1 September 2013

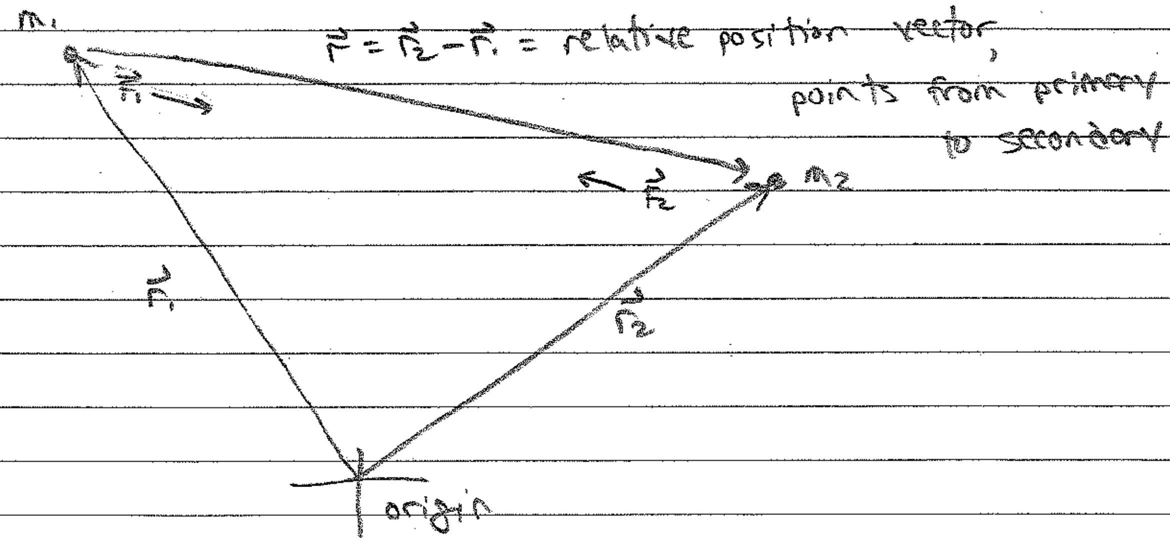
Chap 2:

Two Body Problem

Solve for the motion of 2 gravitating bodies, eg, star + planet or planet + satellite.

Two bodies: m_1 = mass of larger primary

m_2 = mass of smaller secondary



gravitational forces:

Newton's Law of Gravity:

NII

$$\vec{F}_1 = + \frac{G m_1 m_2}{r^3}\vec{r} = m_1 \ddot{\vec{r}}_1 = \text{force on } m_1 \text{ due to } m_2$$

$$\vec{F}_2 = - \frac{G m_1 m_2}{r^3}\vec{r} = m_2 \ddot{\vec{r}}_2 = \text{force on } m_2 \text{ due to } m_1$$

signs are used to indicate direction

$\vec{F}_1 = -\vec{F}_2$ so NIII is obeyed.

equation for the relative motion

$$\ddot{\vec{r}} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 = -\frac{G m_1}{r^3}\vec{r} - \frac{G m_2}{r^3}\vec{r} = -\frac{\mu}{r^3}\vec{r}$$

where $\mu = G(m_1 + m_2)$

note that of the $\vec{r}$ vector rides on m_1, which is accelerated by m_2

are we using an inertial refrence frame?

is this a problem?

Derive this system's <u>integrals of the motion</u>

constants (sometimes useful) that are derived from the equation of motion (EOM)

<u>angular momentum integral</u>

$$\vec{r} \times \ddot{\vec{r}} = \frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) \propto \vec{r} \times \vec{r} = 0$$

$$\Rightarrow \vec{h} = \vec{r} \times \dot{\vec{r}} = \text{constant}$$

aka ang. mom. integral

aka ang mom per mass

aka specific angular momentum

Note that $\vec{h}$ is not the system's total angular momentum: (why?)

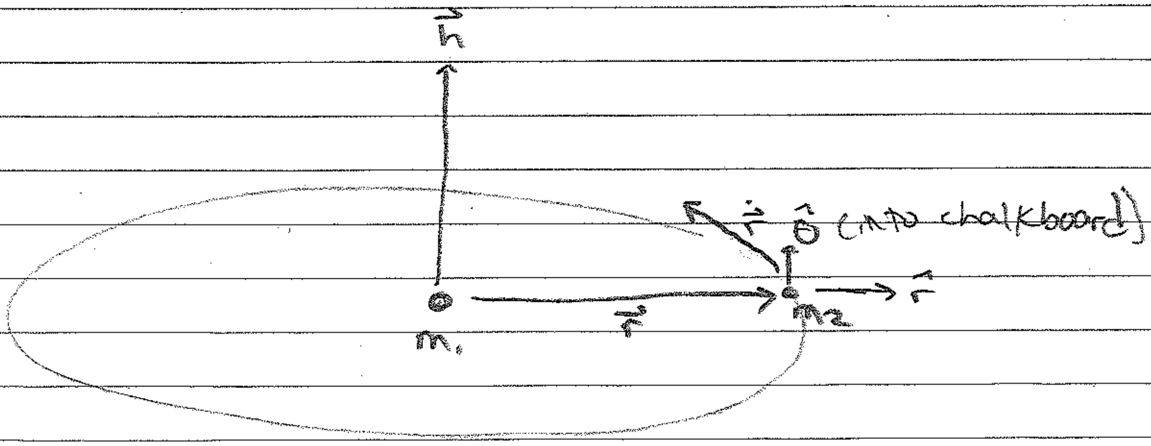
$$\vec{L} = \mu_r \vec{h} \quad \text{where} \quad \mu_r = \frac{m_1 m_2}{m_1 + m_2}$$

= system's reduced mass

which you will confirm in Assignment #2: 2.3

Note that $\vec{h} = \vec{r} \times \dot{\vec{r}}$ = constant, is perpendicular to $\vec{r}$ & $\dot{\vec{r}}$

$\Rightarrow$ this tells us that m_1 and m_2's motion is confined to a plane:



The motion is coplanar, so use cylindrical coords:

$$\vec{r} = r\hat{r} \quad \text{and} \quad \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \quad \text{since } z=0$$

$$\text{so} \quad \vec{h} = \vec{r} \times \dot{\vec{r}} = r^2\dot{\theta}\,\hat{r}\times\hat{\theta} = r^2\dot{\theta}\,\hat{z}$$

$$= h\hat{z}$$

so $h = |\vec{h}| = r^2\dot{\theta}$ is the magnitude of $\vec{h}$

end of Sept 3 lecture

<u>Energy Integral</u>

$$\text{EOM} \quad \ddot{\vec{r}} = -\frac{\mu\vec{r}}{r^3}$$

$$\dot{\vec{r}}\cdot\ddot{\vec{r}} = -\frac{\mu\dot{\vec{r}}\cdot\vec{r}}{r^3} \qquad \vec{r} = r\hat{r}$$

$$\dot{\vec{r}}\cdot\vec{r} = \dot{r}r$$

$$\text{so} \quad \dot{\vec{r}}\cdot\ddot{\vec{r}} + \frac{\mu\dot{r}}{r^2} = 0$$

note that $\frac{1}{2}\frac{dv^2}{dt} = \frac{1}{2}\frac{d}{dt}\left(\dot{\vec{r}}\cdot\dot{\vec{r}}\right) = \dot{\vec{r}}\cdot\ddot{\vec{r}}$

Also note $\frac{d(r^{-1})}{dt} = -\frac{\dot{r}}{r^2}$

$$\text{so} \quad \dot{\vec{r}}\cdot\ddot{\vec{r}} + \frac{\mu\dot{r}}{r^2} = \frac{d}{dt}\left(\frac{1}{2}v^2 - \frac{\mu}{r}\right) = 0$$

$$\Rightarrow \mathcal{E} = \tfrac{1}{2}v^2 - \underbrace{\frac{\mu}{r}}_{} = \text{constant}$$

specific kinetic energy

resembles gravitational potential
ie specific potential energy

This is the system's energy integral,
which is NOT the system's total energy (why?)

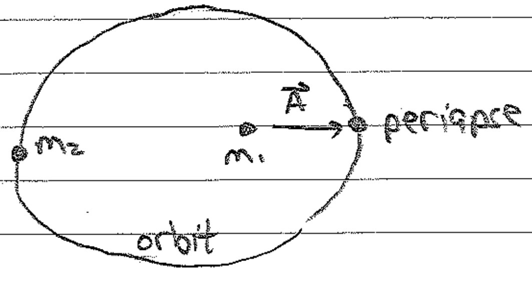
rather, $E = \mu_r \mathcal{E}$ = system's total energy

Assignment #2: problem 2.4

Another useful integral is the Laplace Runge Lenz vector:

$$\vec{A} = \frac{\vec{r}\times\vec{h}}{\mu} - \hat{r}$$

Assignment #2: problem 2.5
after we
2-body problem you will see that
$\vec{A}$ points to the site of m_2's closest approach to m_1
= periapse

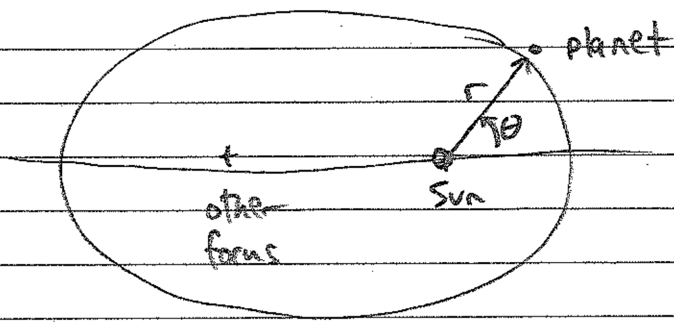


published 1609-1619

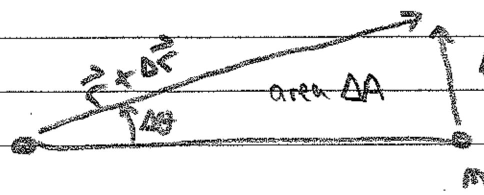
Kepler's 3 empirical Laws of Planetary motion

I a planet travels along an ellipse about the sun, with the sun at 1 focus

eqn for ellipse $r(\theta) = \frac{p}{1+\varepsilon\cos\theta}$



II. The planet's position vector sweeps out equal areas in equal times



$\Delta\vec{r}$ = distance traveled in time Δt

III planet's orbit period^2 $\propto$ semimajor axis^3

All 3 Kepler's Laws follow from

Newton's law's of motion + law of gravitation.

