engineers?
astro?
geology?
undergrads?

28 August 2013

Lecture Notes
for ASE 396
Dynamics of Planetary Systems

go over syllabus.

office hours, when? how? skype, google+?

textbook:

Dynamics of Planetary Systems, ~70% done
will make chapters available via canvas

if you spot typos, errors, confusing text,
send me an email, I want to know!

In this class, planetary dynamics (ex 2-body problem, 3-body prob, planetary rings & satellites, extra-solar planets, etc) = methods of classical mechanics applied to planetary environments. So begin with a quick review of classical mech.

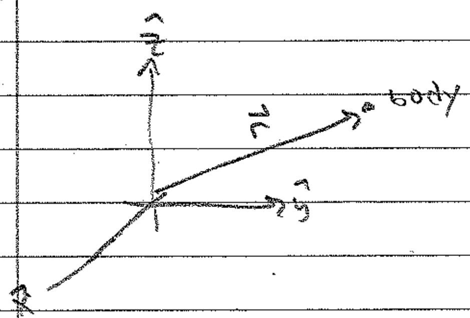
This lecture reviews the relevant parts of classical mechanics that are usually taught to 3rd year physics & astro undergrads. If you see something unfamiliar here, please get a textbook and review that material! I like Thornton & Marion's text, Classical Dynamics of Particles and Systems

Everything derived in this class follows from

Newton's Laws of Motion:

I. A body remains at rest or in uniform motion unless acted upon by a force, ie

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \text{constant} \quad \text{when force } \vec{F} = 0$$



$\vec{r}$ = body's position vector

$\dot{\vec{r}}$ = its velocity

$\ddot{\vec{r}}$ = its acceleration

II. a body subject to force $\vec{F}$ will have its momentum $\vec{p}$ changed at the rate

$$\dot{\vec{p}} = \vec{F} \quad \text{where} \quad \vec{p} = m\dot{\vec{r}}$$

body's mass — velocity

this is Newton's familiar $\vec{F} = m\ddot{\vec{r}}$ law

III. if 2 bodies exert forces on each other,

~~$\vec{F}_{12}$ = force that body 1 exerts on 2~~

~~$\vec{F}_{21}$ = force that #2 exerts on #1~~

then these forces are equal in magnitude and opposite in direction, ie

$$\vec{F}_{12} = -\vec{F}_{21}$$

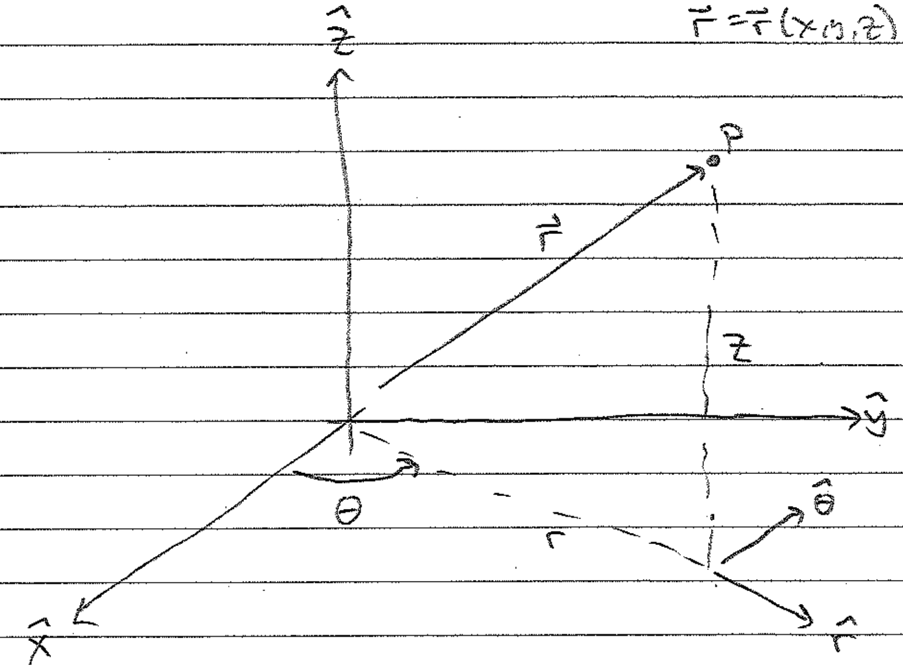
Reference Frames:

reference frame = coordinate grid against which a particle's position, velocity $\vec{r}, \dot{\vec{r}}$ is measured.

Newton's Laws are valid in an inertial reference frame, and Law I implies that an inertial frame can be stationary or moving with constant velocity.

This class will make use of Cartesian and cylindrical coordinate systems:

$\vec{r} = \vec{r}(x,y,z) = \vec{r}(r,\theta,t)$



particle P has position vector $\vec{r}$

in Cartesian coordinates, $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

P's x coordinate

unit vector $\hat{x}$, points in x-direction, has length = 1 in cylindrical coordinate system, $\vec{r} = r\hat{r} + z\hat{z}$

here the unit vector $\hat{r}$ is always confined to the $\hat{x}$-$\hat{y}$ plane.

and length $r = \sqrt{x^2+y^2}$ is the length of $\vec{r}$ when projected to the $\hat{x}$-$\hat{y}$ plane.

