{an}\left(a_r \sin f + 2a_\theta \cos f\right)$$

$$\dot{e} = \frac{\sqrt{1-e^2}}{an}\left[a_r \sin f + a_\theta\left(\cos f + \cos E_c\right)\right]$$

di/dt:



$$\hat{z}_{ref} \cdot \hat{n} = \cos i$$

so $$\frac{d}{dt}\hat{z}_{ref}\cdot\hat{n} = \hat{z}_{ref}\cdot\frac{d\hat{n}}{dt} = -\sin i \frac{di}{dt}$$

$$= \frac{r a_n}{h}\,\hat{z}_{ref}\cdot\left(\hat{r}\times\hat{n}\right)$$

now use vector identity A.13:

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

so $\frac{di}{dt} = -\frac{r a_n}{h \sin i} \hat{r} \cdot \underbrace{(\hat{n} \times \hat{z}_{ref})}_{\text{what is this? use right fingers}}$

$$\hat{n} \times \hat{z}_{ref} = -\sin i \, \hat{\Omega}$$

also note $\hat{r} \cdot \hat{\Omega} = \cos(\omega + f)$

$$\text{so } \frac{di}{dt} = -\frac{r a_n}{h \sin i}(-\sin i)\cos(\omega+f)$$

$$\frac{di}{dt} = -\frac{r\, a_n}{h}\cos(\underbrace{\omega+f}_{\theta})$$

for nearly circular orbits, $h = r^2\dot{\theta} \simeq a^2 n$

$$\text{and } \frac{di}{dt} \simeq \frac{a_n}{an}\cos\theta$$

only normal accelerations a_n drive i-evolution stop Sept 17

<u>calculate $\dot{\Omega}$</u> = rate at which $\hat{\Omega}$ moves

let $\vec{\Delta n}$ = change in $\hat{n}$ during time Δt
due to perturbation $\vec{a}_p$

$\vec{\Delta n}$ = sum of 3 orthogonal parts
$= a\hat{n} + b\hat{\Omega} + c(\hat{n} \times \hat{\Omega})$

where a,b,c are coefficients to be dermined:

$a = \vec{\Delta n} \cdot \hat{n}$ → change in $\Delta\hat{n}$ along $\hat{n}$

$b = \vec{\Delta n} \cdot \hat{\Omega}$

$c = \vec{\Delta n} \cdot (\hat{n} \times \hat{\Omega})$ → corresponds to rotating orbit plane about $\hat{\Omega}$ axis

Does moving $\hat{n}$ along the $\hat{n}$ or $\hat{n} \times \hat{\Omega}$ axes change Ω?

If not, then we can ignore the a,c components, and focus on the

$\vec{\Delta n} \cdot \hat{\Omega}$ component that causes Ω to increment by $\Delta\Omega$ in time Δt

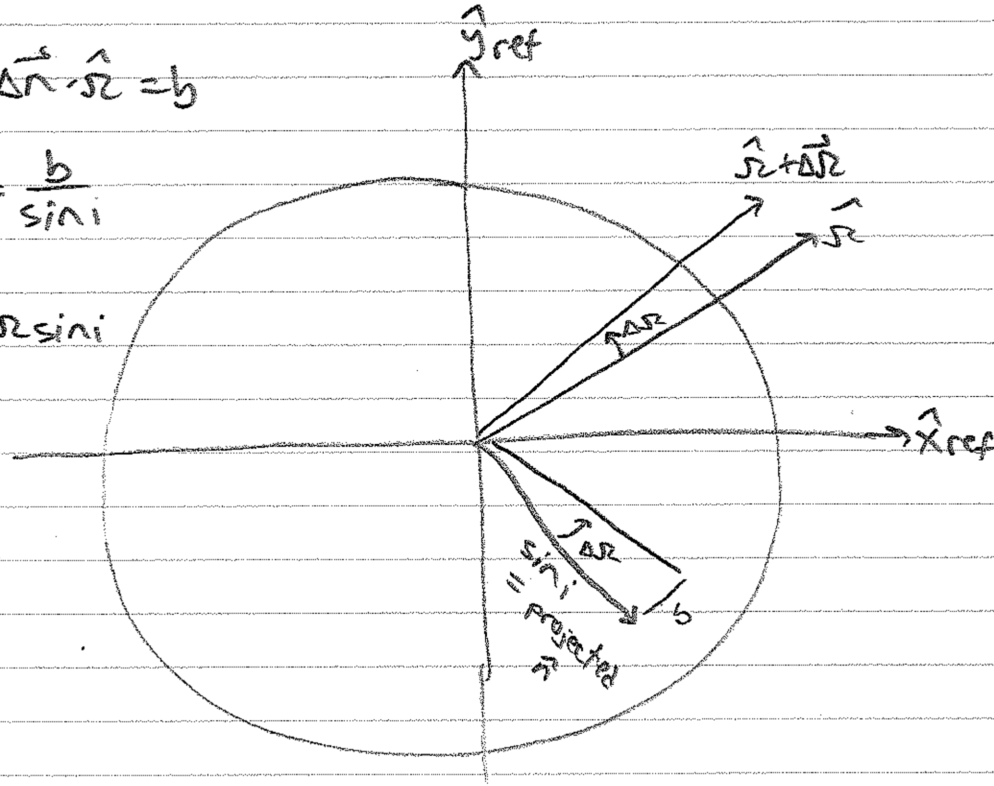
note the projected length of $\hat{n}$ onto $\hat{x}_r$-$\hat{y}_{ref}$ plane is $\sin i$

and $\Delta\vec{n}\cdot\hat{\Omega} = b$

and $\Delta\Omega = \frac{b}{\sin i}$

so $\Delta\vec{n}\cdot\hat{\Omega} = \Delta\Omega \sin i$

and



$$\dot{\Omega} = \frac{\Delta\Omega}{\Delta t} = \frac{\hat{\Omega}\cdot\Delta\vec{n}}{\sin i\,\Delta t} = \frac{1}{\sin i}\,\hat{\Omega}\cdot\frac{d\hat{n}}{dt} = -\frac{r a_n}{h\sin i}\,\hat{\Omega}\cdot\hat{\theta}$$

what is $\hat{\Omega}\cdot\hat{\theta}$?