KII: ΔA = area swept out as planet's position vector $\vec{r}$ advances to $\vec{\Delta r}$ during short time interval Δt

$$= \tfrac{1}{2}\,\text{base} \cdot \text{height} \simeq \tfrac{1}{2} r \Delta\theta \cdot r$$

so $$\frac{dA}{dt} = \frac{\Delta A}{\Delta t}\Big|_{\Delta t \to 0} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h = \text{constant}$$

since $h = r^2\dot{\theta}$ = ang. mom. integral

Kepler's 2^{nd} Law says that the planet's position vector has constant areal velocity $\propto h$

To derive K1, we need to solve the 2-body EOM: $\ddot{\vec{r}} = -\dfrac{\mu \vec{r}}{r^3}$

recall $$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\,\hat{\theta} + \ddot{z}\,\hat{z}$$

in cylindrical coordinates

so radial EOM $$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$$

does anyone know the solution to this DE for $r(t)$ and $\theta(t)$?

trick: use alternate variable

$$u = \frac{1}{r}$$

And assume that $u = u(\theta)$ where $\theta = \theta(t)$

$$r = u^{-1} \quad \text{so} \quad \dot{r} = -\frac{\dot{u}}{u^2} = -u^{-2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

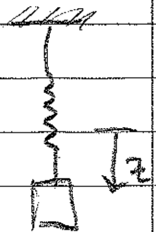
but $h = r^2\dot{\theta}$ so $\dot{\theta} = \frac{h}{r^2} = hu^2$

so $\dot{r} = -h\frac{du}{d\theta}$

and $\ddot{r} = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -h^2u^2\frac{d^2u}{d\theta^2}$

the radial EOM: $-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^{4-1} = -\mu u^2$

$$\text{so} \quad \frac{d^2u}{d\theta^2} = -u + \frac{\mu}{h^2}$$



this EOM resembles that of a mass suspended by spring subject to the downward pull of gravity:

$$\ddot{z} = -\omega_0^2 z + g$$

phase constant

which has solution $z(t) = A\cos(\omega_0 t - \delta) + z_0$

$z_0 = \omega_0^2/g$ = displacement due to grav.

so $u(\theta) = A\cos(\theta - \omega) + B$

where A, B, ω are constants

insert trial solution back into EOM:

$$\frac{d^2u}{d\theta^2} = -A\cos(\theta-\omega) = -A\cos(\theta-\omega) - B + \frac{\mu}{h^2}$$

$$\Rightarrow B = \mu/h^2$$

Also set $A = eB$ where e = another constant

$$u(\theta) = \frac{\mu}{h^2}\left[e\cos(\theta-\omega) + 1\right]$$

$$\text{so } r(\theta) = \frac{h^2/\mu}{1 + e\cos(\theta-\omega)} = \frac{p}{1+e\cos(\theta-\omega)}$$

which is Keplers first law

This is the equation for a <u>conic section</u>

= intersection of a plane & cone

(see text fig 2.3)

where $p = \dfrac{h^2}{\mu}$

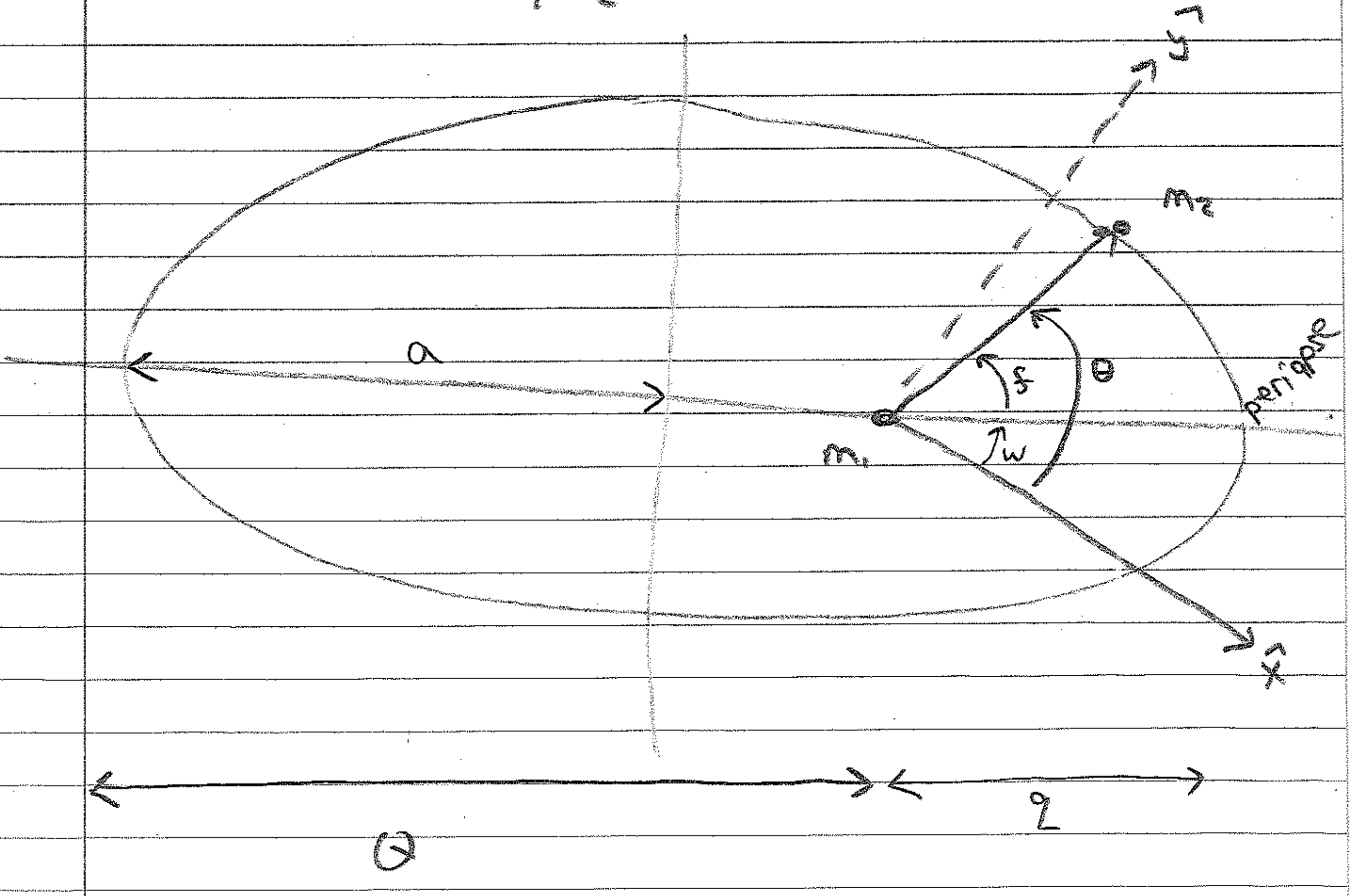
$$r(\theta) = \frac{p}{1+e\cos(\theta-\omega)} =$$ secondary's ~~m_2's orbit (trajectory)~~ about primary m_1

minimum seperation occurs at longitude $\theta = \omega$

$$r(\theta) = q = \frac{p}{1+e} = \text{periapse distance}$$

If m_2 is gravitationally bound to m_1
ie $\mathcal{E} < 0$ Then max separation is at $\theta = \omega + \pi$

$$r(\theta) = Q = \frac{p}{1-e} = \text{apoapse distance}$$



The length of
The ellipse's long axis $= 2 \times$ semimajor axis

$$= 2a = q + Q$$

$$= \frac{p}{1+e} + \frac{p}{1-e}$$

so $p = a(1-e^2) = \frac{h^2}{\mu}$

$$= \frac{p(1-e) + p(1+e)}{(1-e)^2}$$

$$= \frac{2p}{1-e^2}$$

so $h = \sqrt{\mu p} = \sqrt{\mu a(1-e^2)} =$ ang. mom. integral

when written in terms of the orbit elements a, e

and $r(\theta) = \frac{a(1-e^2)}{1+e\cos(\theta-\omega)}$

The orbit's shape depends on its eccentricity e:

when $e=0$, $p=a$, $r(\theta)=a$, orbit = circle

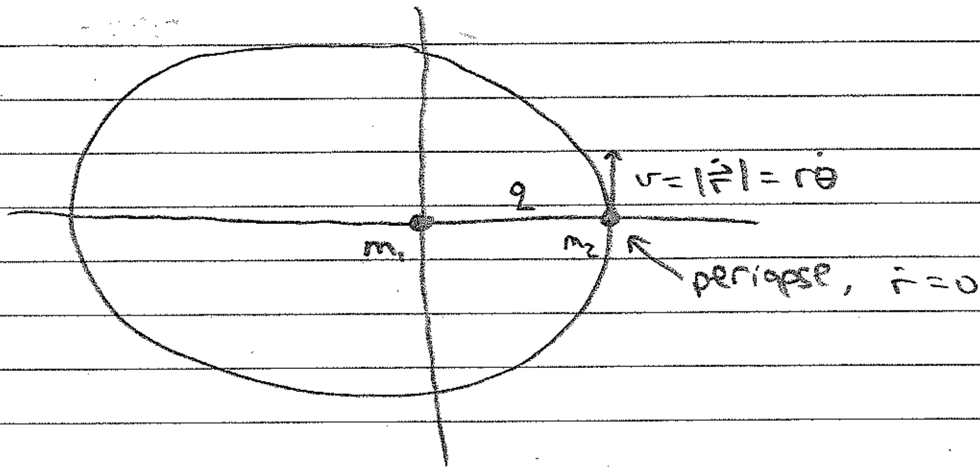
when $0<e<1$,

$$q = a(1-e) = \frac{a(1-e^2)}{1+e} \le r(\theta) \le \frac{a(1-e^2)}{1-e} = a(1+e) = Q$$

apoapse

so $q \le r(\theta) \le Q$, orbit = ellipse

periapse



let's calculate energy integral $\mathcal{E}$ = constant when m_2 is at periapse (closest approach to m_1)