This differs from $|\vec{r}|$ = total length of $\vec{r}$
$= \sqrt{x^2+y^2+z^2}$

particle p's velocity in Cartesian coordinates is

$\dot{\vec{r}} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$ by chain rule,
noting that $\hat{x}$ etc
are <u>constant</u> unit vectors

and acceleration $\ddot{\vec{r}} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$

but in cylindrical coordinates,
the $\hat{r}, \hat{\theta}$ vectors are not static,
They change direction as particle moves

so velocity $\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}$

and acceleration $\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\theta} + \ddot{z}\hat{z}$

(see classical mechanics textbook for proof, such as section 1.14 in Thornton & Marion)

we will return to this eqn. when we tackle the 2-body problem

Useful
Conservation Laws

Conservation of linear momentum:

NII says that $\dot{\vec{p}} = 0$ when total force $\vec{F}$ on particle is $\vec{F} = 0$

so $\vec{p} = m\dot{\vec{r}}$ is conserved (ie constant) when $\vec{F} = 0$

Conservation of Angular Momentum:

the particle's angular momentum is

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \dot{\vec{r}}$$

vector cross product, see appendix A.18 to evaluate.

torque $\vec{T}$ on particle = rate at which $\vec{L}$ changes:

$$\vec{T} = \frac{d\vec{L}}{dt} = \underbrace{m\dot{\vec{r}} \times \dot{\vec{r}}}_{\text{zero}} + m\vec{r} \times \ddot{\vec{r}}$$

so torque $\vec{T} = \dot{\vec{L}} = \vec{r} \times \vec{F}$

$\Rightarrow$ angular momentum is conserved when

i) $\vec{F} = 0$

or ii) when $\vec{F} \propto \hat{r}$ ie force is radial

so $\vec{r} \times \vec{F} = 0$

This is why $\vec{L}$ is conserved in the 2-body problem (eg, star-planet system), because gravity $\vec{g} \propto \hat{r}$

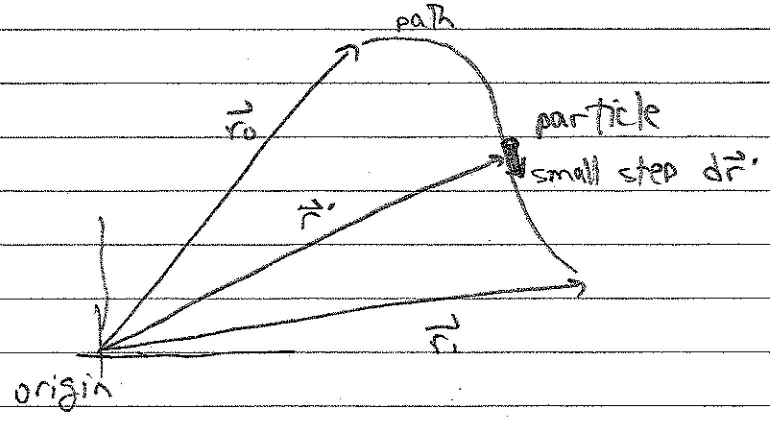
we will invoke $\vec{L}$ conservation often in this class

<u>Work</u> = energy that force $\vec{F}$ deposits on particle P as it moves from $\vec{r}_0$ to $\vec{r}_1$.

$dW = \vec{F} \cdot d\vec{r}\,'$ = tiny bit of work done on particle as it is pushed small distance $d\vec{r}\,'$

vector dot product, A.17

Thus $W = \int_{r_0}^{\vec{r}_1} \vec{F} \cdot d\vec{r}\,'$ = total work done on particle as it travels $\vec{r}_0 \to \vec{r}_1$, by whatever is creating force $\vec{F}$ (such as a star that exerts gravitational pull on orbiting planet)



whoups

Also note that $dW = m\ddot{\vec{r}}\cdot d\vec{r} = m\ddot{\vec{r}}\cdot \frac{d\vec{r}}{dt}\,dt = m\ddot{\vec{r}}\cdot\dot{\vec{r}}\,dt$

$$= \frac{1}{2}m\,d(\dot{\vec{r}}\cdot\dot{\vec{r}}) = \frac{1}{2}m\,d(v^2)$$

where $v^2 = \dot{\vec{r}}\cdot\dot{\vec{r}}$ = speed^2

so total work $W = \frac{1}{2}m\int_{\vec{r}_0}^{\vec{r}_1} d(v^2) = \frac{1}{2}mv^2\Big|_{v(\vec{r}_0)}^{\vec{v}(\vec{r}_1)}$

so $W = \frac{1}{2}m(v_1^2 - v_0^2) = T_1 - T_0$

$\Rightarrow$ work done on particle by force $\vec{F}$
= change in particle's kinetic energy $T_1 = \frac{1}{2}mv_1^2$
as it moves from $\vec{r}_0 \rightarrow \vec{r}_1$

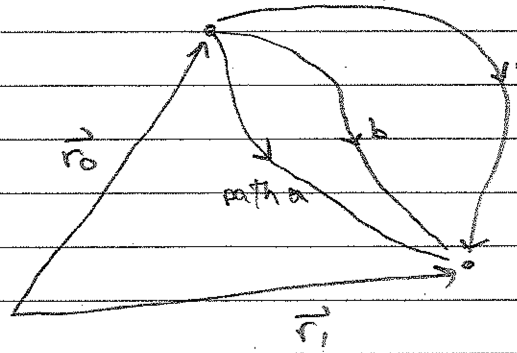
we might also need to know the rate at which force $\vec{F}$ changes the particle's kinetic energy;

$$P = \frac{dW}{dt} = m\ddot{\vec{r}}\cdot\dot{\vec{r}}$$

= power delivered to particle by $\vec{F}$.