$$= |\hat{\Omega}||\hat{\theta}| \times \cos(\text{angle btwn})$$

$$= \cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$$

↑ from trig identity A.6

so $\dot{\Omega} = \frac{r a_n}{h\sin i}\sin\theta$

The derivations of the rates for the remaining elements, $\dot{\omega}$ and $\dot{\tau}$, are laborious.

But they are easy to obtain when $e \ll 1$ and the orbit is nearly circular, which is the regime where we will use these eqn's anyway

To get $\dot{\tilde{\omega}}$, start with $r = a(1 - e\cos E_c)$

$$\simeq a(1 - e\cos f) + \Theta(e^2)$$

so $\dot{r} \simeq \dot{a}(1 - e\cos f) - a\dot{e}\cos f + ea\dot{f}\sin f$

$$= \frac{ean\sin f}{\sqrt{1-e^2}} \quad \text{(from 2-body problem)}$$

$$\simeq ean\sin f$$

long of osc' node

The true anomaly $f = \theta - \tilde{\omega}$

so $\dot{f} = \dot{\theta} - \dot{\tilde{\omega}}$ where $\dot{\theta} \simeq n + \Theta(en)$

$$\simeq n - \dot{\tilde{\omega}} + \Theta(en)$$

insert this + $\dot{a} \simeq \frac{2a_\theta}{n}$ and $\dot{e} \simeq \frac{a_r}{an}\sin f + \frac{2a_\theta}{an}\cos f$

$$\frac{2a_\theta}{n}(1 - \Theta(e)) - \frac{a\cos f}{an}\left(a_r\sin f + 2a_\theta\cos f\right) + ea(n - \dot{\tilde{\omega}})\sin f$$

$$\simeq ean\sin f$$

$$\text{so} \quad \frac{2a_\theta}{n} - \frac{a_r}{n}\sin f\cos f - \frac{2a_\theta}{n}\overset{1-\sin^2 f}{\cos^2 f} = ea\dot{\tilde{\omega}}\sin f$$

$$-\frac{a_r}{n}\sin f\cos f + \frac{2a_\theta}{n}\sin^2 f = ea\dot{\tilde{\omega}}\sin f$$

$$\text{so} \quad \dot{\tilde{\omega}} = \frac{2a_\theta \sin f - a_r\cos f}{ean} \quad \text{to lowest order in } e$$

<u>To get $\dot{\tau}$</u>,

start with $M = n(t-\tau)$ = mean anomaly

$\simeq f + O(e)$ since $f \simeq M + 2e\sin M$

$$\text{so} \quad \dot{M} \simeq \dot{f} = n - \dot{\tilde{\omega}} = \dot{n}(t-\tau) + n(1-\dot{\tau})$$

$$= \dot{n}M/n + n - n\dot{\tau}$$

Note $n = \sqrt{\mu}\, a^{-3/2}$ so $\dot{n} = -\frac{3}{2}\frac{n}{a}\dot{a} \simeq \frac{-3a_\theta}{a}$

$$\text{and} \quad n\dot{\tau} = -\frac{3a_\theta}{an}M + n - n + \dot{\tilde{\omega}}$$

$$\text{so} \quad \dot{\tau} = \frac{2a_\theta \sin f - a_r \cos f}{ean^2} - \frac{3a_\theta}{an}(t-\tau)$$

These formula for $\dot{\tilde{\omega}}$ and $\dot{\tau}$ are only valid when e and i are $\ll 1$. See reference [1] by Burns (1976) for exact derivation of $\dot{\tilde{\omega}}, \dot{\tau}$

Lets examine orbital decay of interplanetary dust, due incident sunlight... aka PR drag

The solar system is full of dust of radii $10 \lesssim R \lesssim 100\ \mu m$
comets come close to sun, $r \lesssim 1.5$ AU,
their icy surfaces sublimate (boil off),
flowing gas drags dust away, producing
cometary comae and tails (see text fig 3.4)

all that comet dust goes into orbit about sun.

Asteroids also collide and produce dust,
first-ever collision detected by Hubble in 2010
(look up P/2010 A2)

The resulting cloud of dust orbits the sun,
you can see it right after sunset or
just before sunrise — the zodical light

These grains orbit the Sun,
but the absorption & re-radiation of sunlight
results in a drag force on the grain —
Poynting-Robertson (PR) drag,
which causes dust to slowly spiral into Sun.