$r = q$ and $v = |\dot{\vec{r}}| = r\dot{\theta}$ since $\dot{r}=0$ at peri

$= a(1-e)$ $\qquad = \frac{h}{r}$

where $h = \sqrt{\mu a(1-e^2)}$ = ang. mom. integral so $$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} = \frac{h^2}{2r^2} - \frac{\mu}{r} = \frac{\mu a(1-e^2)}{2a^2(1-e)^2} - \frac{\mu}{a(1-e)}$$

$$= \frac{\mu(1+e)}{2a(1-e)} - \frac{\mu}{a(1-e)} = \frac{\mu}{a(1-e)}\left(\frac{1+e}{2} - 1\right)$$

$$= \frac{\mu}{a(1-e)}\,\frac{e-1}{2}$$

$$\Rightarrow \mathcal{E} = -\frac{\mu}{2a} = -\frac{G(m_1+m_2)}{2a}$$

so bound elliptical orbits have

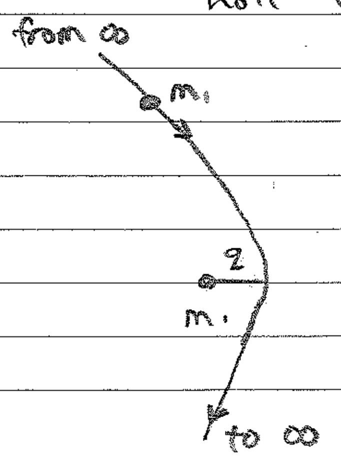
semimajor axis $a>0$
eccentricity $0 \le e < 1$
energy integral $\mathcal{E} < 0$

unbound ie hyperbolic orbits have

$a<0$, $\mathcal{E}>0$, and $e>1$

note that $r = \dfrac{a(1-e^2)}{1+e\cos(\theta-\omega)} \to \infty$

when denominator $\to 0$



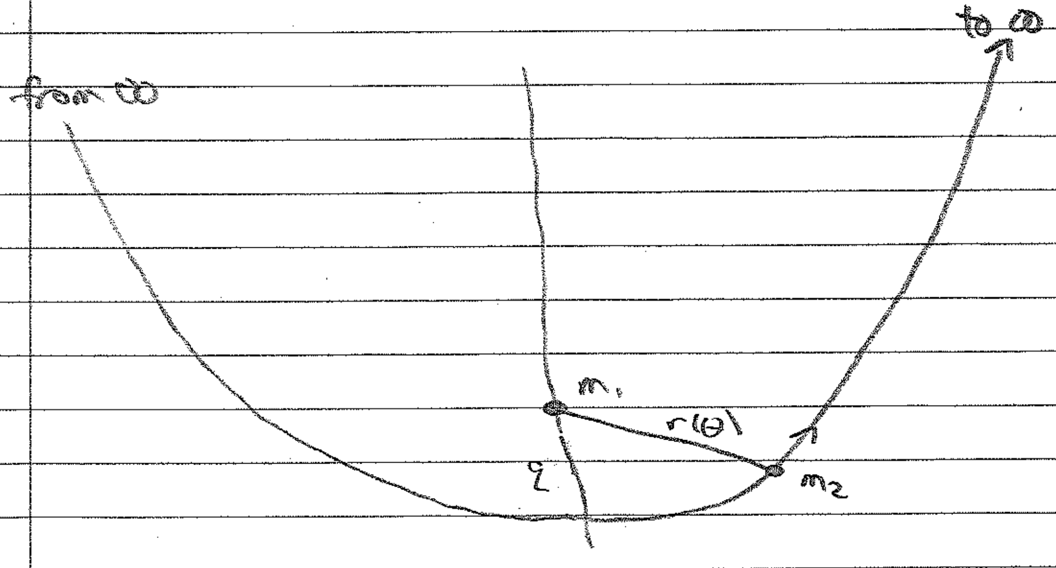
See text section 2.4
for more on hyperbolic
orbits

The $\mathcal{E}=0$ orbit is a parabolic orbit,
(divides elliptic $\mathcal{E}<0$ orbits from hyperbolic $\mathcal{E}>0$ orbits)

start with elliptic orbit and take limit $e \to 1$:

$$r = \frac{a(1-e^2)}{1+e\cos(\theta-\omega)} \longrightarrow \frac{2q}{1+\cos(\theta-\omega)}$$

This orbit is characterized by its periapse distance q



Comets in Oort cloud have $a \sim \mathcal{O}(10^4\ \mathrm{AU})$,
barely bound to solar system,
their orbits are nearly parabolic

The system's total inertial energy is

$$E = \mu_r \mathcal{E} = -\frac{Gm_1m_2}{2a}$$ = KE + PE of 2-body system calculated in $\frac{\text{inertial}}{\text{ref}}$ frame

where $\mu_r = \frac{m_1m_2}{m_1+m_2}$ = reduced mass

we often set $f = \theta - w$ = <u>true anomaly</u>

so $r(f) = \frac{a(1-e^2)}{1+e\cos f}$

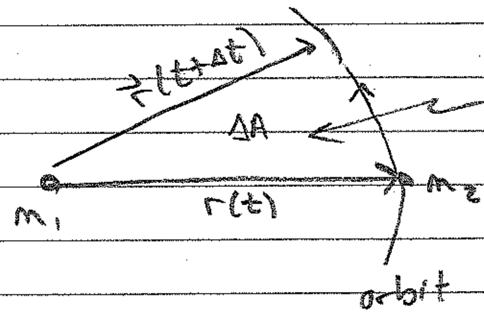
= m_2's longitude relative to periapse

w = argument of periapse
= angle between $\hat{x}$ direction and periapse

Note that $r(f)$ = equation for ellipse with origin (where m_1 is) at one of ellipse's <u>focus</u>

This confirms Kepler's 1^{st} Law of planetary motion confirm Kepler's 3^{rd} Law, that $T^2 \propto a^3$:

recall $\frac{dA}{dt}$ = areal velocity = rate at which $\vec{r}$ sweeps area

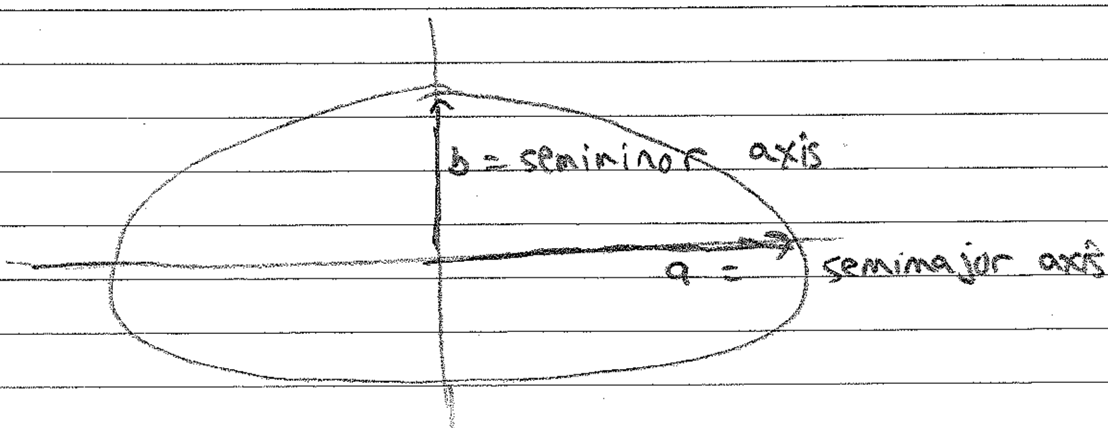


area ΔA is swept by $\vec{r}$ at rate $\frac{dA}{dt}$

set T = orbit period = time for m_2's motion to repeat

so $A = \int_0^T \frac{dA}{dt}\,dt = \frac{1}{2}hT$ = area enclosed by m_2's orbit

area of ellipse: $A = \pi ab$



Assignment #2 prob 2.2: show $b = a\sqrt{1-e^2}$

so $T = \frac{2A}{h} = \frac{2\pi a^2 \sqrt{1-e^2}}{\sqrt{\mu a(1-e^2)}} = 2\pi \sqrt{\frac{a^3}{\mu}}$

or $T = 2\pi \sqrt{\frac{a^3}{G(m_1+m_2)}}$ = orbit period

and $T^2 \propto a^3$ is Kepler's 3rd law

we also use $n(a) = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}$ = mean motion

= angular velocity of circular orbit

(recall our disk problem where $\dot{\theta} = n \propto a^{-3/2}$)

m_2's motion over time

ellipse equation $r(\theta) = \dfrac{a(1-e^2)}{1+e\cos(\theta-\omega)}$

only tells you m_2's distance from m_1 at longitude θ.

we still need to solve for $\theta(t)$ to know where m_2 is at various times t.