We are largely concerned with conservative forces = force where work W is independent of the particle's path:

when $W = \int_{\vec{r}_0}^{\vec{r}_1} \vec{F} \cdot d\vec{r}\,'$



= same when particle takes path a, b, or c

then force $\vec{F}$ is conservative

When W is path independent, and $\vec{F}$ is conservative then $\vec{F}$ can be expressed as the gradient of a scalar function $U(\vec{r})$ that is a function of position $\vec{r}$ only:

$$\vec{F} = -\nabla U$$

where U = system's potential energy

$$\nabla U = \frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z} \quad \text{in Cartesian coords}$$

$$= \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta} + \frac{\partial U}{\partial z}\hat{z} \quad \text{in cylindrical}$$

so $F_x = -\frac{\partial U}{\partial x}$ = $\hat{x}$-component of force on particle

$F_\theta = -\frac{1}{r}\frac{\partial U}{\partial \theta}$ = azimuthal force, etc.

$$W = \int_{\vec{r}_0}^{\vec{r}_1} \vec{F} \cdot d\vec{r}$$

so work $W = -\int_{\vec{r}_0}^{\vec{r}_1} \nabla U \cdot d\vec{r}$

where $U(\vec{r})$ = function of particle's trajectory $\vec{r}(t)$
= particle's path over time

use chain rule: $\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt}$

$$= (\nabla U) \cdot \frac{d\vec{r}}{dt}$$

$\Rightarrow dU = (\nabla U) \cdot d\vec{r}$ = small change in particle's potential energy resulting from travelling small distance $d\vec{r}$

so $W = -\int_{\vec{r}_0}^{\vec{r}_1} dU = -U \Big|_{\vec{r}_0}^{\vec{r}_1} = -(U_1 - U_0)$

where $U_1 = U(\vec{r}_1)$ = particles' potential energy when at position $\vec{r}_1$

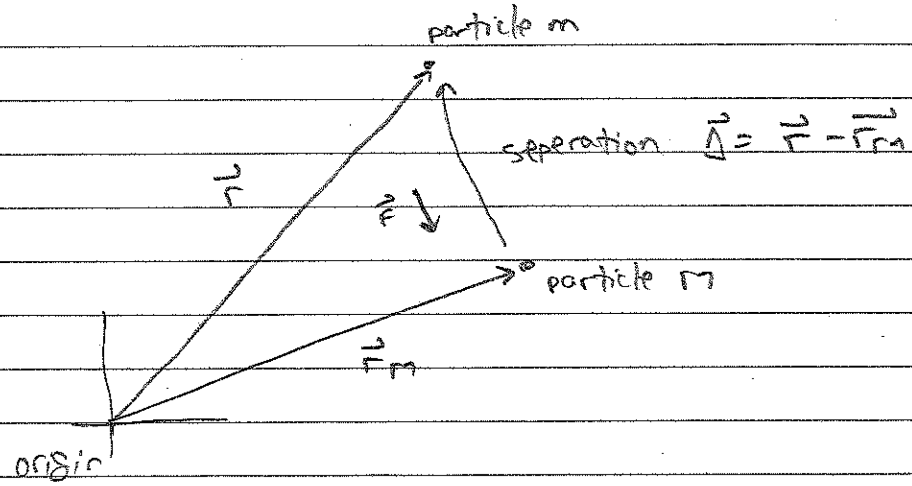
so $W = T_1 - T_0 = -U_1 + U_0$

$\Rightarrow E_1 = T_1 + U_1 = T_0 + U_0 = E_0$

The particle's energy $E = T + U$ is conserved when acted upon by a conservative force.

conservative forces (such as gravity) are frictionless (ie $\vec{F}$ is independent of $\dot{\vec{r}}$) and do not have any explicit time dependence.

<u>example</u>: calculate the potential energy U of 2 gravitating particles:



Newton's Law of gravity

$$\vec{F} = \text{force on } m \text{ due to } M = \underbrace{\frac{GMm}{\Delta^2}}_{\substack{\text{magnitude} \\ \text{of } \vec{F}}} \underbrace{(-\hat{\Delta})}_{\substack{\text{direction,} \\ \vec{F} \text{ pulls } m \text{ towards } M}} = -\frac{GMm(\vec{r} - \vec{r}_M)}{|\vec{r} - \vec{r}_M|^3}$$

G = gravitation constant

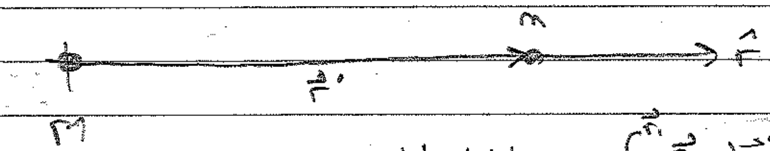
M = mass of <u>source</u> body (M is source of force $\vec{F}$, assumed to reside at fixed position $\vec{r}_M$)

m = mass of field particle, the moving particle of interest

simplify by placing origin on the source mass M,
so $\vec{r}_M = 0$ and

$$\vec{F} = -\frac{GMm}{r^2}\,\hat{r}$$

where $\hat{r}$ points from M to m:



$$W = -U_1 + U_0 = \int_{\vec{r}_0}^{\vec{r}_1} \vec{F}\cdot d\vec{r}'$$

recall $\quad U_1 - U_0 = -W = -\int_{\vec{r}_0}^{\vec{r}_1} \vec{F}(\vec{r}')\cdot d\vec{r}'$

replace $\vec{r}_1 \to r$

call $\vec{r}_0$ = reference site ← This reference site can be anywhere, u get to choose.