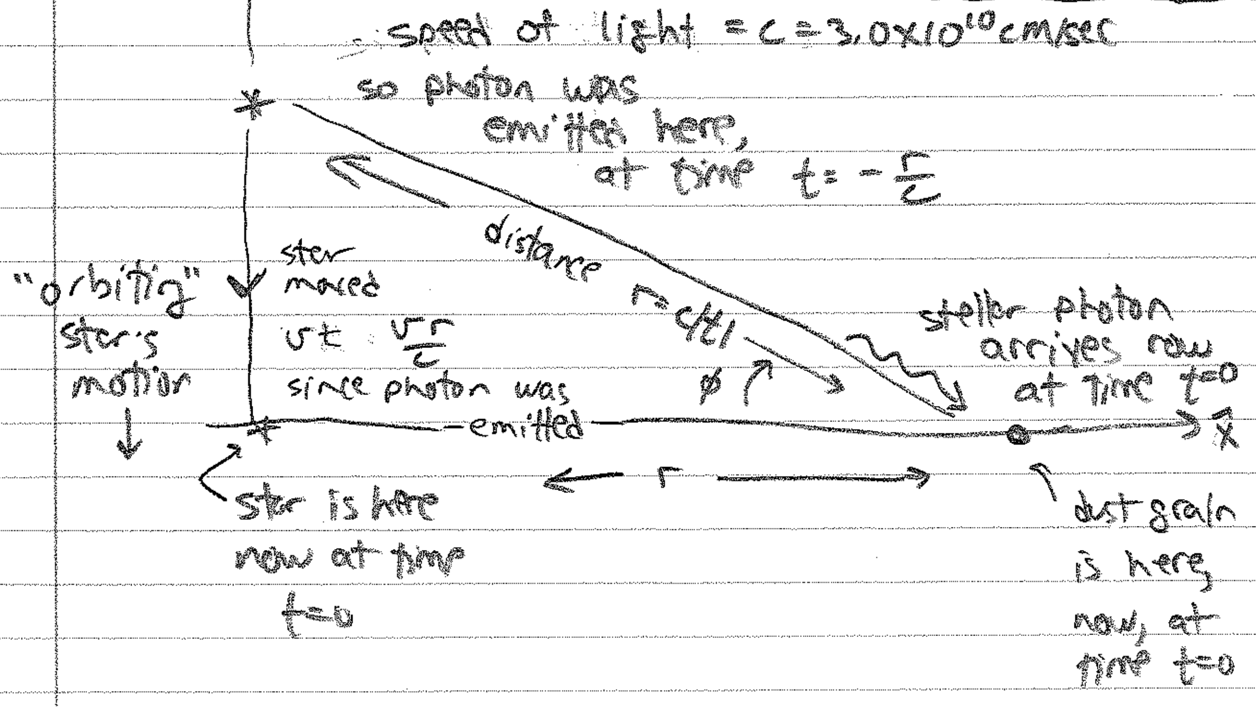
orbit decay due to PR drag

PR drag is analogous to driving in the rain, the faster you drive, the harder the rain seems to pelt the front windshield, less so in the rear.

Ditto for a dust grain travelling thru a storm of photons:

PR drag is due to stellar aberation: the apparent displacement of a star due to an observer's motion (usually an astronomer on an orbiting Earth)

Lets put our coordinate system on the orbiting dust grain. In this coordinate system the star orbits the grain



Suppose you are riding on the grain, and you look back along the photon's path to you — you see a star, but it isn't really located where you are looking

The finite speed of light causes the star to appear to be displaced by angle $\phi = \frac{v}{c}$ (stellar aberration)

Note: to solve this problem exactly, one must use relativistic Lorentz transformations, which we will not due. But when dust grain's orbital speed obeys $v/c \ll 1$, the exact solution reduces to the non-relativistic solution obtained here.

First, note that $F_0 = \frac{L_*}{4\pi r^2}$ would be the flux (eg, rate-per-area) of photon-energy incident upon the grain, where

L_* = star's luminosity (= rate at which star emits energy)

But if the grain has radial velocity $\dot{r}$, Then photons of wavelength λ_0 are Doppler shifted by $\frac{\lambda}{\lambda_0} \simeq 1 + \frac{\dot{r}}{c}$ for non-relativistic orbital speeds, $|\dot{r}| \ll c$

recall that a photon's energy is

$$E = h\nu = \frac{hc}{\lambda}$$

Planck constant

so $E \propto \lambda^{-1}$

and L_* and F get boosted by

factor $(1 + \frac{\dot{\epsilon}}{c})^{-1} \simeq 1 - \frac{\dot{\epsilon}}{c}$

so $F_{ds} = (1 - \dot{\epsilon}/c) F_0$ is flux of Doppler-shifted photons hitting the dust grain

These photons deliver energy to grain

at rate $\dot{E} = F_{ds} A$ where A = grain's cross-sectional area $= \pi R^2$

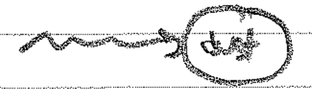
photons carry momentum $p = E/c$

so starlight delivers momentum to grain at rate $\dot{p}_{inc} = \dot{E}/c$

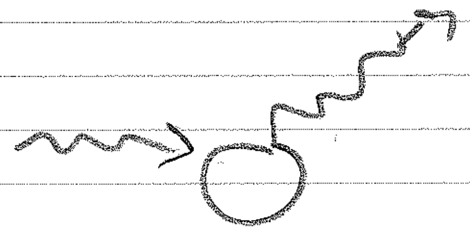
$$= \frac{1}{c}\left(1 - \frac{\dot{\epsilon}}{c}\right) F_0 A$$

A dust grain can do 3 things w/incident photon:

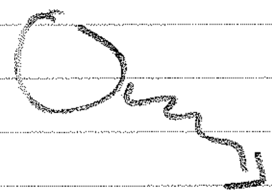
1. it can absorb the photon (which heats grain up)



2. dust can scatter photon away



3. or grain can emit a new photon (eg, radiate in IR), which cools grain



Each event alter's the grain's momentum $\vec{p} = m\dot{\vec{r}}$
So Newton's 2^{nd} law for the grain is

$$m\ddot{\vec{r}} = \dot{\vec{P}}_{abs} + \dot{\vec{P}}_{scat} + \dot{\vec{P}}_{emis}$$

← for radiative processes only, ignoring stellar gravity for now

perturbing force

= rates at which grain's $\vec{p}$ varies due to absorbtion + scattering of starlight + grain's thermal emission.

of course the dust grain is bombarded by countless photons each second, so we should replace $m\ddot{\vec{r}}$ with its time-average:

$$m\ddot{\vec{r}} \rightarrow \langle \dot{\vec{p}}_{abs} + \dot{\vec{p}}_{scat} + \dot{\vec{p}}_{emis} \rangle$$

= average rate of grain's momentum change as it recoils from incident light

We don't know what the RHS looks like in detail, but it probably (but see exception below) has the form:

$$\langle \dot{\vec{p}}_{abs} + \dots \rangle = Q_{pr} \, \dot{p}_{inc} \, \hat{p}$$

where $\dot{p}_{inc}$ = rate at which grain's momentum would change if

Q_{pr} = radiation pressure efficiency = 1

for instance, if the dust grain absorbed all incident sunlight and scattered light isotropically in its rest frame $\langle \dot{\vec{p}}_{scat} \rangle = 0$, and its thermal emission where isotropic $\langle \dot{\vec{p}}_{emis} \rangle = 0$, then $Q_{pr} = 1$ (for perfectly absorbing grain)

But what if the grain was instead a perfect reflector of sunlight?

$Q_{pr} = ?$

lecture stops Sept 19

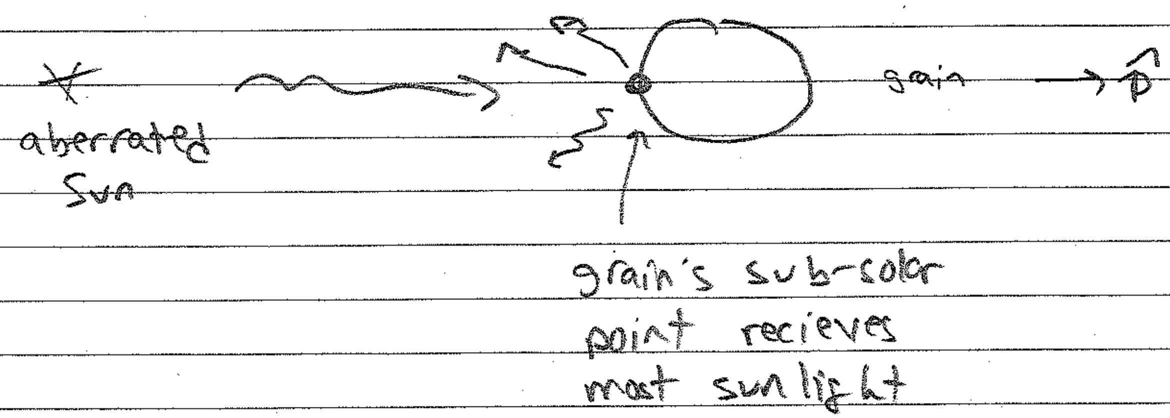
so the remainder will assume that

$$m\ddot{\vec{r}} = \langle \dot{\vec{P}}_{abs} \rangle + \langle \dot{\vec{P}}_{scat} \rangle + \langle \dot{\vec{P}}_{emiss} \rangle = Q_{pr} \hat{P}_{inc} \hat{p}$$

but this is only true under certain conditions. The concern is the

$\langle \dot{\vec{P}}_{emiss} \rangle$ term... This is the rate

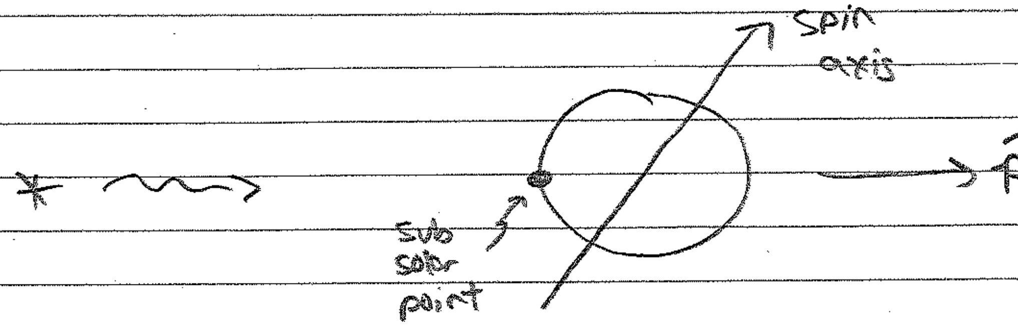
at which the grain recoils due to its thermal emission, the grain is heated by optical sunlight that is later re-radiated at IR wavelengths.



If grain is not rotating (or rotating slowly enough), then the subsolar point acts like a photon jet. Then $\langle \dot{\vec{P}}_{emiss} \rangle \propto \hat{p}$

and the above formulas are ok.