The obvious way to attack this problem is to start with

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{h}{r^2} = \frac{h}{p^2}\left[1+e\cos(\theta(t)-\omega)\right]^2$$

how do you solve this for $\theta(t)$?

again, another math trick:

start with $\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$

where $v^2 = \dot{\vec{r}}\cdot\dot{\vec{r}} = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + \frac{h^2}{r^2}$

so $v^2 = \frac{2\mu}{r} - \frac{\mu}{a} = \dot{r}^2 + \frac{\mu a(1-e^2)}{r^2}$

$$\dot{r}^2 = \frac{\mu}{a}\left[\frac{2a}{r} - 1 - \left(\frac{a}{r}\right)^2(1-e^2)\right]$$ ← yield rad velocity $\dot{r}(r)$

also recall $\mu = n^2a^3$

so $\dot{r}^2 = \left[\left(\frac{ae}{r}\right)^2 - \left(\frac{a}{r} - 1\right)^2\right](an)^2$

Also recall $r(f) = \frac{a(1-e^2)}{1+e\cos f}$

so $\dot{r} = \frac{a(1-e^2)}{(1+e\cos f)^2}\, e\sin f \cdot \dot{f}$

where $\dot{f} = \dot{\theta} = \frac{h}{r^2}$ so $\dot{r} = \frac{r^2 e\sin f}{a(1-e^2)}\frac{h}{r^2}$

$$\dot{r} = \frac{e\sin f\sqrt{\mu a(1-e^2)}}{a(1-e^2)} = \frac{ean\sin f}{\sqrt{1-e^2}}$$ ← $\dot{r}(f)$

m_2's tangential velocity

$$r\dot{f} = r\dot{\theta} = \frac{h}{r} = \frac{\sqrt{\mu a(1-e^2)}}{a(1-e^2)}(1+e\cos f)$$

$$\text{so} \quad r\dot{f} = \frac{an}{\sqrt{1-e^2}}\left(1+e\cos f\right)$$

Note for circular orbit, $e=0$, $r=a$, $\dot{r}=0$

$$\text{and} \quad r\dot{f} = an = \sqrt{\frac{G(m_1+m_2)}{a}} = V_K$$

= circular velocity of Keplerian orbit

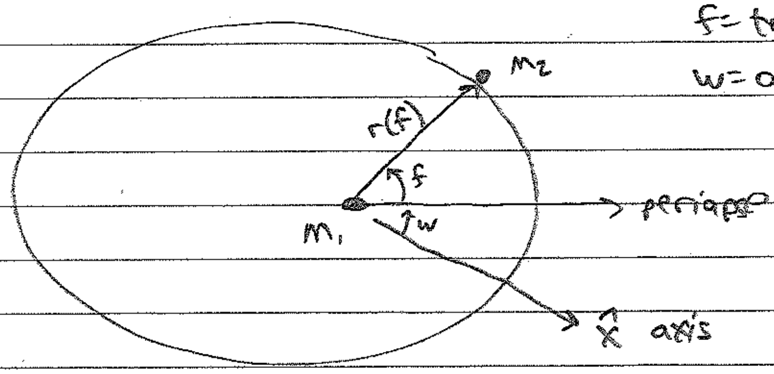
We will need these expressions for m_2's radial $\dot{r}$ and tangential $r\dot{\theta}$ velocities

Kepler's Eqn for elliptical orbit

let's solve for Kepler's eqn, which can be used to compute m_2's longitude $\theta(t)$ at time t.

Assume m_2 is bound to m_1, its orbit is an ellipse, so $\mathcal{E}<0$, $a>0$, $0 \le e < 1$

f = true anomaly
w = arg. of periapse



we know

$$r(f) = \frac{a(1-e^2)}{1+e\cos f} = \frac{a(1-e^2)}{1+e\cos(\theta-w)}$$

if we solve for $\theta(t) = f(t) - w$, ← constant

then we know $r(t)$ and $\theta(t)$, which is the desired solution.

begin by assuming that

$$r = r(E_c) = a(1 - e\cos E_c)$$

where <u>eccentric anomaly</u> $E_c = E_c(t)$

why assume this?

differentiate: $\dot{r} = ae\dot{E}_c \sin E_c$

$$\text{so } \dot{r}^2 = a^2e^2\dot{E}_c^2\sin^2 E_c$$

$$= (an)^2\left[\left(\frac{ae}{r}\right)^2 - \left(\frac{a}{r} - 1\right)^2\right]$$

$$= (an)^2\left[\left(\frac{e}{1 - e\cos E_c}\right)^2 - \left(\frac{1}{1 - e\cos E_c} - 1\right)^2\right]$$

$$= (an)^2\left[\frac{e^2}{(1 - e\cos E_c)^2} - \frac{e^2\cos^2 E_c}{(1 - e\cos E_c)^2}\right]$$

$$= \frac{(ane)^2(1 - \cos^2 E_c)}{(1 - e\cos E_c)^2}$$

$$= \frac{(ean)^2\sin^2 E_c}{(1 - e\cos E_c)^2}$$

$$\text{so } n^2 = (1 - e\cos E_c)^2\dot{E}_c^2$$

where $n = \sqrt{\frac{\mu}{a^3}}$ = mean motion stop Sept 5

all terms in the above are always positive so take square root w/o worrying about sign errors:

$$n = (1 - e\cos E_c)\frac{dE_c}{dt}$$

so $n\,dt = (1 - e\cos E_c)\,dE_c$

integrate over time $\tau \le t' \le t$

$$\underbrace{\int_{\tau}^{t}}_{\text{some reference time}} n\,dt' = \int_{E_c(\tau)}^{E_c(t)} (1 - e\cos E_c')\,dE_c'$$

E_c when $t = \tau$

for convenience, choose τ = time when m_2 is at periapse = when m_2 is closest to m_1

$\Rightarrow$ integration constant τ = time of periapse passage

what is E_c?

$$\text{so} \quad \int_{\tau}^{t} n\,dt' = n(t-\tau) = \int_{0}^{E_c(t)} (1 - e\cos E_c')\,dE_c'$$

$$= \left(E_c' - e\sin E_c'\right)\Big|_{0}^{E_c(t)}$$

$$= E_c - e\sin E_c$$

set $M = n(t-\tau)$ = <u>mean anomaly</u> = some angle that advances linearly w/ time

so $M = n(t-\tau) = E_c - e\sin E_c$ = Kepler's Eqn (KE)

Note that KE is <u>trancendental</u>, must be solved <u>numerically</u>, see Danby's textbook for algorithm.

given time t,
solve KE for E_c,
plug E_c into $r = a(1 - e\cos E_c)$ to get $r(t)$

to finish the solution you could finish up by solving $r(f) = \dfrac{a(1-e^2)}{1+e\cos f}$ for f

but the sign of f will be ambiguous since $\cos(-f) = \cos f$

but the prefered way is to calculate f from

$$\tan\left(\frac{f}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E_c}{2}\right)$$

the sign of f will not be ambiguous

derivation of $\tan\left(\frac{f}{2}\right)$ is on page 33 of Murray & Dermott, 1999, <u>Solar System Dynamics</u>

The derivation of $\tan(f/2) = \ldots$
should appear in my notes or text,
but it's boring algebra, not included...

Also see section 2.5.3 for derivation
of Keplers Eqn for hyperbolic orbits

Kepler's Eqn relates time $t \to M \to E_c \to r, f$
is trancendental, usually must be
solved numerically (see Danby ref
at end of chapter for iterative algorithm)

However analytic solution to KE exists
when $e \ll 1$ and orbit is nearly circular.
we will use the following often in
this class:

expansions for low e:

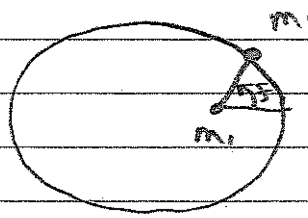
recall $r = a(1 - e\cos E_c)$

$$M = n(t - \tau) = E_c - e\sin E_c$$

so $E_c \simeq M + \mathcal{O}(e)$

so $r(M) \simeq a - ea\cos(M) + \mathcal{O}(e^2)$

where $M = nt$ assuming periapse passage time $\tau = 0$



to get $f(t)$ = true anomaly

$\mu = G(m_1 + m_2) = n^2 a^3$

use $h = r^2 \dot{f} = \sqrt{\mu a(1-e^2)} = na^2\sqrt{1-e^2}$

$$\text{so } \frac{df}{dt} = n\left(\frac{a}{r}\right)^2 \sqrt{1-e^2} \simeq \frac{n\left[1 - \frac{1}{2}e^2 + \mathcal{O}(e^4)\right]}{1 - 2e\cos M + \mathcal{O}(e^2)}$$

by binomial thm A.1
$(1+x)^r \simeq 1 + rx + \mathcal{O}(x^2)$

$$\text{so } \frac{df}{dt} \simeq n\left[1 + 2e\cos M + \mathcal{O}(e^2)\right]$$

$$\text{or } \frac{df}{dM} = 1 + 2e\cos M \quad \text{since } M = nt$$

integrate: $f(M) \simeq M + 2e\sin M + \mathcal{O}(e^2)$

These are the 1^{st} order expressions

for $r(t \text{ or } M)$ and $f(t \text{ or } M)$

Occasionally you need equations for $r(t)$ and $f(t)$ that are correct to $\Theta(e^2)$

Assign #2 prob 2.10

2.2, 2.3, 2.4, 2.5, 2.6, 2.10, 2.12, 2.14
due Tues Sept 24

Guiding Center Approximation:

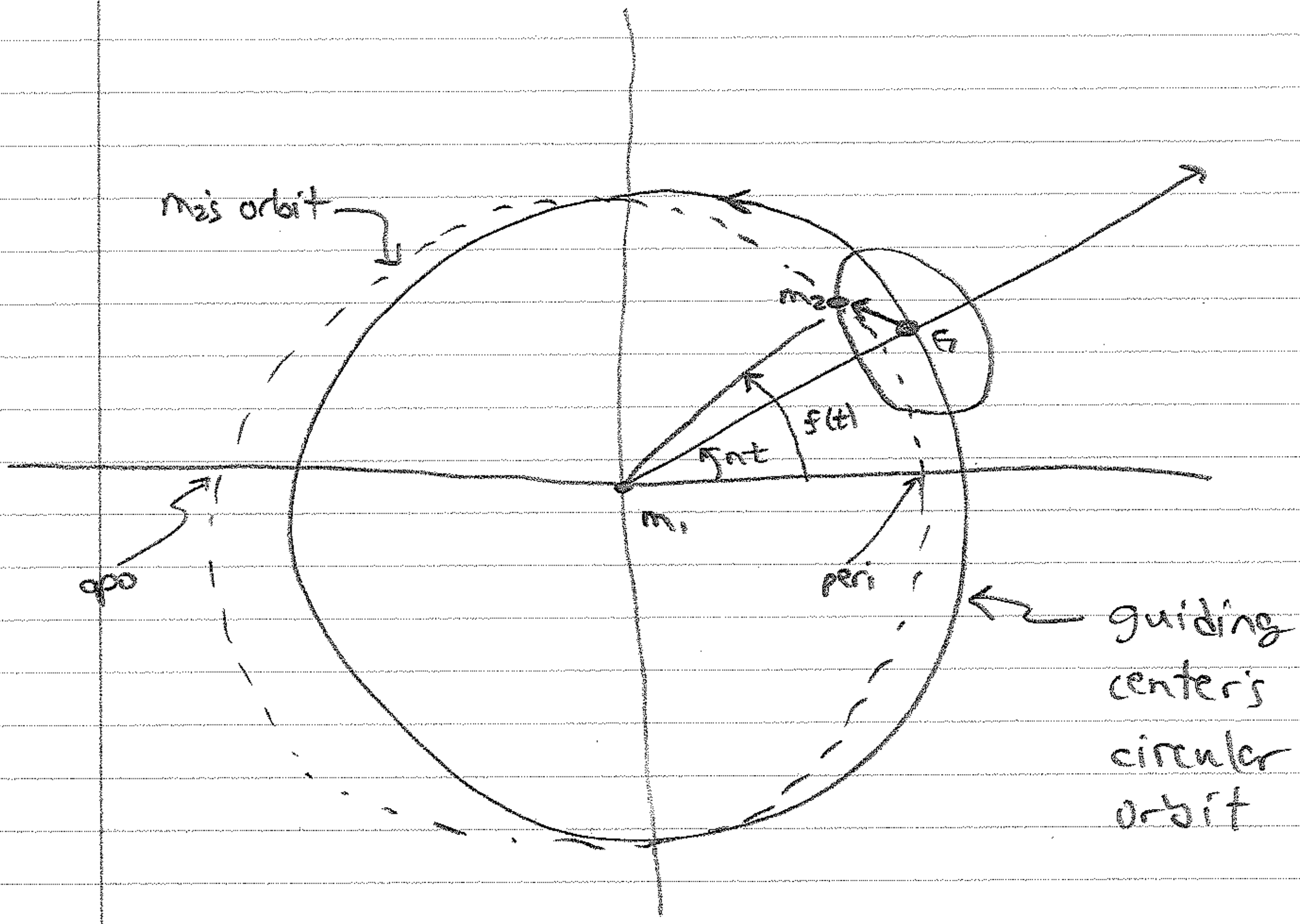
$r(t) = a - ae\cos nt \equiv a + x(t)$

$f(t) = nt + 2e\sin(nt) = nt + \frac{y(t)}{a}$

where $x(t) = -ae\cos nt$

$y(t) = 2ea\sin(nt)$

so m_2's motion = guiding center G
in circular orbit
about m_2 + offset x, y



These expressions for $r(t)$, $f(t)$
are known as the <u>guiding center approximation</u>

m_2's motion about G an <u>epicycle</u>
= small circle that rides about a large circle (deferent)

Epicyclic motion was the Earth-centered model that Greek astronomer Ptolemy (~ 150 AD) used to describe planetary motions

The prevailing cosmology was the Earth-centered Ptolemaic model until Copernicus published his Sun-centered model in 1543,

Galileo's observations of Jupiter's moons and Venus' phases in 1610 cast doubt on Plotemy model and strongly supported Copernicus' Sun-centered model

Kepler's 3 Laws of planetary motions published 1609 & 1619 confirmed Copernican model.

m_2's 2D orbit in 3D space

in general, the orbit is tilted or inclined wrt a generic $\hat{x}$-$\hat{y}$ plane; that $\hat{x}$-$\hat{y}$ plane is called the reference plane

recall that the shape of m_2's orbit in the orbit plane can be specified by 4 orbit elements: a, e, ω, τ

but we need to more elements: i, Ω to specify that orbit in 3D space

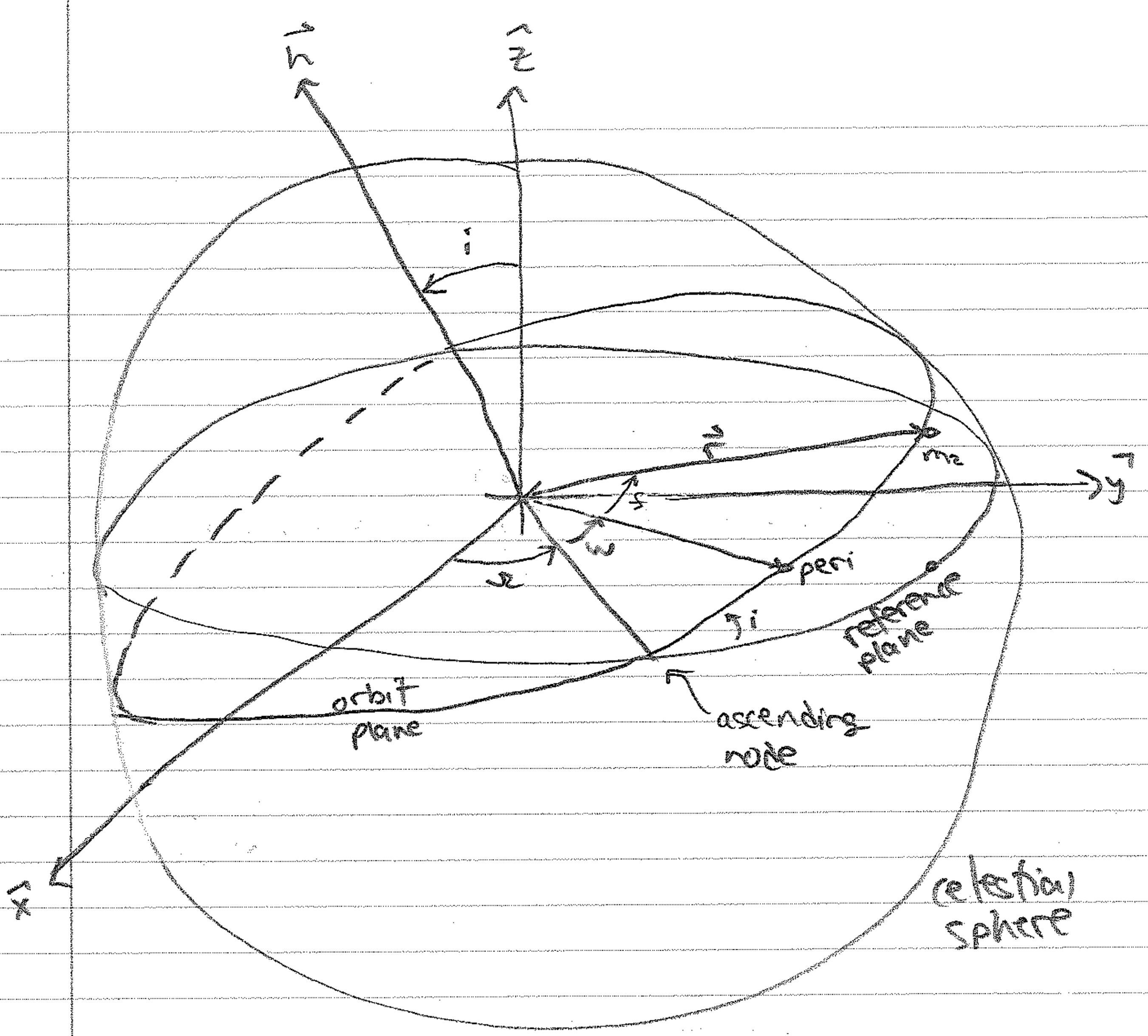
orbit tilt wrt $\hat{x}$-$\hat{y}$ plane

orientation of tilted orbit plane

so we have 6 orbit elements:

$\underbrace{a, e, i, \omega, \Omega, \tau}$

with these 6 orbit elements you can calculate m_2's $\vec{r}$ and $\dot{\vec{r}}$ at any time t