so $U_0 = U(r_0)$ = system's potential energy when m is at $\vec{r}_0$

$$\text{so} \quad U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}\cdot d\vec{r}' + U_0$$

what's a convenient choice for $\vec{r}_0$ and $U_0(\vec{r}_0)$?

put $\vec{r_0}$ out at infinity where $\vec{F} = 0$
and $V_0 = 0$

also $d\vec{r}\,' = dr'\,\hat{r}$ so $\vec{F}(\vec{r}\,')\cdot d\vec{r}\,' = -\frac{GMm}{r'^2}\,dr'$

so $$U(r) = +\int_{\infty}^{r} \frac{GMm}{r'^2}\,dr' = -\frac{GMm}{r'}\Big|_{\infty}^{r} = -\frac{GMm}{r}$$

this is the familiar potential energy
for gravitating 2-body problem

Note that if we had chosen an alternate
reference site $\vec{r_0}$ then

$$U(r) = -\frac{GMm}{r} + C(\vec{r_0})$$

some constant
that depends
on site $\vec{r_0}$

but C is unimportant since dynamics
is governed by

$$\vec{F} = -\nabla U = -\frac{GMm}{r^2}\,\hat{r}$$

<u>The potential</u> = potential energy per unit mass

$$\Phi(\vec{r}) = \frac{U}{m} = -\frac{GM}{r}$$

suppose we are interested in the motion of a zero-mass test particle orbiting star M

so $m=0$ and $U=0$!

So U is not a useful quantity here.

however the star's potential $\Phi(\vec{r}) = -\frac{GM}{r}$ is still useful

and since $\vec{F} = m\ddot{\vec{r}} = -\nabla U = -\nabla(m\Phi)$

$$\Rightarrow \ddot{\vec{r}} = -\nabla\Phi$$

is Newton's 2^{nd} Law again,

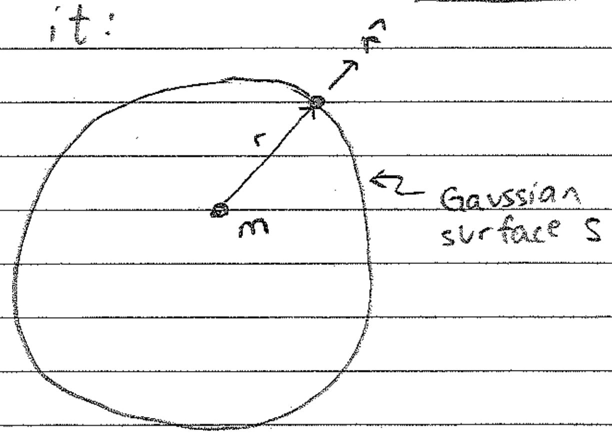
and is the principle equation that we will solve in this class.

Now derive <u>Gauss' Law</u>

which you might recall from your E&M class,

Gravity and the Coulomb force laws have the same form, so we also use GL in planetory dynamics.

start with a gravitating point mass m, and place an imaginary <u>Gaussian surface</u> S around it:



The gravitational acceleration at a point on that surface is

$$\vec{g} = -\nabla\Phi = -\frac{Gm}{r^2}\hat{r}$$

Let $d\vec{a} = da\,\hat{n}$ = small patch of area da on surface S, where $\hat{n}$ is the unit vector that is normal to da, $\hat{n}$ tells you how area $d\vec{a}$ is oriented

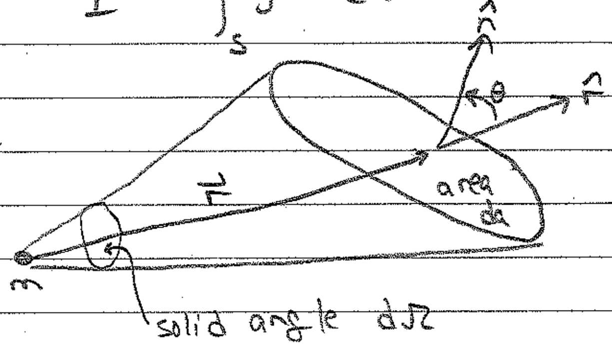
The gravitational flux through $d\vec{a}$ is $\vec{g} \cdot d\vec{a}$

(this is analogous to electrostatic flux)

The total gravitational flux Ψ through surface S is

$$\Psi = \int_S \vec{g} \cdot \hat{n} \, da \quad \text{where} \quad \underset{\uparrow}{\vec{g} \cdot \hat{n}} = \frac{-Gm}{r^2} \cos\theta$$

this dot product = $g n \cos$



$$\text{so} \quad \Psi = -Gm \int_S \cos\theta \, da / r^2$$

Note that $\cos\theta \, da$ = projected area of da as seen by observer at m

so $d\Omega = \cos\theta \, da / r^2$ = solid angle that da subtends

$$\text{so} \quad \Psi = -Gm \underbrace{\int_S d\Omega}_{\substack{\text{solid angle} \\ \text{of sphere} = 4\pi}} = -4\pi G m$$