But what if the grain is rotating?



$\langle \vec{F}_{emiss} \rangle$ does NOT necessarily point anti-sunward along $\hat{p}$

⇒ the grain's warmer side acts like a photon jet that drives additional orbit evolution that is not accounted for by the following equations; that evolution is known as the Yorkovsky effect (YE)

YE is significant: high-precision monitoring of spacecraft (LAGEOS, for example)

And asteroid orbit evolution, especially asteroids of sizes

$$0.1 \lesssim R \lesssim 1 \text{ km}$$

ex: R ~ 100 m - sized asteroids in asteroid belt slowly drift into orbital resonance w/ Jupiter, which can kick asteroid into Mars - crossing orbit which can then kick the asteriod down to perigpse $q < 1.3$ AU, and the asteroid is then classified as a Near Earth Object (NEO).

NEOs are now the preferred destination for NASA missions (manned & unmanned)

Most NEOs likely spent billions of years in asteroid belt at $2 \lesssim a \lesssim 3$ AU, but only recently drifted into Near-Earth space via YE

NEOs (and comets) are also impact hazards, tho that threat is usually over blown... No-one has died from asteroid impact during all of recorded history.

...end of digression.

But a small $R \sim 100\,\mu m$ dust grain likely does not suffer YE.

When a grain absorbs a photon, that causes a Thermal wave to propagate across the grain. If the grain is small enough, the Thermal wave can cross the grain "soon enough" to equalize the grain's surface temperature, which then makes

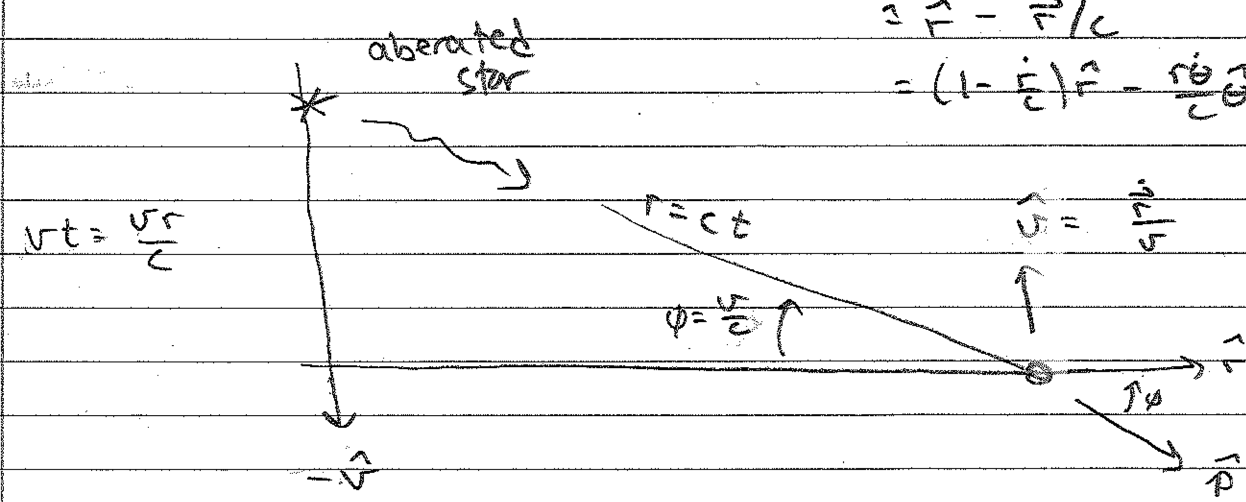
$$\langle \vec{P}_{emiss} \rangle = 0$$

ie, ~~depends on~~ grain's thermal conductivity & Thermal inertia

This is why $R \sim 100\,m$ asteroids suffer YE,
They are too large for thermal conduction
to equalize Their surface temperatures
so $\langle \dot{\vec{p}}_{emiss} \rangle \neq 0$.

But for The dust grain,

$$m\ddot{\vec{r}} = Q_{pr}\,\dot{p}_{inc}\,\hat{p} \quad \text{where} \quad \hat{p} = \cos\phi\,\hat{r} - \sin\phi\,\hat{v}$$
$$\simeq \hat{r} - \dot{\vec{r}}/c$$
$$= \left(1 - \frac{\dot{r}}{c}\right)\hat{r} - \frac{r\dot{\theta}}{c}\hat{\theta}$$



and $\dot{p}_{inc} = (1 - \dot{r}/c)\,F_0 A/c$

so $$\ddot{\vec{r}} = \frac{Q_{pr}}{m}\frac{F_0 A}{c}\left(1 - \frac{\dot{r}}{c}\right)\left[\left(1 - \frac{\dot{r}}{c}\right)\hat{r} - \frac{r\dot{\theta}}{c}\hat{\theta}\right]$$

small terms
since $v \ll c$

= perturbing acceleration due to sunlight.

so $$\ddot{\vec{r}} \simeq \frac{F_0 A Q_{pr}}{mc}\left[\left(1 - \frac{2\dot{r}}{c}\right)\hat{r} - \frac{r\dot{\theta}}{c}\hat{\theta}\right]$$

to first order in The small quantities

set $\vec{\ddot{r}} = \vec{a}_{rad} + \vec{a}_{PR}$

velocity-independent part = radiation pressure

velocity-dependent part = Poynting Robertson drag.