Ω = longitude of ascending node
= long. where m_2's orbit carries it up through reference $\hat{x}$-$\hat{y}$ plane

w = argument of periapse
= angle between ascending node (which is a spot on the celestial sphere aka 'sky') and periapse

f = true anomaly = angle between asc. node and periapse
(f is NOT an orbit element)

i = orbit inclination = angular tilt between $\hat{z}$ and ang. mom. integral $\vec{h}$

another set of orbit elements: $a, e, i, \Omega, \omega, M$

sometimes we replace τ with $M = n(t-\tau)$, which is handy because time t is embedded in M

and another set: $a, e, i, \Omega, \tilde{\omega}, \tau$

where $\tilde{\omega} = \omega + \Omega$ = longitude of periapse
= "dog-leg" angle since ω, Ω are not necessarily coplanar

transform orbit-plane coordinates r, f
→ reference plane coordinates x, y, z

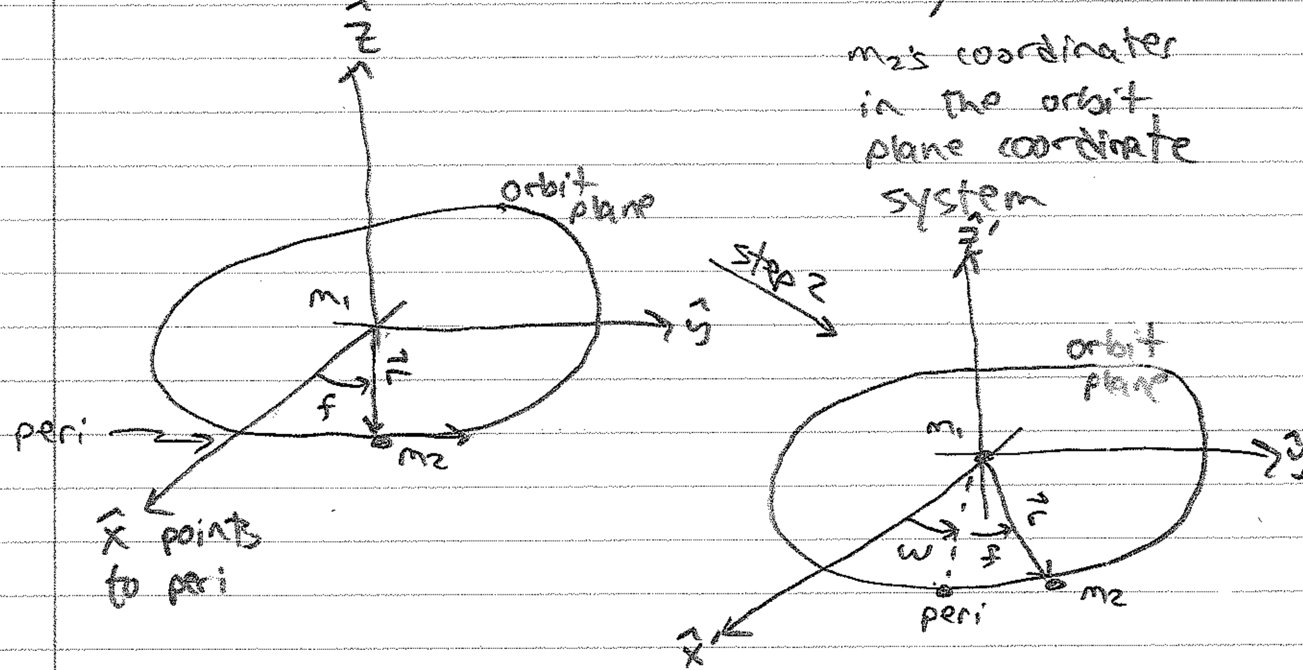
suppose we know m_2's orbit elements $a, e, i, \omega, \Omega, M$, which includes time-info, so where m_2 is in its orbit

converting elements → cartesian coordinates requires 4 steps:

step 1: solve Kepler's equation for r, f, and write its cartesian coordinates as the column vector

$$\vec{r} = r\begin{pmatrix}\cos f\\ \sin f\\ 0\end{pmatrix} = \begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

m_2's coordinates in the orbit plane coordinate system



II: <u>rotate the coordinate system by angle $-w$ about $\hat{z}$ axis</u>, where w = arg. of peri

use right-hand rule to do these rotations, ie right thumb points along rotation axis while fingers curl towards <u>positive</u> rotation angles this rotation can be described via matrix math:

$$\vec{r}\,' = R_z(-\omega)\,\vec{r}$$

m_2's position vector in rotated coordinate system

rotation matrix, this rotates the coordinate system about $\hat{z}$ axis by angle $-\omega$

m_2's position in unrotated coordinate system

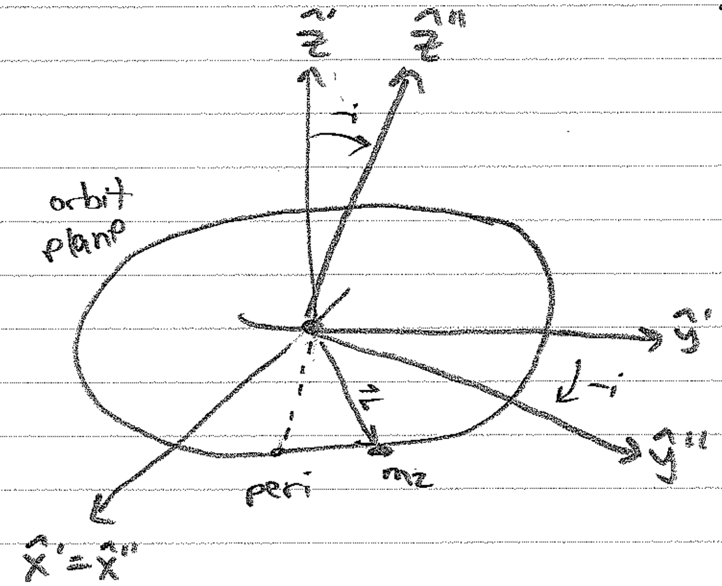
see appendix B

$$\vec{r}\,' = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} r \begin{pmatrix} \cos f \\ \sin f \\ 0 \end{pmatrix}$$

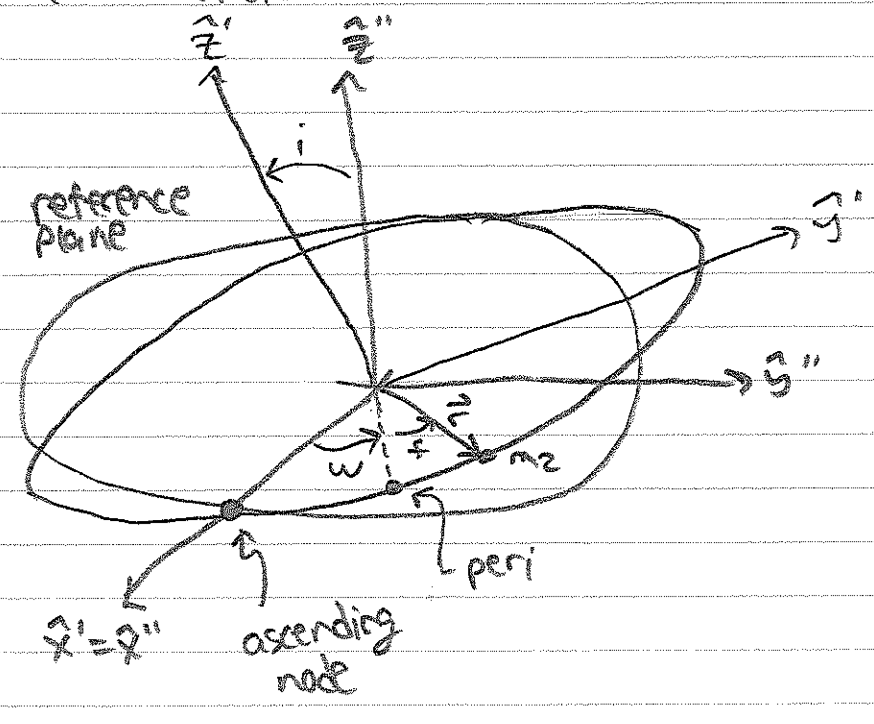
$$= r \begin{pmatrix} \cos\omega\cos f - \sin\omega\sin f \\ \sin\omega\cos f + \cos\omega\sin f \\ 0 \end{pmatrix}$$

step III: rotate coord system about $\hat{x}$ by angle $-i$, so $\vec{r}'' = R_x(-i)\vec{r}'$

$$= R_x(-i)R_z(-\omega)\vec{r}$$



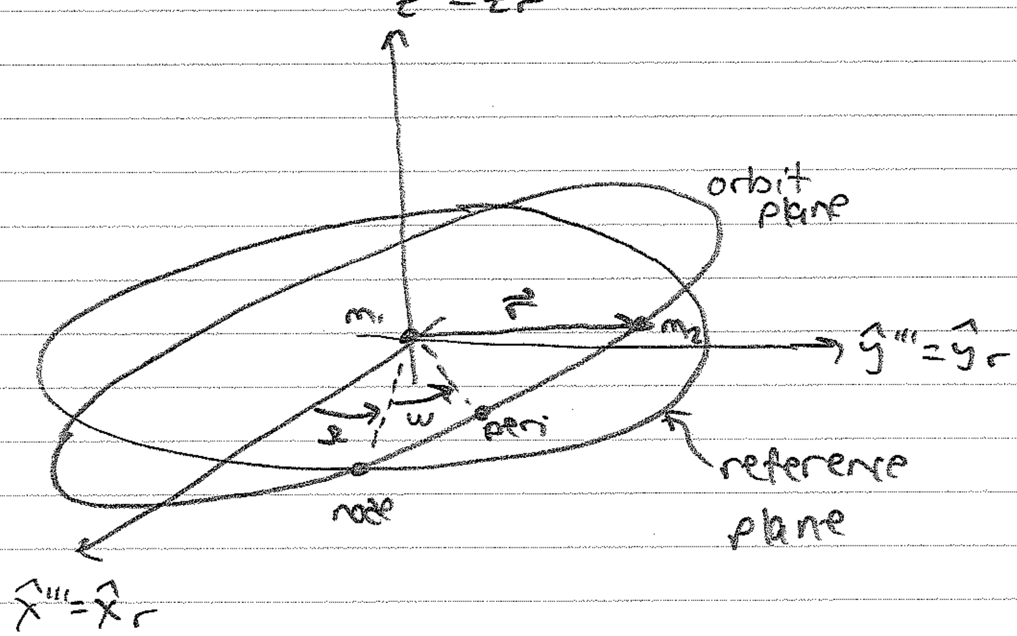
redraw the above with $\hat{x}''$-$\hat{y}''$ plane horizontal:



step 4: what's last rotation?

we need Ω = angle between $\hat{x}'''$ and ascending node

$\Rightarrow$ rotate about $\hat{z}''$ by angle $-\Omega$



$$\vec{r}_r = R_z(-\Omega)R_x(-i)\,R_z(-\omega)\vec{r}$$

$$\text{so}\quad \begin{pmatrix} x_r \\ y_r \\ z_r \end{pmatrix} = r\begin{pmatrix} \cos\Omega\cos(\omega+f) - \cos i\sin\Omega\sin(\omega+f) \\ \sin\Omega\cos(\omega+f) + \cos i\cos\Omega\sin(\omega+f) \\ \sin i\sin(\omega+f) \end{pmatrix}$$

which you will confirm in problem 2.12

where $\vec{r}_r = (x_r, y_r, z_r)$ = m_2's cartesian coordinates in the so-called 'reference' coordinate system.