Note that S can have <u>any</u> shape,
it doesn't have to be a <u>sphere</u>,
$\int_S d\Omega = 4\pi$ regardless

and if S contains multiple masses m_i

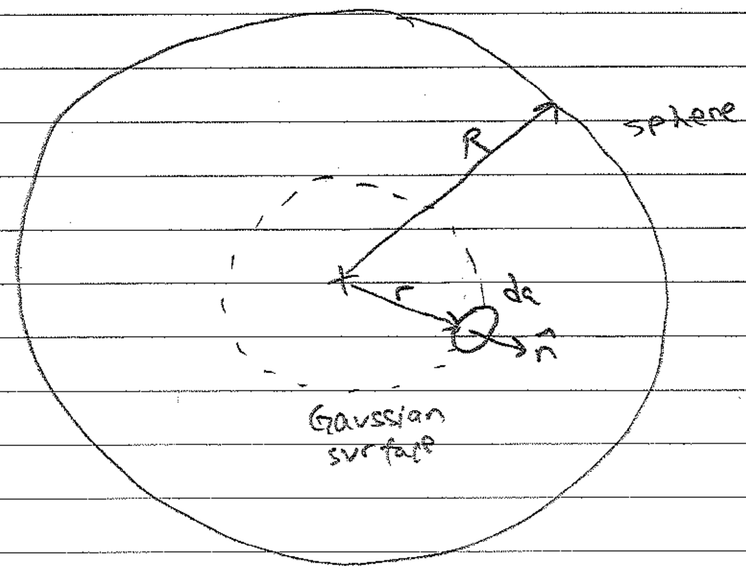
then $m \to \sum_i m_i = M_{enc}$

and gravitational flux $\Psi = \int_S \vec{g} \cdot \hat{n}\, da = -4\pi G M_{enc}$

The following example shows that Gauss' Law provides a handy way to calculate $\vec{g}$ for bodies that have lots of symmetry:

example: a sphere of radius R has constant density ρ. Calculate it's potential $\Phi(\vec{r})$.

The body is spherical, so $\vec{g} = g(r)\hat{r}$, and use a spherical Gaussian surface:



The surface normal $\hat{n} = \hat{r}$ so

$$\Psi = \int_S \vec{g} \cdot \hat{r}\, da = \int g(r)\, da = g(r)\, 4\pi r^2 = -4\pi G M_{enc}$$

where mass enclosed by S $= M_{enc}(r) = \begin{cases} \frac{4\pi}{3}\rho r^3 & \text{when } r < R \\ \frac{4\pi}{3}\rho R^3 \equiv M & \text{when } r \geq R \end{cases}$

so $g(r) = -\dfrac{G M_{enc}(r)}{r^2} = \begin{cases} -\frac{4\pi}{3}\rho r & \text{inside sphere at } r < R \\ = -\frac{GM}{r^2} & \text{outside at } r \geq R \end{cases}$

as expected

But we want the sphere's gravitational potential

$\vec{F}/m = \vec{g} = \text{acceleration}.$

$$\Phi(\vec{r}) = \frac{U}{m} = -\int_{\vec{r}_0}^{\vec{r}} \frac{F(\vec{r}\,')}{m} \cdot d\vec{r}\,'$$

m = mass of some small test particle

so $\Phi(\vec{r}) = -\displaystyle\int_{\vec{r}_0}^{\vec{r}} \vec{g}(\vec{r}\,') \cdot d\vec{r}\,'$

reference point is arbitrary, so set $\vec{r}_0 \rightarrow$ infinity

lecture #1 stopped here

so $\Phi(r>R) = \underset{\text{sphere}}{\text{potential external to}} = -\int_{\infty}^{r}\left(-\frac{GM}{r'^2}\right)dr' = -\frac{GM}{r}$

= potential of point mass, as expected

and $\Phi(r\leq R) = \underset{\text{sphere}}{\overset{\text{sphere's gravitational}}{\text{potential, inside the}}} = -\int_{\infty}^{r} g(r')\,dr'$

$$= -\int_{\infty}^{R} \underbrace{g(r'>R)}_{\text{external}}\,dr' - \int_{R}^{r} \underbrace{g(r'<R)}_{\text{internal}}\,dr'$$

$$= -\frac{GM}{R} + \frac{2\pi}{3}\rho\left(r^2-R^2\right)$$

<u>Poisson's Equation</u> = differential form of Gauss' Law

suppose Gaussian surface S instead encloses a distribution of matter whose density is $\rho(\vec{r})$. Then

$$M_{enc} = \int_V \rho(\vec{r})\, dV$$

V = volume enclosed by S

$$\text{so} \quad \Psi = \int_S \vec{g}\cdot\hat{n}\, da = -4\pi G M_{enc} = -4\pi G \int_V \rho(\vec{r}')\, dV$$

invoke the divergence theorem of vector calculus, Eqn (A.24a), which converts the area integral into a volume integral

$$\Psi = \int_S \vec{g}\cdot\hat{n}\, da = \int_V \nabla\cdot\vec{g}\, dV$$

V volume that encloses surface S

but $\vec{g} = -\nabla\Phi$ so $\nabla\cdot\vec{g} = -\nabla\cdot(\nabla\Phi) = \underbrace{-\nabla^2\Phi}_{\text{Laplacian of }\Phi}$

$\vec{g}$: grav. acceleration due to ρ

Φ: gravitational potential due to ρ

$$\text{so} \quad \Psi = -\int_V \nabla^2\Phi\, dV = -\int_V 4\pi G\rho\, dV$$

$$\text{so} \quad \int_V \left(\nabla^2\Phi - 4\pi G\rho\right) dV = 0$$

This integral must be zero for any arbitrary volume V, which tells us That the integrand itself must be zero, so

$$\nabla^2 \Phi = 4\pi G \rho \quad \text{is Poisson's Eqn}$$

it relates matter distribution $\rho(\vec{r})$ to its gravitational potential $\Phi(\vec{r})$,

This equation is widely used in astrophysical dynamics, like formation of stars and galaxies, and to study gravitational instabilities.