$\vec{a}_{rad} = a_{rad}\,\hat{r}$ $\qquad a_{rad} = \dfrac{F_0 A Q_{pr}}{mc}$

and $\vec{a}_{PR} = -a_{rad}\left(\dfrac{2\dot{r}}{c}\hat{r} + \dfrac{r\dot{\theta}}{c}\hat{\theta}\right)$

radiation pressure:

set $g = -\dfrac{GM_*}{r^2}$ = gravity on grain due to primary star

set $\beta = \dfrac{a_{rad}}{|g|} = \dfrac{r^2}{GM_*}\underbrace{\left(\dfrac{L_*}{4\pi r^2}\right)}_{F_0}\dfrac{\overbrace{\pi R^2}^{A}\,Q_{pr}}{\underbrace{\frac{4\pi}{3}\rho R^3}_{m}}$

$= \dfrac{3L_* Q_{pr}}{16\pi G M_* \rho R c}$ assuming spherical grain of radius R density ρ.

$= \dfrac{\text{radiation pressure}}{\text{stellar gravity}}$

= constant, independant of r

$c = 3\times10^{10}$ cm/sec

For dust orbiting the sun having

$\rho \sim 3$ gm/cm^3 $\quad Q_{pr} \sim 1$

$$\Rightarrow \beta \sim 0.2 \left(\frac{R}{\mu m}\right)^{-1}$$

$G = 6.67\times10^{-8}$ cm^3/gm/sec^2

$M_\star \simeq 2\times10^{33}$ gm

$L_\star \simeq 4\times10^{33}$ ergs/sec

so small micron-sized dust experience significant radiation pressure.

note: astro-types use CGS units,

energy unit $\quad$ 1 erg = gm·cm^2/sec^2 = 10^{-7} joules

The above is valid in the geometric optics limit, ie the dust grain must be significantly larger than wavelength of the incident light, ie

$$R \gg \lambda \qquad \text{where } \lambda \sim 0.6\,\mu m \text{ for sunlight.}$$

if $R \lesssim \lambda$, you need to use light-scattering theory (ie Mie theory) to calculate $\beta(R)$.

But interplanetary dust have $10 \lesssim R \lesssim 100\mu m$

so our derivation is appropriate.

So the equation of motion for a dust grain that suffers radiation pressure only is

$$\ddot{\vec{r}} = -\frac{GM_*}{r^2}(1-\beta)\hat{r}$$

so this grain is in a simple Keplerian orbit, but that grain behaves as if in orbit about a star whose mass is reduced by factor $1-\beta$

Assignment #3 due ?
problem 3.5

Orbit decay due to PR drag:

$\vec{a}_{PR} = a_r\hat{r} + a_\theta\hat{\theta}$ = acceleration of dust due to PR drag

where $a_r = -\frac{2a_{rad}\dot{r}}{c}$ and $a_\theta = -a_{rad}r\dot{\theta}/c$

a = grain's sma

Note $a_{rad} = \beta|\vec{g}| = \frac{\beta GM_*}{r^2} = \frac{\beta\mu}{r^2} = \beta\frac{n^2a^3}{r^2}$

The grain's radial, tangential velocities are

$$\dot{r} = \frac{ean\sin f}{\sqrt{1-e^2}} \simeq ean\sin(M) + O(e^2)$$

where $f \simeq M + O(e)$ and $M = nt$

true, mean anomaly (we can set peri-passage time $\tau = 0$)

and $$r\dot{\theta} = r\dot{f} = \frac{an(1+e\cos f)}{\sqrt{1-e^2}} \simeq an(1+e\cos M)$$

and $r \simeq a(1-e\cos M)$

so $$a_r \simeq -2\beta n^2 a\left(\frac{a}{r}\right)^2 \frac{ean}{c}\sin M$$
$$\simeq -2\beta e\left(\frac{an}{c}\right)\sin M\, an^2$$

to lowest order in small quantities; ex $\frac{an}{c}$

and $$a_\theta \simeq -\beta an^2\left(\frac{a}{r}\right)^2 \frac{an}{c}(1+e\cos M)$$

by binomial thm

$$= -\beta\left(\frac{an}{c}\right)(1+2e\cos M)((1+e\cos M))\, an^2$$

so $$a_\theta \simeq -\beta\left(\frac{an}{c}\right)\left[1+3e\cos M + O(e^2)\right] an^2$$

to the same precision, ie,

the smallest terms in a_θ are $O\left(e\frac{an}{c}\right)$

When Taylor expansions to approximate the equations of motion, always be <u>consistent</u> when dropping small terms.

If we keep only $O\left(\frac{ean}{c}\right)$ terms and drop smaller $e^2\frac{an}{c}$ terms in a_r,

we <u>must</u> do the same when simplifying a_θ

For instance, if kept only the lowest order term in a_θ ie

$$a_\theta \simeq -\beta\left(\frac{an}{c}\right)\left[1 + O(e)\right]$$

we will probably arrive at incorrect result since our a_r is correct to $O(e^1)$ while our a_θ is only correct to $O(e^0)$

$\Rightarrow$ be consistent when dropping higher-order e^2 terms.

To calculate grain's orbit decay rate, insert a_r and a_θ into Gauss' planetary Eqn for a:

$$\dot{a} = \frac{2}{n\sqrt{1-e^2}}\left[a_r e\sin f + a_\theta(1+e\cos f)\right]$$

which is obviously a complicated mess of sinusoids

But we aren't interested in the instaneous value of $\dot{a}(t)$ which might vary widely during time t.