Your choice of coordinate system depends on the orbit you are interested in.

If you are studying the motion of an Earth-orbiting satellite, then your $\hat{x}_r - \hat{y}_r$ plane is likely Earth's equatorial plane with $\hat{z}_r$ pointing to N pole

but if you are instead a JPL navigator and calculating a trajectory for a NASA probe to another planet, your $\hat{x}_r - \hat{y}_r$ plane is likely the ecliptic plane = mean plane of Earth's orbit about the sun.

but if you are a planetary scientist that has a camera onboard that probe, your camera's line-of-sight is likely specified by angles (α, δ) = (RA, DEC) that are measured wrt Earth's equatorial plane, because that is the coordinate system used by astronomers also, your spacecraft determines it's orientation wrt a map of bright stars whose coordinates α, δ are measured wrt Earth's equatorial plane

Velocities

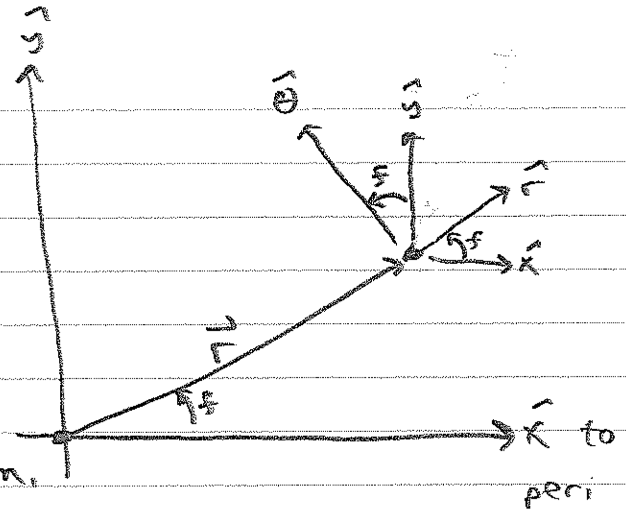
we will also need to convert m_2's velocity in the orbit-plane-coordinate system to the ref-plane-coord-sys:

recall $\vec{v} = \dot{r}\hat{r} + r\dot{f}\hat{\theta}$ = m_2's velocity in the orbit plane, where

$$\dot{r} = \frac{ean\sin f}{\sqrt{1-e^2}}$$

$$r\dot{f} = \frac{an}{\sqrt{1-e^2}}(1+e\cos f)$$

in polar coords,
but we need v_x, v_y, v_z



so $\hat{r} = \cos f\, \hat{x} + \sin f\, \hat{y}$

$\hat{\theta} = -\sin f\, \hat{x} + \cos f\, \hat{y}$

$$\text{so} \quad \vec{v} = \begin{pmatrix} \dot{r}\cos f - r\dot{f}\sin f \\ \dot{r}\sin f + r\dot{f}\cos f \\ 0 \end{pmatrix}$$

= m_2's velocity in cart. coords. in the orbit-plane coord. system

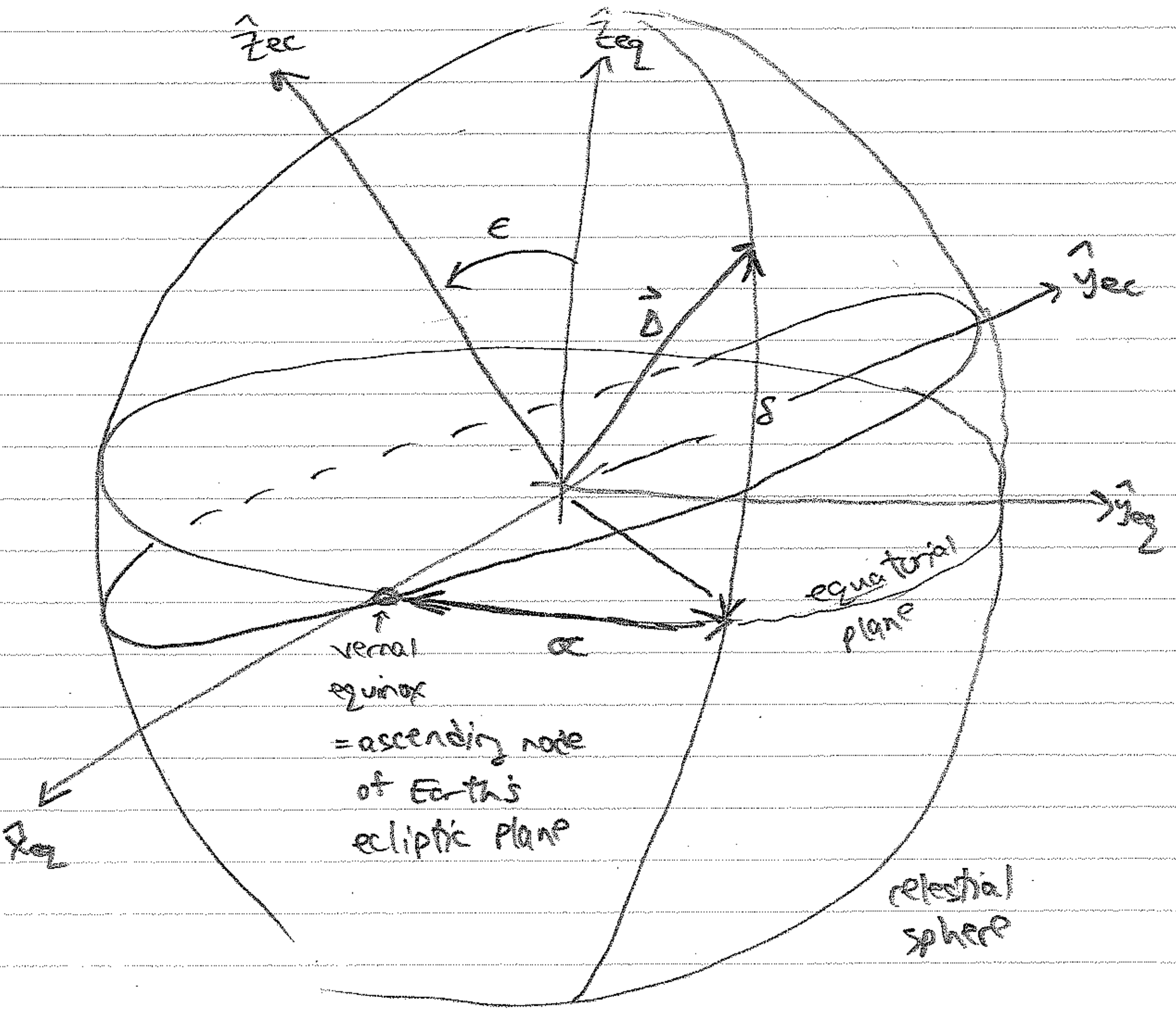
and $\vec{v}_r = R_z(-\Omega) R_x(-i) R_z(-\omega) \vec{v}$

= m_2's velocity in the reference coordinate system.

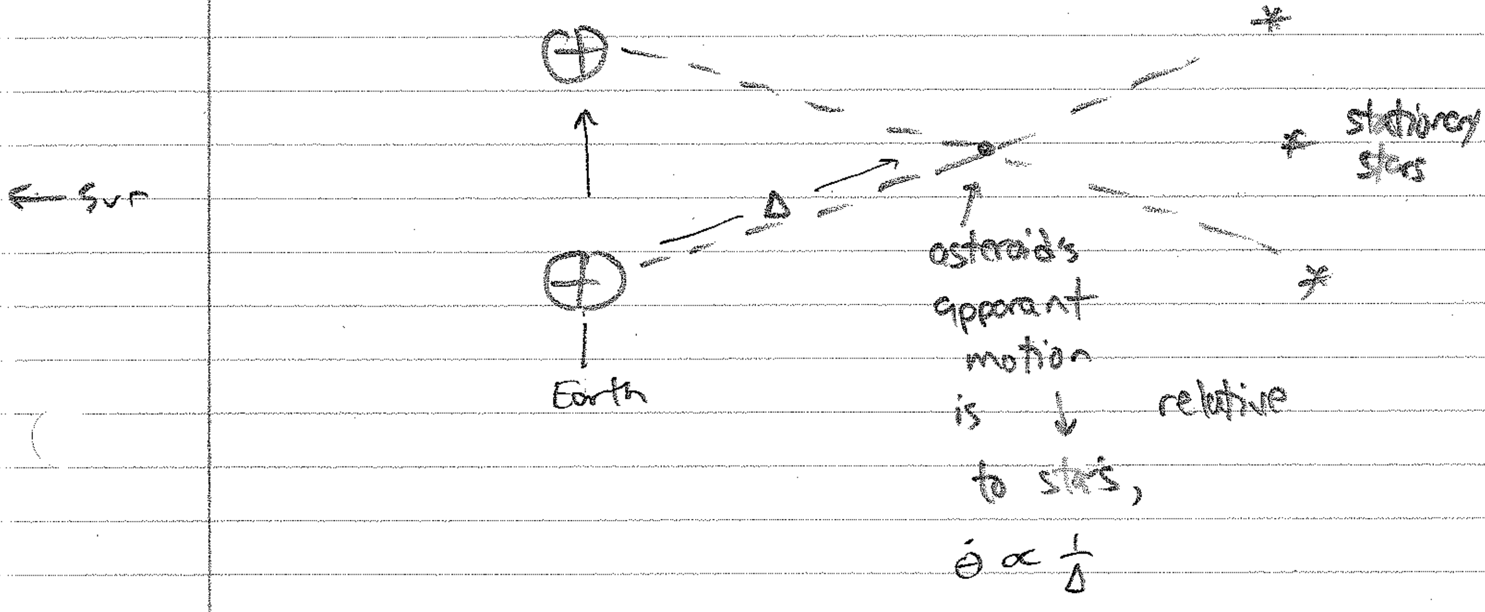
Ecliptic & Equatorial Coordinates

The ecliptic coord system has $\hat{x}_{ec}$-$\hat{y}_{ec}$ plane in Earth's orbit; this is the coord system ordinarily used when calculating spacecraft trajectories