Laplace's Equation:

in free space where $\rho = 0 \Rightarrow \nabla^2 \Phi = 0$

Assignment #1: problems 1.2, 1.3, 1.4, 1.5, 1.6 from text chapter 1 go to canvas, click FILES on left

due? Tues Sept 10?

S.G spotted error in my notes & lecture:

in derivation of $W = T_1 - T_0$. The corrected derivation is:

$$dW = m\ddot{\vec{r}}\cdot d\vec{r} = \text{work done on } m \text{ during step } d\vec{r}$$

$$= m\ddot{\vec{r}}\cdot\frac{d\vec{r}}{dt}\,dt = m\ddot{\vec{r}}\cdot\dot{\vec{r}}\,dt$$

$$= \tfrac{1}{2}m\, d(\dot{\vec{r}}\cdot\dot{\vec{r}}) = \tfrac{1}{2}m\, d(v^2)$$

I think I had extra dt in earlier expression

so total work $W = \frac{1}{2}m\int_{\vec{r}_0}^{\vec{r}_1} d(v^2) = \frac{1}{2}m(v_1^2 - v_0^2)$

$= T_1 - T_0$

where $T_i = \frac{1}{2}mv_i^2 = KE$ at site $\vec{r}_i$

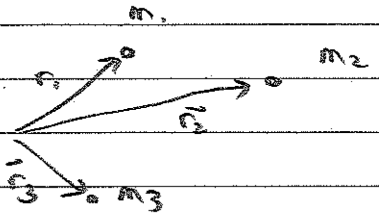
Thanks SG!

If you spot an error in class, please alert me!

$\boxed{\text{1 September 2013}}$

continuing the review of classical mechanics:

suppose we have a system of N particles having masses m_j and positions $\vec{r}_j$



we could calculate say the system's center of mass

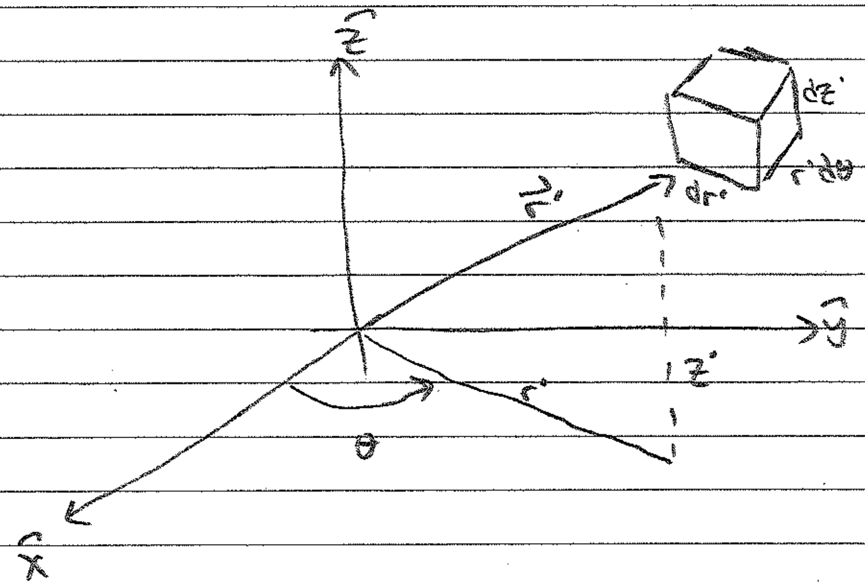
$$\vec{R} = \frac{1}{M} \sum_{j=1}^{N} m_j \vec{r}_j$$

where $M = \sum_{j=}^{N} m_j$ = total mass what if the system is instead a continuous blob of matter, rather than discreet particles?

replace sum → integral over volume elements

$m_j \to \rho(\vec{r}')\, dV'$ ← small volume

$\vec{R} \to \frac{1}{M}\int_V \rho(\vec{r}')\, dV'$ where $dV' = dx'\,dy'\,dz'$ in cartesian coords



but in cylindrical coords

$dV' = r'\,dr'\,d\theta'\,dz'$

2D objects (eg plane or shell) are handled similarly:

$$\text{mass element } \rho(\vec{r}')dV' \rightarrow \underbrace{\sigma(\vec{r}')}_{\text{mass surface density}}\ \underbrace{dA}_{\text{area element}}$$

ex: thin flat disk has radius R
constant surface density σ,
and is in Keplerian rotation about its center,
with Ω = angular velocity at its outer edge.