Instead we want the time-averaged value of $\langle\dot{a}\rangle$ which is the slow & steady rate at which PR drag damps the grain's orbit

To proceed, write $\dot{a}$ to order $O(e^1)$ (ie, keep terms proportional to e^0 & e^1 but drop the e^2 and smaller terms). This will give us a result that is then easily time-averaged:

$$e\sin f \simeq e\sin M + O(e^2)$$

$$\text{and } (1-e^2)^{-1/2} \simeq 1 + O(e^2)$$

$$\text{so } \dot{a} \simeq \frac{2}{n}(-\beta)\left(\frac{an}{c}\right)an^2\Big[2e^2\sin^2 M$$

$$+ (1+3e\cos M)(1+e\cos M)\Big]$$

what terms in [ ] should be preserved and neglected, consistent with our earlier approximations?

so $\left[\quad\right] \simeq \left[1 + 4e\cos M + 2e^2\sin^2 M + 3e^2\cos^2 M\right]$

$= \left[1 + 4e\cos nt + e^2 + e^2\cos^2 nt\right]$

since $M = nt$

so $\dot{a} = -2\beta\left(\frac{an}{c}\right)\left[1 + 4e\cos nt + O(e^2)\right] an$

lets average $\dot{a}$ over 1 orbit:

$\langle\dot{a}\rangle = -2\beta\left(\frac{an}{c}\right)\langle 1 + 4e\cos nt\rangle an$

what is the time average of

$\langle 4e\cos nt\rangle = \frac{1}{T}\int_0^T 4e(t')\cos(nt')\,dt'$

where T = particle's orbit period

looks complicated since grain's eccentricity
e = funtion of time.

But if PR drag is weak, then
$e(t)$ varies very little during 1 orbit,
we can treat $e(t')$ = constant, so

$\langle 4e\cos(nt)\rangle = 4e\langle\cos(nt)\rangle = ?$

so $\langle \dot{a} \rangle = -2\beta\left(\frac{an}{c}\right)an\left[1+\Theta(e^2)\right]$

this means that this approximate result is correct to first-order in e, and that terms neglected by our approximations are of order e^2 or smaller.

so $\dot{a} \simeq -2\beta\left(\frac{an}{c}\right)an$ = grain's sma decay rate

The orbit decay time scale is

$$\tau_a = \left|\frac{a}{\dot{a}}\right| = \frac{c/an}{2\beta n}$$

but T = orbit period $= \frac{2\pi}{n}$

so $\tau_a = \frac{c/an}{4\pi\beta} T$

For a $R = 1\mu m$ grain, $\beta \simeq 0.2$

if it orbits at $a = 1$ AU, $T = 1$ yr

$an = \sqrt{\frac{GM_*}{a}} = 30$ km/sec $= 3\times10^4$ cm/sec

= Kepler speed at 1 AU

since $c = 3\times10^{10}$ cm/sec then $\frac{c}{a n} = \frac{3\times10^{10}}{3\times10^{6}} = 10^{4}$

$\Rightarrow \quad \tau_a \simeq 4\times10^{3}$ yrs

This is roughly the time it takes a $R = 1\,\mu m$ grain at $a = 1$ AU to spiral into the sun.

To solve for the exact orbit decay timescale we need to integrate the DE

$$\frac{da}{dt} = -2\beta\frac{a^2n^2}{c}$$

but the above expression for $\tau_a = \left|\frac{a}{\dot{a}}\right|$

provides an easy esimate of decay time scale for PR drag.

Note that $\tau_a \propto \beta^{-1} \propto R$

so an $R = 10$ or $100\,\mu m$ grain (typical of interplanetary dust)
decays into Sun in

$\tau \sim 4\times10^{4\text{ or }5}$ yrs

<u>eccentricity damping</u> by PR drag

Gauss Planetary Eqn for e:

$$\dot{e} = \frac{\sqrt{1-e^2}}{an}\left[a_r \sin f + a_\theta \underbrace{(\cos f + \cos E_c)}_{\text{tricky terms}}\right]$$

$$= -\frac{\sqrt{1-e^2}}{an}\beta\left(\frac{an}{c}\right)an^2\left[2e\sin M \sin f + (1+3e\cos M)\underbrace{(\cos f + \cos E_c)}\right]$$

Again, time-average over an orbit period T:

$$\langle \dot{e} \rangle = \frac{1}{T}\int_0^T \dot{e}(t')\,dt'$$

assuming e(t), a(t) are constant during this short time interval.

and preserve on the RHS the non-zero time-averaged terms that are lowest-order in e:

Note that the tricky terms are NOT simply

$$\cos f + \cos E_c \simeq 2\cos M$$

Assignment #3 problem 2.11

show $\cos E_c \simeq \cos M + \frac{1}{2} e (\cos 2M - 1) + O(e^2)$

$\cos f \simeq \cos M + e(\cos 2M - 1) + O(e^2)$

so $\cos f + \cos E_c \simeq 2\cos M + \frac{3}{2} e (\cos 2M - 1)$

ie the $\cos f + \cos E_c \simeq 2\cos M$ approximation

neglects a constant $-\frac{3}{2}e$ on RHS

inserting this into $\langle \dot{e} \rangle$ to $O(e^1)$ yields:

$\langle \dot{e} \rangle \simeq -B\left(\frac{an}{c}\right) n \Big\langle 2e\sin^2 M$

$+ (1 + 3e\cos M)\left(2\cos M + \frac{3}{2}e\cos 2M - \frac{3}{2}e\right)\Big\rangle$

what is $\langle \sin^2 M \rangle$?