But the equatorial coordinate system is Earth-centered, with an $\hat{x}_{eq}$-$\hat{y}_{eq}$ plane in Earth's equator. This is the coordinate system used by astronomers.



Suppose an astronomer discovers a new asteroid, that astronomer will report that objects α, δ over time, as well as its distance Δ from Earth (which you get by measuring the objects parallax = objects apparent angular motion due to Earth's motion about the sun



Suppose you want to calculate this new asteroids orbit elements. Its position vector in geocentric equatorial Cartesian coordinates is

$$\vec{\Delta} = \Delta \begin{pmatrix} \cos\delta \cos\alpha \\ \cos\delta \sin\alpha \\ \sin\delta \end{pmatrix} = \begin{pmatrix} x_{eq} \\ y_{eq} \\ z_{eq} \end{pmatrix}$$

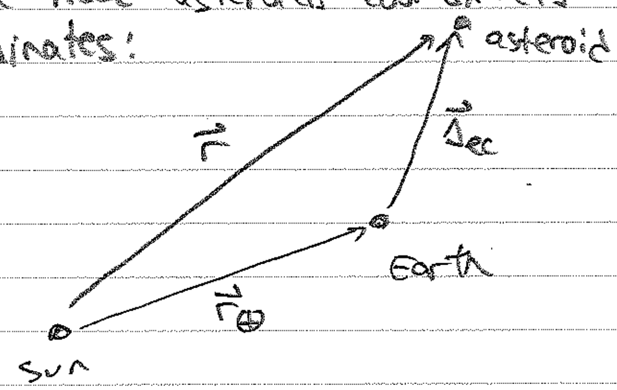
we want to rotate our coordinate system towards the ecliptic coordinate system ... which rotation should be used?

$$\vec{\Delta}_{ec} = R_x(+\epsilon)\vec{\Delta}$$

← only approximately true ... why?

= asteroid's coordinates in Earth-centered ecliptic coordinates

but we need asteroid's coordinates in Sun-centered coordinates!



$$\vec{r} = \vec{\Delta}_{ec} + \vec{r}_\oplus$$

↑ Earth's position vector in Sun-centered ecliptic coordinate system

You look up $\vec{r}_\oplus(t)$ using a <u>planetary ephemeris</u>; I like to use JPL's <u>Horizons</u> website for this

Generating orbit elements from reference plane coordinates.

let's assume the astronomer that discovered the new asteroid provided enough data to deduce the asteroid's position $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

and velocity $\vec{v} = \dot{\vec{r}} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$

(which is not always easy, esp. for distant, faint, slow-moving objects)

anyway, transform $\vec{r}, \vec{v} \rightarrow$ 6 orbit elements via following steps:

1. Use energy integral to get a:

$$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$\Rightarrow a = \left(\frac{2}{r} - \frac{v^2}{\mu}\right)^{-1} = \text{semimajor axis}$$

2. use ang. mom. integral to solve for e:

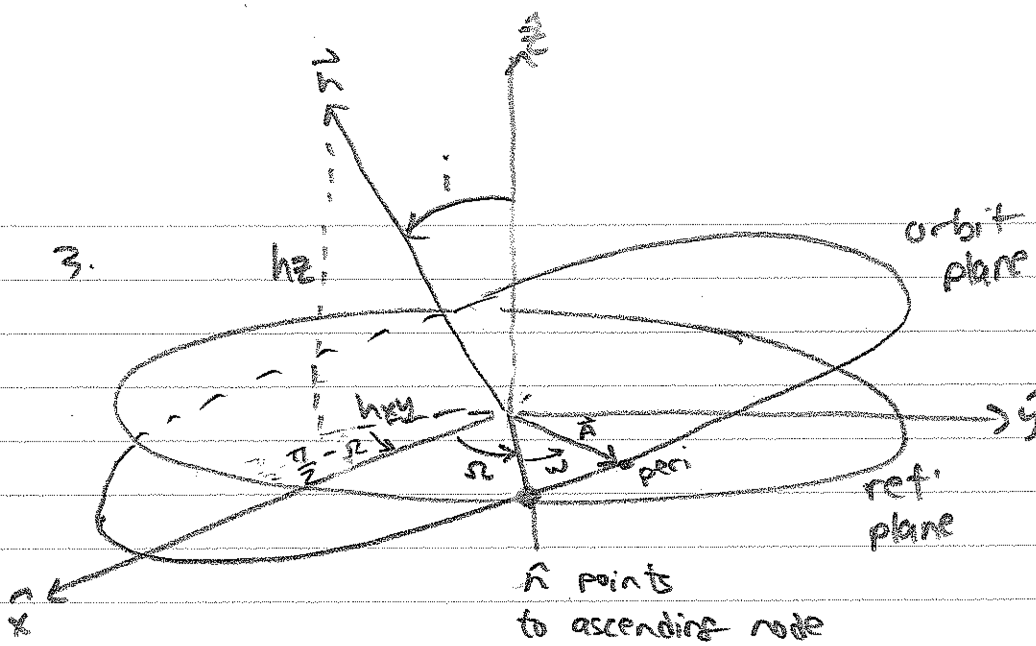
$$\vec{h} = \vec{r} \times \dot{\vec{r}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}$$

$$= \underbrace{(y\dot{z} - z\dot{y})}_{h_x}\hat{x} - \underbrace{(x\dot{z} - z\dot{x})}_{h_y}\hat{y} + \underbrace{(x\dot{y} - y\dot{x})}_{h_z}\hat{z}$$

where $h = |\vec{h}| = \sqrt{h_x^2 + h_y^2 + h_z^2} = \sqrt{\mu a(1-e^2)}$

so $e = \sqrt{1 - \frac{h^2}{\mu a}}$ eccentricity

3.

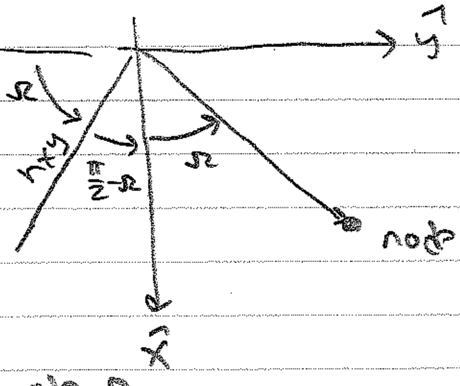


so $\cos i = \dfrac{h_z}{h}$ inclination i

4. note $h_{xy} = \sqrt{h_x^2 + h_y^2}$ = projection of $\vec{h}$ onto reference $\hat{x}$-$\hat{y}$ plane

so $h_x = h_{xy} \cos \Omega$

$h_y = -h_{xy} \sin \Omega$



so $\tan \Omega = \dfrac{-h_y}{h_x}$

← yields ascending node Ω

or $\Omega = \arctan(h_x, -h_y)$ in computer code

dont use $\Omega = \arctan(-h_x/h_y)$, result will be ambiguous by $\pm\pi$

5. recall Laplace-Runge-Lenz vector

$$\vec{A} = \frac{\dot{\vec{r}} \times \vec{h}}{\mu} - \hat{r}$$ points to periapse

and $|\vec{A}| = e$

set $\hat{n}$ = unit vector pointing towards ascending node: $\hat{n} = \cos\Omega\,\hat{x} + \sin\Omega\,\hat{y}$

also $|\vec{A} \times \hat{n}| = e \sin w$

$\vec{A} \cdot \hat{n} = e \cos w$

so $\tan w = \dfrac{|\vec{A} \times \hat{n}|}{\vec{A} \cdot \hat{n}}$ provide argument of periapse

6. if $a > 0$ then the orbit is an ellipse.

Recall $r = a(1 - e\cos E_c)$

so $e \cos E_c = 1 - \frac{r}{a}$

from notes pages 23-24: $n = \dfrac{r\dot{E}_c}{a}$

and $\dot{r} = ae\dot{E}_c \sin E_c$

$= ae\frac{a}{r} n \sin E_c$

so $e \sin E_c = \dfrac{r\dot{r}}{a^2 n} = \dfrac{\vec{r} \cdot \dot{\vec{r}}}{\sqrt{\mu a}}$

so $\tan E_c = \dfrac{e\sin E_c}{e\cos E_c} = \sqrt{\dfrac{a}{\mu}}\ \dfrac{\vec{r}\cdot\dot{\vec{r}}}{a-r}$ yield eccentric anomaly E_c

7. mean anomaly M

$$M = E_c - e\sin E_c = n(t-\tau)$$

yields τ

or τ = time of periapse passage.