Calculate the disk's total angular momentum

When we solve the 2-body problem,
we will learn that Keplerian rotation
means angular velocity varies as $\dot{\theta} \propto r^{-3/2}$

so $\dot{\theta}(r) = \Omega \times (r/R)^{-3/2}$ = disk's angular velocity

disk's motion is azimuthal (ie radial $\dot{r}=0$)

so $\dot{\vec{r}} = r\dot{\theta}\,\hat{\theta}$

the mass of an area element in this disk
is $dm = \sigma\, dA$

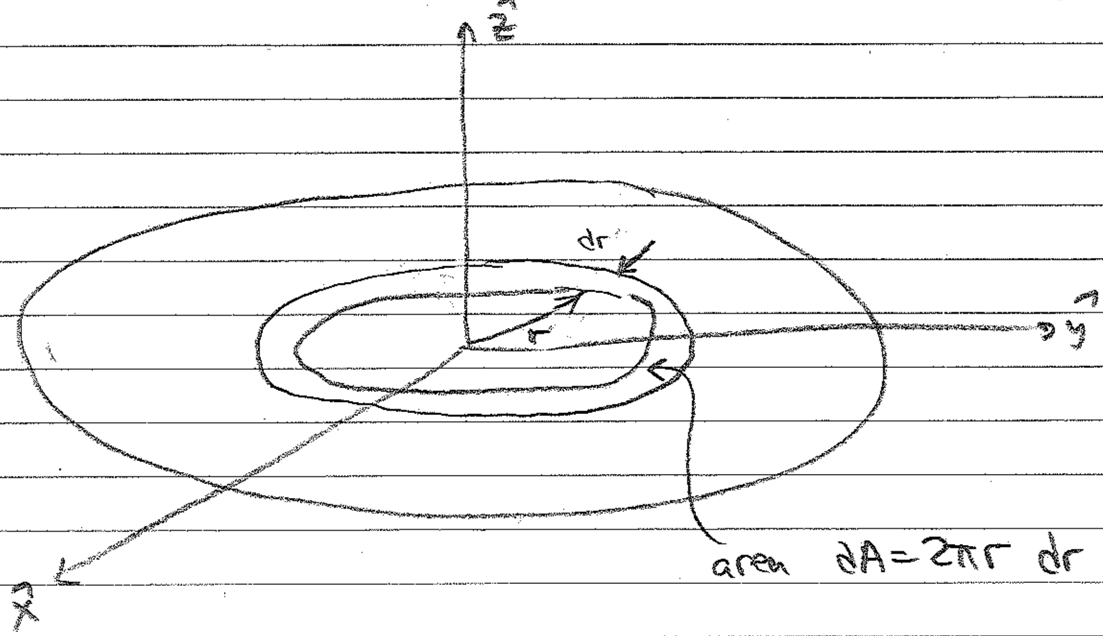
so $d\vec{L} = dm\, \vec{r} \times \dot{\vec{r}}$ = angular momentum
$= \sigma\, dA\, r^2 \dot{\theta}\, \hat{z}$ of area dA
in disk

we want $\vec{L} = \int d\vec{L} = \hat{z} \int_A \sigma r^2 \dot{\theta}\, dA$

= disk's total angular moment

what next?

the easiest way to evaluate this integral
is to split the disk up into concentric rings:



so $\vec{L} = \hat{z} \int_0^R \sigma r^2 \Omega \left(\frac{r}{R}\right)^{3/2} 2\pi r \, dr$

$$= 2\pi\sigma\Omega R^{3/2} \hat{z} \underbrace{\int_0^R r^{3/2} \, dr}_{\frac{2}{5} R^{5/2}}$$

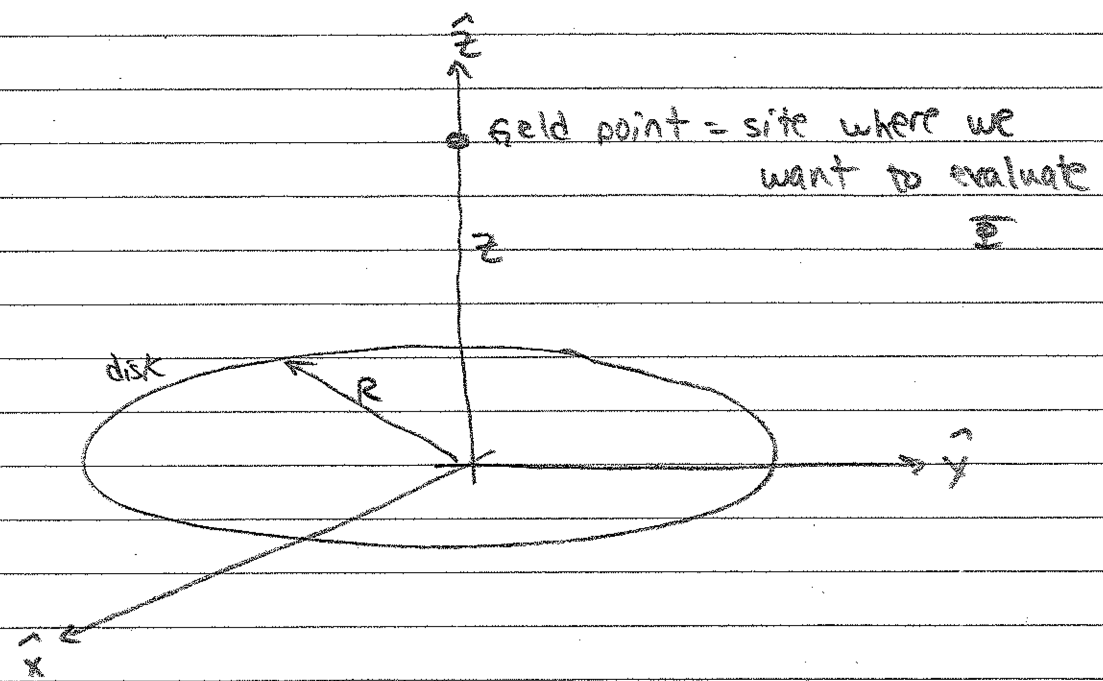
so $\vec{L} = \frac{4}{5}\pi\sigma R^4 \Omega \hat{z}$

dont forget $\vec{L}$ is a vector!