$$= \frac{1}{T}\int_0^T \sin^2(nt')\,dt'$$

Eqn A.2

$$= \frac{1}{T}\int_0^T \left(\frac{1}{2} - \frac{1}{2}\cos 2nt'\right) dt'$$

so $\langle \sin^2 M \rangle = \frac{1}{2}$

we also need

$$\left\langle 2\cos M + \tfrac{3}{2}e\cos 2M - \tfrac{3}{2}e + 6e\cos^2 M + \mathcal{O}(e^2) \right\rangle$$

$$= 0 + 0 - \tfrac{3}{2}e + 3e = \tfrac{3}{2}e$$

so $\langle \dot{e} \rangle = -\beta\left(\frac{an}{c}\right)n\left(e + \frac{3}{2}e\right)$

$$= -\frac{5}{2}\beta\left(\frac{an}{c}\right)en$$

= grain's e-damping rate due to PR drag

so $\tau_e = \left|\frac{e}{\langle \dot{e} \rangle}\right| = \frac{2(c/an)}{5\beta n} = \frac{c/an}{5\pi\beta}T$

$$= \frac{2}{5}\tau_a = \text{e-damping timescale}$$

so PR drag causes $a, e \to 0$
among $10 < R < 100\mu m$ dust
in $\tau \sim 10^{4 \text{ or } 5}$ yrs

so interplanetary dust quickly drains into the sun via PR drag.

The fact that the SS is filled w/ dust tells us the interplanetary dust complex is replenished by outgassing comets and colliding asteroids

Another calculation using Gauss' Planetary Eqns.

text prob 3.8:

Planets are not perfect spheres; they are oblate due to rotational flattening; they can also have tidal bulges raised by orbiting satellites. The gravitational potential of a rotating gas giant planet can be written

fatter at equator

$$\Phi(\vec{r}) = -\frac{GM_p}{r}\left[1 - \sum_{n=1}^{\infty} J_n \left(\frac{R_p}{r}\right)^n P_n(\sin\phi)\right]$$

coefficient, usually smaller for higher n

R_p = planet's mean radius

$P_n(\sin\phi)$ = Legendre polynomial

ϕ = latitude above/below planet's equator

J_n = zonal harmonic for even n.

for gas giants, the J_n where n=odd is usually very small, so for satellites in/near planet's equator at $r > R_p$

$$\Phi(r) \simeq \Phi_0 + \Phi_2 + \text{negligable terms}$$

$$\Phi_0 = -\frac{GM_p}{r} = \text{Kepler potential}$$

$$\Phi_2 = \frac{GM_p}{r} J_2 \left(\frac{R_p}{r}\right)^2 P_2(0) = -\frac{GM_p J_2 R_p^2}{2r^3}$$

Legendre polynomial $P_2(0) = -\frac{1}{2}$, see A.26

so Φ_2 = gravitational potential of the non-spherical part of the planet's gravity

$$a_r = -\frac{\partial \Phi_2}{\partial r} = -\frac{3GM_p J_2 R_p^2}{2r^4}$$

= radial acceleration due to oblate planet's J_2

Saturn is the most oblate planet in solar System, has $J_2 \simeq 0.016$

for a close satellite having r just above R_p so $r \simeq R_p$,

$$|a_r| \simeq \frac{3}{2} J_2 \left(\frac{GM_p}{r^2}\right)$$

$\Rightarrow$ gravity due to Saturn's oblate figure is $\simeq 2\%$ of planet's $\frac{GM_p}{r^2}$

Plug a_r into $\dot{\tilde{\omega}}$ = rate at which satellite's longitude of perape varies, ie its precession rate

$$\dot{\tilde{\omega}} = \frac{2a_\theta \sin f - a_r \cos f}{ean}$$

$a_\theta = ?$

so $\dot{\tilde{\omega}} = +\frac{\cos f}{ean}\frac{3GM_p J_2 R_p^2}{2r^4}$

from problem 2.11

where $\cos f \simeq \cos M + e(\cos 2M - 1) + O(e^2)$

$r \simeq a(1 - e\cos M) + O(ae^2)$

so $\dot{\tilde{\omega}} = \frac{3GM_p J_2 R_p^2 (\cos M + O(e))}{2ea^5 n(1 - e\cos M)^4}$

$GM_p = n^2 a^3$

$$\dot{\tilde{\omega}} \simeq \frac{3n J_2 R_p^2}{2e a^2}\left[\cos M + e(\cos 2M - 1)\right]\left(1 + 4e\cos M\right)$$

which again involves many periodic terms, but the sinusoidal contributions to $\dot{\tilde{\omega}}$ sum to zero every orbit

So time average $\dot{\tilde{\omega}}$ to get the <u>secular</u> rate at which $\tilde{\omega}$ varies

$$\langle \dot{\tilde{\omega}} \rangle = \frac{3J_2}{2e}\left(\frac{R_p}{a}\right)^2 n \left\langle \cos M + 4e\cos^2 M + e(\cos 2M - 1) + O(e^2) \right\rangle$$

what is $\langle \cos M \rangle$?

$\langle \cos 2M \rangle$?

so $\langle \, \rangle = e$

and $\langle \dot{\tilde{\omega}} \rangle = \frac{3}{2} J_2 \left(\frac{R_p}{a}\right)^2 n$

= rate at which the satellite's long of peri advances due to planet's oblateness