check your units: $L \sim (\sigma R^2) \cdot R \cdot R\Omega$

$\sim$ mass $\times$ r $\times$ velocity ✓

ex: calculate the disk's gravitational potential a distance z above/below its center:

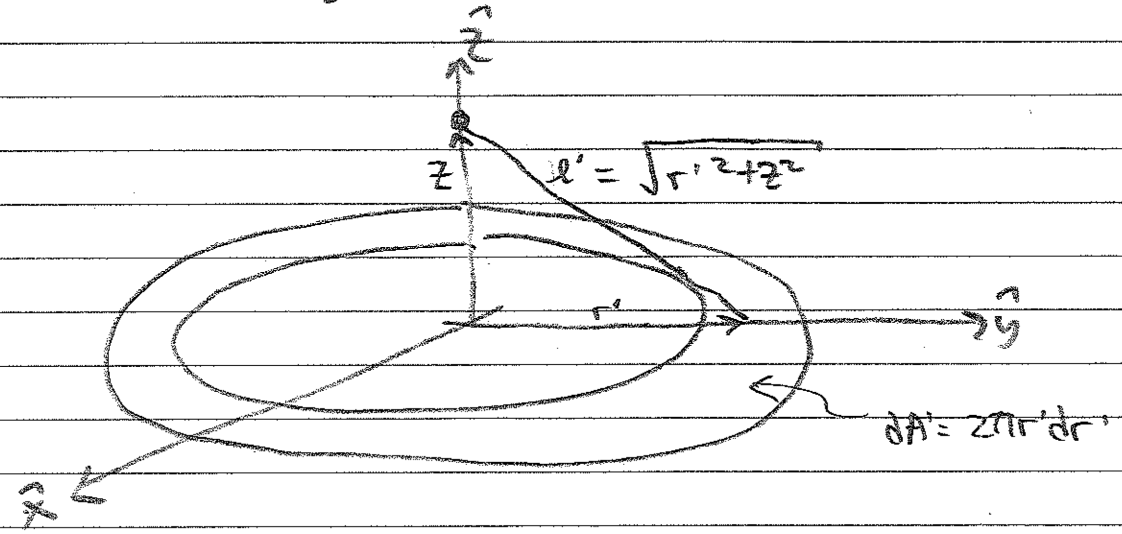


$d\Phi = -\frac{G\,dm'}{r'}$ = gravitational potential of small mass $dm' = \sigma\, dA'$ that lies r' away from field point.

whats the best way to break up this disk into numerous areas dA'?

ie whats the easiest way to solve the integral $\Phi = -\int_A \frac{G\sigma\, dA'}{r'}$

break the disk into annuli that are everywhere equidistant from z:



$$d\Phi = -\frac{G\sigma\, 2\pi r' dr'}{\sqrt{r'^2+z^2}}$$ = grav pot. due to single annulus in disk

so $$\Phi = -2\pi G\sigma \int_0^R \frac{r' dr'}{\sqrt{r'^2+z^2}}$$

solve by u-substitution: $u = \sqrt{r'^2+z^2}$

$$du = \tfrac{1}{2}(r'^2+z^2)^{-1/2}\, 2r' dr'$$

so $$\Phi = -2\pi G\sigma \int_{|z|}^{\sqrt{R^2+z^2}} du = -2\pi G\sigma \left[\sqrt{R^2+z^2} - |z|\right]$$

note the abs value

ex: use earlier result to calculate Φ

for a sheet that extends to infinity

How?

Calculate Φ in the limit that $R \gg |z|$

first rewrite $\Phi = -2\pi G\sigma R\left[\sqrt{1+\left(\frac{z}{R}\right)^2} - \frac{|z|}{R}\right]$

since $(z/R)^2$ is small, use the binomial

expansion, A.1:

$(1+x)^r \simeq 1 + rx + \frac{1}{2!}r(r-1)x^2 + \cdots$ for small $|x|$

so $\left[1+\left(\frac{z}{R}\right)^2\right]^{1/2} \simeq 1 + \frac{1}{2}\left(\frac{z}{R}\right)^2 + O\left(\frac{z}{R}\right)^4$

so when $R \gg |z|$,

$\Phi \simeq -2\pi G\sigma R + 2\pi G\sigma|z|$ + other term smaller by factor of $\left|\frac{z}{R}\right|$

does this term affect the system's dynamics?

ignoring the unimportant constant term,

$\Phi = 2\pi G\sigma|z| =$ grav. potential a distance $|z|$ from large (ie infinite) sheet of matter

ex: calculate acceleration of test particle a distance z from the infinite plane

recall $\ddot{\vec{r}} = -\nabla \Phi = -\frac{\partial}{\partial z} 2\pi G\sigma s_z z \, \hat{z}$

where $|z| = s_z z$

so $s_z = \mathrm{sgn}(z) = \pm 1$

so $\ddot{\vec{r}} = -s_z 2\pi G\sigma \hat{z}$

need to keep track of signs to get correct answer, ie particle is drawn to sheet